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Coefficient Bounds for Subclasses of Analytic and Bi-Univalent Functions

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Abstract. In this paper, various coefficient bounds of functions in some classes which are defined by subordination are estimated. Some special consequences of the main results are also presented. Moreover, it is pointed out that the given bounds improve and generalize some of the pervious results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.

Using the concept of subordination, Ma and Minda [24] introduced the subclasses of starlike and convex functions in which either of the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. To this goal, they defined an analytic univalent function φ with positive real part in $\mathbb{U}$, $\varphi(\mathbb{U})$ is symmetric with respect to the real axis and starlike with the conditions $\varphi(0) = 1$ and $\varphi'(0) > 0$. They defined the classes consisting of several well-known classes as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathbb{U} \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\ z \in \mathbb{U} \right\},$$

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where $\prec$ stands for the usual subordination for analytic functions in $\mathbb{U}$. For example, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ for $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ reduce to the classes $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$ of Janowski starlike and Janowski convex functions, respectively. Note that if $0 \leq \alpha < 1$, then $\mathcal{S}^*[1 - 2\alpha, -1] =: \mathcal{S}^*(\alpha)$, the classes of starlike functions of order α and $\mathcal{K}[1 - 2\alpha, -1] =: \mathcal{K}(\alpha)$ the convex functions of order α. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the well-known classes of starlike functions and of convex functions in $\mathbb{U}$, respectively.

Setting $\varphi(z) = \sqrt{1+z}$ we get the class $\mathcal{S}^*_l$ of consisting of functions f such that $w = zf'(z)/f(z)$ lies in the region bounded by the right half of the lemniscate of Bernoulli given by $\left|w^2 - 1\right| < 1$ which was investigated by Sokół and Stankiewicz [32]. In [31], Sokół generalized this class by introducing a more general class $\mathcal{S}^*_{l_\lambda} =: \mathcal{S}^*(\sqrt{1 + \lambda z})$, $\lambda \in (0, 1]$. Moreover, the feature of the class $\mathcal{S}^*_s := \mathcal{S}^*(1 + \sin z)$ such that the quantity $w = zf'(z)/f(z)$ lies in an eight-shaped region in the right-half plane was studied by Cho et al. in [13]. Raina and Sokół [29] considered the class $\mathcal{S}^*_{\mathbb{C}} := \mathcal{S}^*_q$, where $q(z) = z + \sqrt{1 + z^2}$. They have proved that $f \in \mathcal{S}^*_{\mathbb{C}}$ if and only if $zf'(z)/f(z) \in \mathcal{R}$ where $\mathcal{R} := \{z \in \mathbb{C} : \left|w^2 - 1\right| < 2|w|\}$.

Further, in a survey-cum-expository article [33] by Srivastava, it was indicated that the recent and future applications and importance of the classical q-calculus and the fractional q-calculus in geometric function theory of complex analysis motivate researchers to study many of these and other related subjects in this filed.

It is known that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius 1/4. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z$ $(z \in \mathbb{U})$, and $f\left(f^{-1}(w)\right) = w$ $(|w| < r_0(f);\ r_0(f) \geq 1/4)$, where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots =: w + \sum_{n=2}^{\infty} c_n w^n. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$. Lewin [23] studied the bi-univalent function class Σ and obtained the bound for the second Taylor-Maclaurin coefficient $|a_2|$. A brief historical overview of functions in the class Σ can be found in the work of Srivastava et al. [37], which is a fundamental research on the bi-univalent function class Σ, as well as in the references cited therein. In a number of sequels to [37], bounds for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of different subclasses of bi-univalent functions were given, for example, see [5, 9, 16, 34, 35, 41]. In fact, the study of analytic and bi-univalent functions was successfully revived by the pioneering work of Srivastava et al. [37] in recent years regarding the numerous papers on the subject.

Finding the upper bound for coefficients have been one of the central topic of research in geometric function theory as it gives several properties of functions. In particular, bound for the second coefficient gives growth and distortion theorems for functions in the class $\mathcal{S}$. According to [37], many authors put effort to review and study various subclasses of the class Σ of bi-univalent functions in recent years, for example, see [9, 16, 34, 35, 41]. In the literature, several authors used the Faber polynomial expansions to determine the general coefficient bounds of $|a_n|$ for the analytic bi-univalent functions [6, 18, 20, 36, 38–40, 45, 46]. It is remarkable that Faber polynomials play an important act in geometric function theory which was introduced by Faber [15].

Thorough this paper, we assume that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$ and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ such that it has series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \quad (B_1 > 0). \tag{3}$$

A function $\omega : \mathbb{U} \to \mathbb{C}$ is called a *Schwarz function* if ω is a analytic function in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Clearly, a Schwarz function ω is the form

$$\omega(z) = w_1z + w_2z^2 + \cdots .$$

We denote by Ω the set of all Schwarz functions on $\mathbb{U}$. Denote by $f * \Theta$ the Hadamard product (or convolution) of the functions f and Θ, that is, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$$(f * \Theta)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Recently, Bulut [9] introduced a comprehensive subclass of analytic bi-univalent functions and obtained non-sharp estimates of first two coefficients of functions in this class as follows.

Definition 1.1. *[9] Let the function f, defined by* (1)*, be in the class $\mathcal{A}$ and let*

$$\Theta \in \Sigma \qquad \text{and} \qquad \Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (b_n > 0).$$

We say that

$$f \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta) \qquad (\lambda \geq 1,\ \mu \geq 0),$$

if the following conditions are satisfied:

$$f \in \Sigma,$$

$$(1-\lambda)\left(\frac{(f*\Theta)(z)}{z}\right)^{\mu} + \lambda (f*\Theta)'(z)\left(\frac{(f*\Theta)(z)}{z}\right)^{\mu-1} \prec \varphi(z) \quad (z \in \mathbb{U})$$

and

$$(1-\lambda)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu} + \lambda \left((f*\Theta)^{-1}\right)'(w)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu-1} \prec \varphi(w) \quad (w \in \mathbb{U}),$$

where the function $(f*\Theta)^{-1}$ is given by

$$(f*\Theta)^{-1}(w) = w - a_2b_2w^2 + \left(2a_2^2b_2^2 - a_3b_3\right)w^3 - \left(5a_2^3b_2^3 - 5a_2b_2a_3b_3 + a_4b_4\right)w^4 + \cdots.$$

Definition 1.2. *[21] For $0 \leq \gamma \leq 1$, the class $\mathcal{V}_{\gamma}(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:*

$$\gamma f'(z) + (1-\gamma)\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \mathbb{U}).$$

Note that $\mathcal{V}_0(\varphi) =: \mathcal{S}^*(\varphi)$ and $\mathcal{V}_1(\varphi) =: \mathcal{C}(\varphi) = \{f \in \mathcal{A} : f'(z) \prec \varphi(z),\ z \in \mathbb{U}\}$ is a subclass of close-to-convex function. Thus, this class provides a continuous passage from a subclass of starlike functions to the subclass of close-to-convex functions when γ varies from 0 to 1.

Theorem 1.3. *[46] For $\lambda \geq 1$ and $\mu \geq 0$, let $f \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta)$ be given by* (1)*. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{B_1}{[\mu + (n-1)\lambda]b_n} \quad (n \geq 3).$$

The present paper is motivated essentially by the recent works ([3, 28, 37, 46]) and the aim of this paper is to study the coefficient estimates of some classes. In Section 2, we use the Faber polynomial expansions to derive bounds for the coefficients $|a_n|$ for the functions of a general class that our results improve some of the previously ones. In Section 3, we obtain the sharp bounds on the functional $|a_2a_3 - a_4|$ for the class $\mathcal{V}_{\gamma}(\varphi)$, which can be studied in the third Hankel determinant $H_3(1)$ for various classes of analytic and univalent functions, [7, 8, 11, 12]. We remark in passing that the Hankel determinants (like the second Hankel determinant, see [17]) play an important role in the study of the singularities and power series with integral coefficients, [26, 27].

2. Coefficient bounds $|a_n|$ of the class $\mathcal{H}_\Sigma^{\lambda,\mu}(\varphi;\Theta)$

In this section, we find a smaller upper bound and more accurate estimation of coefficients $|a_n|$ $(n \geq 3)$ of analytic bi-univalent functions in the class $\mathcal{H}_\Sigma^{\lambda,\mu}(\varphi;\Theta)$. To prove of our results, we need the following lemmas.

Lemma 2.1. *[1, 2] Let $f \in \mathcal{S}$ be given by* (1), *the coefficients of its inverse map $g = f^{-1}$ are given in terms of the Faber polynomials of f with*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n, \tag{4}$$

where

$$\begin{aligned} K_{n-1}^{-n} =& \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \dots, a_n$, and the expressions such as (for example) $(-n)!$ are to be interpreted symbolically *by*

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots, \quad \text{with} \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{N} := \{1, 2, 3, \cdots\}.$$

In particular, the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right) \quad \text{and} \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, for any real number p the expansion of K_n^p is given below (see for details, [1]; see also [2, p. 349])

$$K_n^p = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n, \tag{5}$$

where $D_n^p = D_n^p(a_2, a_3, \dots, a_{n+1})$ (see for details [42]). We also have

$$D_n^m(a_2, a_3, \dots, a_{n+1}) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\mu_1} \cdot \ldots \cdot (a_{n+1})^{\mu_n}}{\mu_1! \cdot \ldots \cdot \mu_n!}, \tag{6}$$

where the sum is taken over all nonnegative integers $\mu_1, \dots, \mu_n$ satisfying the conditions

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that $D_n^n(a_2, a_3, \dots, a_{n+1}) = a_2^n$.

Lemma 2.2. *[46] Let $f \in \mathcal{H}_\Sigma^{\lambda,\mu}(\varphi;\Theta)$. Then we have the following expansion:*

$$\begin{aligned} &(1-\lambda)\left(\frac{f * \Theta(z)}{z}\right)^{\mu} + \lambda (f * \Theta)'(z) \left(\frac{f * \Theta(z)}{z}\right)^{\mu-1} \\ =& 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2 b_2, a_3 b_3, \cdots, a_n b_n) z^{n-1}, \end{aligned}$$

where

$$F_{n-1}(a_2b_2, a_3b_3, \cdots, a_nb_n) = \left(\frac{\mu + (n-1)\lambda}{\mu}\right) K_{n-1}^{\mu}(a_2b_2, a_3b_3, \cdots, a_nb_n)$$
$$= [\mu + (n-1)\lambda](\mu-1)! \times \left[\sum_{i_1+\cdots+(n-1)i_{n-1}=n-1} \frac{(a_2b_2)^{i_1}(a_3b_3)^{i_2}\cdots(a_nb_n)^{i_{n-1}}}{i_1!i_2!\cdots i_{n-1}![\mu-(i_1+i_2+\cdots+i_{n-1})]!} \right].$$

Lemma 2.3. *[46] Let $f \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta)$. Then we have the following expansion:*

$$(1-\lambda)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu} + \lambda((f*\Theta)^{-1})'(w)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu-1}$$
$$=1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \cdots, A_n)w^{n-1},$$

where

$$F_{n-1} = \left(\frac{\mu + (n-1)\lambda}{\mu}\right) K_{n-1}^{\mu} \ \text{ and } \ A_n = \frac{1}{n} K_{n-1}^{-n}(a_2b_2, a_3b_3, \cdots, a_nb_n).$$

Lemma 2.4. *[46] Let $f \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta)$. Then*

$$F_{n-1}(a_2b_2, a_3b_3, \cdots, a_nb_n) = \sum_{k=1}^{n-1} B_k D_{n-1}^{k}(p_1, p_2, \cdots, p_{n-1}) \tag{7}$$

and

$$F_{n-1}(A_2, A_3, \cdots, A_n) = \sum_{k=1}^{n-1} B_k D_{n-1}^{k}(q_1, q_2, \cdots, q_{n-1}), \tag{8}$$

where F_{n-1} and A_n are given by Lemma 2.3.

Lemma 2.5. *[3] Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$; $(n \geq 2)$ be a univalent function in $\mathbb{U}$ and*

$$f^{-1}(w) = w + \sum_{k=n}^{\infty} c_k w^k; \ (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}).$$

Then

$$c_{2n-1} = na_n^2 - a_{2n-1} \qquad \text{and} \qquad c_k = -a_k \quad \text{for} \quad (n \leq k \leq 2n-2).$$

Theorem 2.6. *Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta)$ be given by* (1). *Then*

$$|a_n| \leq \min\left\{ \frac{B_1}{[\mu + (n-1)\lambda]b_n}, \sqrt{\frac{2B_1}{n[\mu + (2n-2)\lambda]b_{2n-1}}} \right\} \qquad (n \geq 3) \tag{9}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{B_1}{[\mu + (2n-2)\lambda]b_{2n-1}}.$$

Proof. Let $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{H}_{\Sigma}^{\lambda,\mu}(\varphi;\Theta)$. Then there are two functions $u, v \in \Omega$ with $u(z) = \sum_{n=1}^{\infty} p_n z^n$ and $v(z) = \sum_{n=1}^{\infty} q_n z^n$, respectively, so that

$$(1-\lambda)\left(\frac{f*\Theta(z)}{z}\right)^{\mu} + \lambda(f*\Theta)'(z)\left(\frac{f*\Theta(z)}{z}\right)^{\mu-1} = \varphi(u(z))$$

and

$$(1-\lambda)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu} + \lambda((f*\Theta)^{-1})'(w)\left(\frac{(f*\Theta)^{-1}(w)}{w}\right)^{\mu-1} = \varphi(v(w)),$$

where by the relations (6) and (3) we have,

$$\varphi(u(z)) = 1 + B_1 p_1 z + (B_1 p_2 + B_2 p_1^2) z^2 + \cdots = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(p_1, p_2, \cdots, p_n) z^n$$

and

$$\varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(q_1, q_2, \cdots, q_n) w^n.$$

Since $a_k = 0$ for $2 \le k \le n-1$, so from Lemma 2.3, by the definition of K_n^p in (5), we obtain $A_n = -a_n b_n$ and since $B_1 > 0$ we have $p_1 = \cdots = p_{n-2} = 0,\ \ q_1 = \cdots = q_{n-2} = 0$. From Lemmas 2.2-2.4 according to (7) and (8), we have

$$[\mu + (n-1)\lambda] a_n b_n = B_1 p_{n-1}$$

and

$$-[\mu + (n-1)\lambda] a_n b_n = B_1 q_{n-1}.$$

Now, taking the absolute values of the above equalities and applying Lemma 3.1 with $|p_{n-1}| \le 1 - |p_1|^2 \le 1,\ |q_{n-1}| \le 1 - |q_1|^2 \le 1$, it yields

$$|a_n| \le \frac{B_1}{[\mu + (n-1)\lambda] b_n} \qquad (n \ge 3). \tag{10}$$

On other the hand, using Lemma 2.5, we have $c_n = -a_n$ for $k = n$ and so according to the above inequality, it yields

$$|c_n| \le \frac{B_1}{[\mu + (n-1)\lambda] b_n} \qquad (n \ge 3). \tag{11}$$

Also, in view of Lemma 2.5, using the inequalities (10) and (11), we obtain

$$|a_n| \le \sqrt{\frac{|a_{2n-1}| + |c_{2n-1}|}{n}} \le \sqrt{\frac{2B_1}{n[\mu + (2n-2)\lambda] b_{2n-1}}} \qquad (n \ge 3). \tag{12}$$

From (10) and (12) we conclude that the inequality (9) holds. Also, by (11) and using Lemma 2.5, we get

$$|n a_n^2 - a_{2n-1}| = |c_{2n-1}| \le \frac{B_1}{[\mu + (2n-2)\lambda] b_{2n-1}}.$$

This completes the proof. □

In particular, we get the following corollaries.

Corollary 2.7. *Let* $f(z) = z + \sum\limits_{k=n}^{\infty} a_k z^k \in \mathcal{H}_{\Sigma}^{\lambda,\mu}\left(\frac{1+(1-2\beta)z}{1-z}; \frac{z}{1-z}\right)$ *be given by* (1). *Then*

$$|a_n| \leq \min\left\{\frac{2(1-\beta)}{[\mu+(n-1)\lambda]}, \sqrt{\frac{4(1-\beta)}{n[\mu+(2n-2)\lambda]}}\right\} \qquad (n \geq 3)$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{2(1-\beta)}{[\mu+(2n-2)\lambda]}.$$

Proof. For

$$\Theta(z) = \frac{z}{1-z}$$

and

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1,\ z \in \mathbb{U}),$$

where $B_1 = 2(1-\beta)$ in Theorem 2.6, it gives the result stated in the corollary. □

Remark 2.8. *(i) The bound for* $|a_n|$ *in Theorem 2.6 is a improvement of the estimation given in Theorem 1.3.*

(ii) From Corollary 2.7, the bound for $|a_n|$ *is smaller than the estimate obtained in [10, Theorem 1].*

(iii) Letting $\lambda = \mu = 1$, $\lambda = 1$ *and* $\mu = 1$ *in Corollary 2.7, we obtain an improvement of the estimates obtained in [20, Theorem 2.1], [18, Theorem 1] and [19, Theorem 2.1], respectively.*

Corollary 2.9. *Let* $f(z) = z + \sum\limits_{k=n}^{\infty} a_k z^k \in \mathcal{H}_{\Sigma}^{\lambda,\mu}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}; \frac{z}{1-z}\right)$ *be given by* (1). *Then*

$$|a_n| \leq \min\left\{\frac{2\alpha}{[\mu+(n-1)\lambda]}, \sqrt{\frac{4\alpha}{n[\mu+(2n-2)\lambda]}}\right\} \qquad (n \geq 3)$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{2\alpha}{[\mu+(2n-2)\lambda]}.$$

Proof. For

$$\Theta(z) = \frac{z}{1-z}$$

and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

where $B_1 = 2\alpha$ in Theorem 2.6, it gives the required result. □

3. Coefficient bounds $|a_2a_3 - a_4|$ of the $\mathcal{V}_\gamma(\varphi)$

In this section, we obtain the sharp bounds on the functional $|a_2a_3 - a_4|$ for the class $\mathcal{V}_\gamma(\varphi)$. For this goal, we need the following lemmas.

Lemma 3.1. *[25, p. 172] Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ for all $z \in \mathbb{U}$. Then $|w_1| \le 1$ and*

$$|w_n| \le 1 - |w_1|^2 \quad \text{for all } n \in \mathbb{N} \text{ with } n \ge 2.$$

Lemma 3.2. *[4, 28] If $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ $(z \in \mathbb{U})$, then for any real numbers q_1 and q_2, the following sharp estimate holds:*

$$|p_3 + q_1 w_1 w_2 + q_2 w_1^3| \le H(q_1; q_2),$$

where

$$H(q_1;q_2) = \begin{cases} 1 & \text{if} \quad (q_1,q_2) \in D_1 \cup D_2 \cup \{(2,1)\}, \\ |q_2| & \text{if} \quad (q_1,q_2) \in \cup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{if} \quad (q_1,q_2) \in D_{12}, \end{cases}$$

and the sets D_k, $k = 1, 2, \ldots, 12$ are defined as follows:

$$D_1 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le 1\right\},$$

$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\ \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1) \le q_2 \le 1\right\},$$

$$D_3 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ q_2 \le -1\right\},$$

$$D_4 = \left\{(q_1,q_2) : |q_1| \ge \frac{1}{2},\ q_2 \le -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 = \{(q_1,q_2) : |q_1| \le 2,\ q_2 \ge 1\},$$

$$D_6 = \left\{(q_1,q_2) : 2 \le |q_1| \le 4,\ q_2 \ge \frac{1}{12}(q_1^2+8)\right\},$$

$$D_7 = \left\{(q_1,q_2) : |q_1| \ge 4,\ q_2 \ge \frac{2}{3}(|q_1|-1)\right\},$$

$$D_8 = \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\ -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{4}{27}\left((|q_1|+1)^3\right) - (|q_1|+1)\right\},$$

$$D_9 = \left\{(q_1,q_2) : |q_1| \ge 2,\ -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10} = \left\{(q_1,q_2) : 2 \le |q_1| \le 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11} = \left\{(q_1,q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12} = \left\{(q_1,q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \le q_2 \le \frac{2}{3}(|q_1|-1)\right\}.$$

Theorem 3.3. *If the function $f \in \mathcal{V}_\gamma(\varphi)$ has the power expansion series given by (1), then*

$$|a_2a_3 - a_4| \le \frac{B_1}{3+\gamma} H(q_1;q_2), \tag{13}$$

where $H(q_1; q_2)$ is given by Lemma 3.2 and

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{B_1[3(1-\gamma)-(3+\gamma)]}{(1+\gamma)(2+\gamma)}, \\ q_2 &= \frac{(3(1-\gamma)-(3+\gamma))\left(B_2(1+\gamma)^2 + B_1^2(1-\gamma)\right)}{(1+\gamma)^3(2+\gamma)} - \frac{B_1^2(1-\gamma)}{(1+\gamma)^3} + \frac{B_3}{B_1}. \end{aligned} \tag{14}$$

The bound (13) *is sharp.*

Proof. If $f \in \mathcal{V}_\gamma(\varphi)$ has the form (1), by the definition of the subordination there exists $\nu \in \Omega$, with $\nu(z) = \sum_{n=1}^{\infty} d_n z^n$, $z \in \mathbb{U}$, such that

$$\begin{aligned} &\gamma f'(z) + (1-\gamma)\frac{zf'(z)}{f(z)} = \varphi(\nu(z)) = 1 + B_1 d_1 z + \left(B_1 d_2 + B_2 d_1^2\right) z^2 \\ &+ \left(B_1 d_3 + 2d_1 d_2 B_2 + B_3 d_1^3\right) z^3 + \cdots \quad (z \in \mathbb{U}). \end{aligned}$$

Comparing the corresponding coefficients of the above relation, it follows that

$$\begin{cases} a_2(1+\gamma) = B_1 d_1, \\ (2+\gamma)a_3 - (1-\gamma)a_2^2 = B_1 d_2 + B_2 d_1^2, \\ (3+\gamma)a_4 - (1-\gamma)\left(3a_2a_3 - a_2^3\right) = B_1 d_3 + 2d_1 d_2 B_2 + B_3 d_1^3. \end{cases}$$

Then from above equations we get

$$\begin{cases} a_2 = \frac{B_1 d_1}{1+\gamma}, \\ a_3 = \frac{1}{2+\gamma}\left[B_1 d_2 + d_1^2\left(B_2 + \frac{B_1^2(1-\gamma)}{(1+\gamma)^2}\right)\right], \\ a_4 = \frac{1}{3+\gamma}\left[B_1 d_3 + \left(2B_2 + \frac{3B_1^2(1-\gamma)}{(1+\gamma)(2+\gamma)}\right)d_1 d_2 + \left(\frac{3B_1(1-\gamma)\left(B_2(1+\gamma)^2 + B_1^2(1-\gamma)\right)}{(1+\gamma)^3(2+\gamma)} - \frac{B_1^3(1-\gamma)}{(1+\gamma)^3} + B_3\right)d_1^3\right]. \end{cases}$$

Therefore

$$\begin{aligned} a_4 - a_2 a_3 =& \frac{B_1}{3+\gamma}\Bigg(d_3 + \left[\frac{2B_2}{B_1} + \frac{B_1[3(1-\gamma)-(3+\gamma)]}{(1+\gamma)(2+\gamma)}\right] d_1 d_2 \\ &+ \left[\frac{(3(1-\gamma)-(3+\gamma))\left(B_2(1+\gamma)^2 + B_1^2(1-\gamma)\right)}{(1+\gamma)^3(2+\gamma)} - \frac{B_1^2(1-\gamma)}{(1+\gamma)^3} + \frac{B_3}{B_1}\right] d_1^3\Bigg). \end{aligned}$$

Now by Lemma 3.2 we have

$$\begin{aligned} |a_2 a_3 - a_4| =& \frac{B_1}{3+\gamma}\Bigg|d_3 + \left[\frac{2B_2}{B_1} + \frac{B_1[3(1-\gamma)-(3+\gamma)]}{(1+\gamma)(2+\gamma)}\right] d_1 d_2 \\ &+ \left[\frac{(3(1-\gamma)-(3+\gamma))\left(B_2(1+\gamma)^2 + B_1^2(1-\gamma)\right)}{(1+\gamma)^3(2+\gamma)} - \frac{B_1^2(1-\gamma)}{(1+\gamma)^3} + \frac{B_3}{B_1}\right] d_1^3\Bigg| \\ \le& \frac{B_1}{3+\gamma} H(q_1; q_2), \end{aligned}$$

where q_1 and q_2 are given in (14). This completes the proof. □

For $\gamma = 0$ in Theorem 3.3, we obtain the following result.

Corollary 3.4. *If the function $f \in \mathcal{S}^*(\varphi)$ has the power expansion series given by* (1), *then*

$$|a_2a_3 - a_4| \le \frac{B_1}{3} H(q_1; q_2),$$

where $H(q_1; q_2)$ is given by Lemma 3.2, $q_1 = 2B_2/B_1$ and $q_2 = (B_3 - B_1^3)/B_1$. The bound is sharp.

Corollary 3.5. *[11, Theorem 2.1] If the function $f \in \mathcal{S}^*(\alpha)$ has the power expansion series given by* (1), *then*

$$|a_2a_3 - a_4| \le \begin{cases} \frac{2}{3}(1-\alpha)[4(1-\alpha)^2 - 1], & 0 \le \alpha \le 1 - \sqrt{3}/2, \\ \frac{2(1-\alpha)}{3\sqrt{1-(1-\alpha)^2}}, & 1 - \sqrt{3}/2 < \alpha < 1. \end{cases}$$

The bounds are sharp.

Proof. For

$$\varphi(z) = \frac{1 + (1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots \quad (0 \le \alpha < 1,\ z \in \mathbb{U}),$$

where $B_1 = B_2 = B_3 = 2(1-\beta)$ and $q_1 = 2,\ q_2 = 1 - 4(1-\alpha)^2$ (with respect to D_4 and D_8) in Corollary 3.4, we obtain the required result. □

Corollary 3.6. *[14, Theorem 2.1(2)] If the function $f \in \mathcal{S}_q^*$ has the power expansion series given by* (1), *then*

$$|a_2a_3 - a_4| \le \frac{4\sqrt{6}}{27}.$$

The bound is sharp.

Proof. For

$$\varphi(z) = z + \sqrt{1+z^2} = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + z + \frac{z^2}{2} - \frac{z^4}{8} + \cdots \quad (z \in \mathbb{U}),$$

where $q_1 = 1,\ q_2 = -1$ (with respect to D_8) in Corollary 3.4, it gives the result stated in the corollary. □

Corollary 3.7. *[14, Theorem 3.1(2)] If the function $f \in \mathcal{S}_{l_\lambda}^*$ has the power expansion series given by* (1), *then*

$$|a_2a_3 - a_4| \le \frac{\lambda}{6}.$$

The bound is sharp.

Proof. If we take

$$\varphi(z) = \sqrt{1+\lambda z} = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + \frac{\lambda}{2}z - \frac{\lambda^2}{8}z^2 + \frac{\lambda^3}{16}z^3 + \cdots \quad (\lambda \in (0,1],\ z \in \mathbb{U}),$$

where $q_1 = -\lambda/2,\ q_2 = -\lambda^2/8$ (with respect to D_1) in Corollary 3.4, then we have the required result. □

Corollary 3.8. *[22, Theorem 3.2] If the function $f \in \mathcal{S}_B^*$ has the power expansion series given by* (1), *then*

$$|a_2a_3 - a_4| \le \frac{2}{3}\sqrt{\frac{6}{17}}.$$

The bound is sharp.

Proof. Letting

$$\varphi(z) = \mathrm{e}^{\mathrm{e}^z-1} = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + z + z^2 + \frac{5}{6}z^3 + \cdots \quad (z \in \mathbb{U}),$$

where $q_1 = 2,\ q_2 = -1/6$ (with respect to D_8) in Corollary 3.4, it gives the result stated in the corollary. □

Taking

$$\varphi(z) = 1 + \sin z = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + z - \frac{z^3}{6} + \cdots \quad (z \in \mathbb{U}),$$

where $q_1 = 0,\ q_2 = -7/6$ (with respect to D_3) in Corollary 3.4, we have a correction of the result obtained in [44, Theorem 3] in the next result:

Corollary 3.9. *If the function $f \in \mathcal{S}^*_s$ has the power expansion series given by* (1)*, then*

$$|a_2a_3 - a_4| \leq \frac{7}{18}.$$

The bound is sharp.

Corollary 3.10. *[43, Corollary 2] If the function $f \in \mathcal{S}^*_e$ has the power expansion series given by* (1)*, then*

$$|a_2a_3 - a_4| \leq \frac{8}{9\sqrt{7}}.$$

The bound is sharp.

Proof. Setting

$$\varphi(z) = \mathrm{e}^z = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \quad (z \in \mathbb{U}),$$

where $q_1 = 1,\ q_2 = -5/6$ (with respect to D_8) in Corollary 3.4, we have the required result. □

Remark 3.11. *(i) For $\lambda = 1$ in Corollary 3.7, we obtain the result given in [30, Theorem 2.5].*

(ii) Setting $\varphi(z) = [(1+z)/(1-z)]^\alpha$ $(0 < \alpha \leq 1)$ in Corollary 3.4, we obtain the estimate obtained in [12, Theorem 2.1].

(iii) Takeing $\gamma = 1$ and $\varphi(z) = (1+z)/(1-z)$ in Theorem 3.3, we obtain the estimate obtained in [8, Theorem 2.1].

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