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Extragradient Methods With CQ Technique for Fixed Point Problems and Equilibrium Problems

Zhangsong Yao[a], Yeong-Cheng Liou[b], Li-Jun Zhu[c]

[a] *School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China*
[b] *Department of Healthcare Administration and Medical Informatics, and Research Center of Nonlinear Analysis and Optimization, Kaohsiung Medical University, and Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung 807, Taiwan*
[c] *School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China and The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China*

Abstract. In this paper, we study iterative algorithms for solving fixed point problems and equilibrium problems in Hilbert spaces. We present an extragradient algorithm with CQ technique for finding a common element of the fixed points of pseudocontractive operators and the solutions of pseudomonotone equilibrium problems. Strong convergence result of the proposed algorithm is proved.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem, in the sense of Blum and Oettli [5] aims to find a point $\tilde{q} \in C$ such that

$$f(\tilde{q}, p) \geq 0, \ \forall p \in C. \tag{1}$$

By $EP(f, C)$, we denote the solution set of equilibrium problem (1).

Now, it is well known that the equilibrium problem (1) has been applied to solve a variety of mathematical models, such as variational inequalities ([4, 6, 11, 14, 16, 21, 22, 26, 30–37]), optimization problems, saddle point problems, fixed point problems ([7, 8, 27–29, 38]), Nash equilibrium in noncooperative games theory ([3, 5, 9, 10, 17, 23]). An important method for solving (1) is proximal point method which was originally introduced by Martinet [15] and further developed by Rockafellar [19] for finding a zero of maximal monotone operators. Particularly, in [5, 9], the resolvent of bi-function f was used to solve (1). For every $\tau > 0$ and $x \in H$, there exists a point $z \in C$ such that

$$f(z, y) + \frac{1}{\tau}\langle z - x, y - x \rangle \geq 0, \forall y \in C.$$

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Corresponding author: Yeong-Cheng Liou.
Email addresses: yaozhsong@163.com (Zhangsong Yao), simplex_liou@hotmail.com (Yeong-Cheng Liou), zhulijun1995@sohu.com (Li-Jun Zhu)

Consequently, Tada and Takahashi [20] introduced an iterative algorithm for solving equilibrium problem (1) and a fixed point problem of nonexpansive mappings:

$$\begin{cases} p_n \in C \text{ such that } \langle f(p_n, p) + \frac{1}{\tau_n}\langle p - p_n, p_n - x_n\rangle \geq 0, \ \forall p \in C, \\ x_{n+1} = (1-\mu_n)x_n + \mu_n T p_n, n \geq 0. \end{cases} \tag{2}$$

However, we note that some strong monotonicity assumptions are needed to impose on f in order to guarantee the existence of the iterates. But, if f is pseudomonotone, the iterates generated by (2) may not be well-defined. To overcome this difficulty, Tran et al. [23] applied extragradient method to solve equilibrium problem (1) when f is pseudomonotone and satisfies certain Lipschitz-type condition. They proposed the following iterative procedure: for given x_0, compute the sequence $\{x_{n+1}\}$ by the form

$$\begin{cases} v_n = \arg\min_{z\in C}\{f(u_n, z) + \dfrac{1}{2\tau_n}\|u_n - z\|^2\}, \\ u_{n+1} = \arg\min_{z\in C}\{f(v_n, z) + \dfrac{1}{2\tau_n}\|u_n - z\|^2\}, \end{cases} \tag{3}$$

where $\tau_n \in (0, \min\{\frac{1}{2\tau_1}, \frac{1}{2\tau_2}\})$ with τ_1 and τ_2 being the Lipschitz constants of f.

Recently, Vuong, Strodiot and Nguyen [24] suggested an extragradient method for solving equilibrium problem (1) and a fixed point problem of nonexpansive mappings:

Step 0. Choose the sequences $\{\alpha_n\} \subset [0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{\tau_n\} \subset (0, 1]$.

Step 1. Let $x_0 \in C$. Set $n = 0$.

Step 2. Compute the sequences $\{y_n\}$ and $\{z_n\}$ by

$$\begin{cases} y_n = \min_{y^\dagger\in C}\{\tau_n f(x_n, y^\dagger) + \frac{1}{2}\|x_n - y^\dagger\|^2\}, \\ z_n = \min_{y^\dagger\in C}\{\tau_n f(y_n, y^\dagger) + \frac{1}{2}\|x_n - y^\dagger\|^2\}. \end{cases}$$

Step 3. Compute $t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)Sz_n]$. If $y_n = x_n$ and $t_n = x_n$, then stop. Otherwise, go to step 4.

Step 4. Compute $x_{n+1} = P_{C_n\cap D_n}[x_0]$, where

$$C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}$$

and

$$D_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}.$$

Step 5. Set $n := n + 1$ and go to Step 2.

Very recently, iterative algorithms for solving (1) and fixed point problems have been future studied in the literature, see, for instance [2, 12, 13, 25].

Motivated and inspired by the above work in the literature, the main purpose of this paper is to investigate fixed point problem of pseudocontractive operators and the pseudomonotone equilibrium problem. We suggest an iterative algorithm for finding a common solution of the pseudomonotone equilibrium problem and fixed point of pseudocontractive operators. Strong convergence analysis of the proposed procedure is given.

2. Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $h : C \to (-\infty, +\infty]$ be a function.

- h is said to be convex if $h(\alpha u^\ddagger + (1-\alpha)v^\ddagger) \leq \alpha h(u^\ddagger) + (1-\alpha)h(v^\ddagger)$ for every $u^\ddagger, v^\ddagger \in C$ and $\alpha \in [0, 1]$.

- h is said to be ρ-strongly convex ($\rho > 0$) if

$$h(\alpha u^{\ddagger} + (1-\alpha)v^{\ddagger}) + \frac{\rho}{2}\alpha(1-\alpha)\|u^{\ddagger} - v^{\ddagger}\|^2 \le \alpha h(u^{\ddagger}) + (1-\alpha)h(v^{\ddagger})$$

for every $u^{\ddagger}, v^{\ddagger} \in C$ and $\alpha \in (0,1)$.

Let $h : C \to (-\infty, +\infty]$ be a convex function. The subdifferential ∂h of h is defined by

$$\partial h(u) := \{v^{\ddagger} \in H : h(u) + \langle v^{\ddagger}, u^{\ddagger} - u\rangle \le h(u^{\ddagger}), \forall u^{\ddagger} \in C\} \tag{4}$$

for each $u \in C$.

Recall that an operator $T : C \to C$ is said to be pseudocontractive if

$$\|Tu - Tu^{\ddagger}\|^2 \le \|u - u^{\ddagger}\|^2 + \|(I-T)u - (I-T)u^{\ddagger}\|^2$$

for all $u, u^{\ddagger} \in C$ and T is called L-Lipschitz if

$$\|Tu - Tu^{\ddagger}\| \le L\|u - u^{\ddagger}\|$$

for all $u, u^{\ddagger} \in C$.

For fixed $z \in H$, there exists a unique $z^{\dagger} \in C$ satisfying

$$\|z - z^{\dagger}\| = \inf\{\|z - \tilde{z}\| : \tilde{z} \in C\}.$$

Denote $z^{\dagger}$ by $proj_C[z]$.

The following inequality is an important property of projection $proj_C$: for given $x \in H$,

$$\langle x - proj_C[x], y - proj_C[x]\rangle \le 0, \ \forall y \in C. \tag{5}$$

The following symbols are needed in the paper.

- $x_n \rightharpoonup p^{\dagger}$ indicates the weak convergence of x_n to $p^{\dagger}$ as $n \to \infty$.
- $x_n \to p^{\dagger}$ implies the strong convergence of x_n to $p^{\dagger}$ as $n \to \infty$.
- $Fix(T)$ means the set of fixed points of T.
- $\omega_w(x_n) = \{p^{\dagger} : \exists\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^{\dagger}(i \to \infty)\}$.

Lemma 2.1 ([3]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let a function $h : C \to \mathbb{R}$ be subdifferentiable. Then $u^{\dagger}$ is a solution to the following minimization problem*

$$\min_{x \in C} h(x)$$

if and only if $0 \in \partial h(u^{\dagger}) + N_C(u^{\dagger})$, where $N_C(u^{\dagger})$ means the normal cone of C at $u^{\dagger}$ defined by

$$N_C(u^{\dagger}) = \{\omega \in H : \langle \omega, u - u^{\dagger}\rangle \le 0, \forall u \in C\}. \tag{6}$$

Lemma 2.2 ([18]). *Let H be a real Hilbert space. Then, the following equalities hold*

(i) $2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2, \ \forall x, y, u, v \in H.$

(ii) $\|\kappa u + (1-\kappa)u^{\dagger}\|^2 = \kappa\|u\|^2 + (1-\kappa)\|u^{\dagger}\|^2 - \kappa(1-\kappa)\|u - u^{\dagger}\|^2, \ \forall u, u^{\dagger} \in H, \forall \kappa \in [0,1].$

(iii) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle, \ \forall u, v \in H.$

Lemma 2.3 ([40]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^{\ddagger} \in Fix(T)$, we have*

$$\|u^{\ddagger} - T[(1-\beta)\tilde{u} + \beta T\tilde{u}]\|^2 \le \|\tilde{u} - u^{\ddagger}\|^2 + (1-\beta)\|\tilde{u} - T[(1-\beta)\tilde{u} + \beta T\tilde{u}]\|^2,$$

where $0 < \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Lemma 2.4 ([1]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies assumptions (A1)-(A4) stated in Section 3. Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence satisfying $\tau_n \in [\underline{\rho}, \overline{\rho}] \subset (0, 1]$. For given $x_n \in C$, let y_n be the unique solution of the following strongly convex program*

$$\min_{u^\ddagger \in C} \left\{ f(x_n, u^\ddagger) + \frac{1}{2\tau_n} \|x_n - u^\ddagger\|^2 \right\}.$$

If $\{x_n\}$ is bounded, then $\{y_n\}$ is also bounded.

Lemma 2.5 ([39]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. If the operator $T : C \to C$ is continuous pseudocontractive, then*

(i) the fixed point set $Fix(T) \subset C$ is closed and convex;

(ii) T satisfies demi-closedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $Tu_n \to z^\ddagger$ as $n \to \infty$ imply that $T\tilde{z} = z^\ddagger$.

Lemma 2.6 ([3]). *For given a sequence $\{u_n\} \subset H$ and a fixed point $u \in H$, if $\omega_w(u_n) \subset C$ and $\|u_n - u\| \le \|u - P_C[u]\|$ for all $n \in \mathbb{N}$, then $u_n \to P_C[u]$.*

3. Main results

In this section, we introduce an iterative algorithm for solving the fixed point problems and pseudomonotone equilibrium problems. Consequently, we show the convergence analysis of the suggested algorithm.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontractive operator with Lipschitz constant $L > 0$. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies the following assumptions:

(A1): $f(z^\ddagger, z^\ddagger) = 0$ for all $z^\ddagger \in C$;

(A2): f is pseudomonotone on C, i.e., $f(u^\ddagger, u) \ge 0$ implies $f(u, u^\ddagger) \le 0$ for all $u, u^\ddagger \in C$;

(A3): f is jointly sequently weakly continuous on $C \times C$ (recall that f is called jointly sequently weakly continuous on $C \times C$, if $x_n \rightharpoonup x^\ddagger$ and $y_n \rightharpoonup y^\ddagger$, then $f(x_n, y_n) \to f(x^\ddagger, y^\ddagger)$);

(A4): $f(z^\ddagger, \cdot)$ is convex and subdifferentiable for all $z^\ddagger \in C$;

(A5): f satisfies the Lipschitz-type condition: $\exists \mu_1, \mu_2 > 0$ such that

$$f(x^\ddagger, y^\ddagger) + f(y^\ddagger, z^\ddagger) \ge f(x^\ddagger, z^\ddagger) - \mu_1 \|x^\ddagger - y^\ddagger\|^2 - \mu_2 \|y^\ddagger - z^\ddagger\|^2, \ \forall x^\ddagger, y^\ddagger, z^\ddagger \in C.$$

Let $\{\tau_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ be three sequences satisfying the following restrictions:

(C1): $\tau_n \in [\underline{\tau}, \overline{\tau}]$, where $0 < \underline{\tau} \le \overline{\tau} < \min\{\frac{1}{2\mu_1}, \frac{1}{2\mu_2}\}$;

(C2): $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < \beta_n < \overline{\beta} < \dfrac{1}{\sqrt{1 + L^2} + 1}, \forall n \ge 0$.

Algorithm 3.1. *Step 0. (Initialization) Fix $x_0 \in C$.*

Step 1. (Fixed point step) For given $\{x_n\}$, compute the sequence $\{z_n\}$ by

$$z_n = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n]. \tag{7}$$

Step 2. (Extragradient technique) Solve the successively strong convex programs

$$\min \left\{ f(z_n, x^\ddagger) + \frac{1}{2\tau_n} \|z_n - x^\ddagger\|^2 : x^\ddagger \in C \right\} \tag{8}$$

and

$$\min\left\{f(y_n, x^\ddagger) + \frac{1}{2\tau_n}\|z_n - x^\ddagger\|^2 : x^\ddagger \in C\right\}, \tag{9}$$

to achieve their unique solutions y_n and u_n, respectively.

Step 3. (CQ technique) Construct the following two half-spaces to cut C:

$$C_n = \{q^\ddagger \in C : \|u_n - q^\ddagger\| \leq \|x_n - q^\ddagger\|\} \tag{10}$$

and

$$Q_n = \{q^\ddagger \in C : \langle x_n - q^\ddagger, x_0 - x_n\rangle \geq 0\}. \tag{11}$$

Step 4. (Projection technique) Compute the sequence $\{x_{n+1}\}$ by the following projection method

$$x_{n+1} = P_{C_n \cap Q_n}[x_0]. \tag{12}$$

Step 5. Set $n := n + 1$ and return to Step 1.

Theorem 3.2. *Suppose that $Fix(T) \cap EP(f, C) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by* (12) *converges strongly to $u^\ddagger = P_{Fix(T)\cap EP(f,C)}[x_0]$.*

Proof. Pick any $p \in Fix(T) \cap EP(f, C)$. Then, $f(p, y_n) \geq 0$. By virtue of the pseudomonotonicity (A2) of f, we deduce

$$f(y_n, p) \leq 0. \tag{13}$$

By (8) and Lemma 2.1, we have

$$0 \in \partial_2\left\{f(z_n, \cdot) + \frac{1}{2\tau_n}\|z_n - \cdot\|^2\right\}(y_n) + N_C(y_n).$$

It follows that there exists $w_n \in \partial_2 f(z_n, \cdot)(y_n)$ such that

$$\frac{1}{\tau_n}(z_n - y_n) - w_n \in N_C(y_n). \tag{14}$$

Thanks to the definition (6) of the normal cone N_C, we get

$$N_C(y_n) = \{\omega \in H : \langle \omega, u - y_n\rangle \leq 0, \ \forall u \in C\}. \tag{15}$$

Combining (14) and (15), we have

$$\left\langle \frac{1}{\tau_n}(z_n - y_n) - w_n, u - y_n \right\rangle \leq 0, \ \forall u \in C,$$

which yields

$$\langle w_n, u - y_n\rangle \geq \frac{1}{\tau_n}\langle z_n - y_n, u - y_n\rangle, \ \forall u \in C. \tag{16}$$

According to the definition (4) of subgradient of $f(z_n, \cdot)$ at y_n, we obtain

$$f(z_n, u) - f(z_n, y_n) \geq \langle w_n, u - y_n\rangle, \ \forall u \in C. \tag{17}$$

In the light of (16) and (17), we deduce

$$f(z_n, u) - f(z_n, y_n) \geq \frac{1}{\tau_n}\langle z_n - y_n, u - y_n\rangle, \ \forall u \in C. \tag{18}$$

Similarly, we can show

$$f(y_n, u) - f(y_n, u_n) \geq \frac{1}{\tau_n}\langle u_n - z_n, u_n - u\rangle, \ \forall u \in C. \tag{19}$$

Setting $u = p$ in (19) and combining with (13), we obtain

$$f(y_n, u_n) \le \frac{1}{\tau_n}\langle u_n - z_n, p - u_n\rangle. \tag{20}$$

Applying the Lipschitz property (A5) of f, it results

$$f(y_n, u_n) \ge f(z_n, u_n) - f(z_n, y_n) - \mu_1\|z_n - y_n\|^2 - \mu_2\|y_n - u_n\|^2. \tag{21}$$

By virtue of (20) and (21), we deduce

$$\begin{aligned}\frac{1}{\tau_n}\langle u_n - z_n, p - u_n\rangle \ge{}& f(z_n, u_n) - f(z_n, y_n) - \mu_1\|z_n - y_n\|^2\\ &- \mu_2\|y_n - u_n\|^2.\end{aligned} \tag{22}$$

Setting $u = u_n$ in (18), we have

$$f(z_n, u_n) - f(z_n, y_n) \ge \frac{1}{\tau_n}\langle z_n - y_n, u_n - y_n\rangle. \tag{23}$$

In terms of (22) and (23), we get

$$\begin{aligned}\langle u_n - z_n, p - u_n\rangle \ge{}& \langle z_n - y_n, u_n - y_n\rangle - \mu_1\tau_n\|z_n - y_n\|^2\\ &- \mu_2\tau_n\|y_n - u_n\|^2.\end{aligned} \tag{24}$$

Applying Lemma 2.2 (i), it yields

$$2\langle u_n - z_n, p - u_n\rangle = \|z_n - p\|^2 - \|u_n - z_n\|^2 - \|u_n - p\|^2. \tag{25}$$

Combining (24) and (25), we derive

$$\|z_n - p\|^2 - \|u_n - z_n\|^2 - \|u_n - p\|^2 \ge 2\langle z_n - y_n, u_n - y_n\rangle - 2\mu_1\tau_n\|z_n - y_n\|^2 - 2\mu_2\tau_n\|y_n - u_n\|^2,$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 \le{}& \|z_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle z_n - y_n, u_n - y_n\rangle\\ &+ 2\mu_1\tau_n\|z_n - y_n\|^2 + 2\mu_2\tau_n\|y_n - u_n\|^2\\ ={}& \|z_n - p\|^2 - (1 - 2\mu_2\tau_n)\|u_n - y_n\|^2 - (1 - 2\mu_1\tau_n)\|y_n - z_n\|^2.\end{aligned} \tag{26}$$

On the basis of (7) and Lemma 2.2 (ii) and Lemma 2.3, we obtain

$$\begin{aligned}\|z_n - p\|^2 ={}& \|(1 - \alpha_n)(x_n - p) + \alpha_n(T[(1 - \beta_n)x_n + \beta_n Tx_n] - p)\|^2\\ ={}& (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|T[(1 - \beta_n)x_n + \beta_n Tx_n] - x_n\|^2\\ &+ \alpha_n\|T[(1 - \beta_n)ux_n + \beta_n Tx_n] - p\|^2\\ \le{}& (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|T[(1 - \beta_n)x_n + \beta_n Tx_n] - x_n\|^2\\ &+ \alpha_n(\|x_n - p\|^2 + (1 - \beta_n)\|x_n - T[(1 - \beta_n)x_n + \beta_n Tx_n]\|^2)\\ ={}& \|x_n - p\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - T[(1 - \beta_n)x_n + \beta_n Tx_n]\|^2\\ \le{}& \|x_n - p\|^2.\end{aligned} \tag{27}$$

Substituting (27) into (26), we have

$$\begin{aligned}\|u_n - p\|^2 \le{}& \|x_n - p\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - T[(1 - \beta_n)x_n + \beta_n Tx_n]\|^2\\ &- (1 - 2\mu_2\tau_n)\|u_n - y_n\|^2 - (1 - 2\mu_1\tau_n)\|y_n - z_n\|^2.\end{aligned} \tag{28}$$

Now, we prove $Fix(T) \cap EP(f, C) \subset C_n \cap Q_n$ for all $n \ge 0$. By (28), we deduce that $\|u_n - p\| \le \|x_n - p\|$ which implies that $p \in C_n$. Therefore, $Fix(T) \cap EP(f, C) \subset C_n$ for all $n \ge 0$.

Next, we show that $Fix(T) \cap EP(f, C) \subset Q_n$ for all $n \ge 0$. First, it is obvious that $Fix(T) \cap EP(f, C) \subset Q_0$. Assume that $Fix(T) \cap EP(f, C) \subset Q_k$. By (12) and the property (6) of projection, we get $\langle q^\ddagger - x_{k+1}, x_0 - x_{k+1}\rangle \le 0$

for all $q^\ddagger \in Fix(T) \cap EP(f, C)$ because of $Fix(T) \cap EP(f, C) \subset C_k \cap Q_k$. Thus, $Fix(T) \cap EP(f, C) \subset Q_{k+1}$. Therefore, $Fix(T) \cap EP(f, C) \subset C_n \cap Q_n (\forall n \geq 0)$ by induction.

According to (12), we get

$$\langle x_0 - x_{n+1}, x_{n+1} - q^\ddagger \rangle \geq 0, \quad \forall q^\ddagger \in C_n \cap Q_n. \tag{29}$$

Since $Fix(T) \cap EP(f, C) \subset C_n \cap Q_n (\forall n \geq 0)$, from (29), we have

$$\langle x_0 - x_{n+1}, x_{n+1} - u \rangle \geq 0, \quad \forall u \in Fix(T) \cap EP(f, C). \tag{30}$$

At the same time, we have

$$\begin{aligned} \langle x_0 - x_{n+1}, x_{n+1} - u \rangle &= \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_{n+1}\|^2 + \langle x_0 - x_{n+1}, x_0 - u \rangle \\ &\leq -\|x_0 - x_{n+1}\|^2 + \|x_0 - x_{n+1}\|\|x_0 - u\|. \end{aligned}$$

This together with (30) implies that

$$\|x_{n+1} - x_0\| \leq \|x_0 - u\|, \quad \forall u \in Fix(T) \cap EP(f, C). \tag{31}$$

Therefore, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{z_n\}$ and $\{u_n\}$ are also bounded by (26) and (27). Applying Lemma 2.4, we deduce that the sequence $\{y_n\}$ is bounded.

Noting that $x_n = P_{Q_n}(x_0)$ by (11) and $x_{n+1} \in Q_n$ by (12), we obtain

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists due to the boundedness of the sequence $\{x_n\}$.

By Lemma 2.2 (iii), we deduce

$$\|x_{n+1} - x_n\|^2 = \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle.$$

Since $x_{n+1} \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0.$$

It follows that

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|u_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,$$

and hence

$$\begin{aligned} \|x_n - u_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| \\ &\leq 2\|x_{n+1} - x_n\| \\ &\to 0. \end{aligned} \tag{32}$$

By (28), we have

$$\begin{aligned} &\alpha_n(\beta_n - \alpha_n)\|x_n - T[(1 - \beta_n)x_n + \beta_n T x_n]\|^2 + (1 - 2\mu_2\tau_n)\|u_n - y_n\|^2 + (1 - 2\mu_1\tau_n)\|y_n - z_n\|^2 \\ &\leq \|x_n - p\|^2 - \|u_n - p\|^2 \\ &\leq \|x_n - u_n\|[\|x_n - p\| + \|u_n - p\|]. \end{aligned} \tag{33}$$

According to (32), (33) and the restrictions (C1) and (C2), we get

$$\lim_{n\to\infty} \|x_n - T[(1 - \beta_n)x_n + \beta_n T x_n]\| = 0 \tag{34}$$

and

$$\lim_{n\to\infty} \|u_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{35}$$

From (7), we derive

$$\|z_n - x_n\| = \alpha_n \|x_n - T[(1-\beta_n)x_n + \beta_n T x_n]\|,$$

which together with (34) and $\liminf_{n\to\infty} \alpha_n > 0$ (by (C2)) implies that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{36}$$

On the other hand, using the Lipschitz property of T, we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T[(1-\beta_n)x_n + \beta_n T x_n]\| + \|T[(1-\beta_n)x_n + \beta_n T x_n] - Tx_n\| \\ &\le \|x_n - T[(1-\beta_n)x_n + \beta_n T x_n]\| + L\beta_n \|x_n - Tx_n\|.\end{aligned}$$

It follows that

$$\|x_n - Tx_n\| \le \frac{1}{1 - L\beta_n} \|x_n - T[(1-\beta_n)x_n + \beta_n T x_n]\|. \tag{37}$$

Since $\liminf_{n\to\infty} \beta_n < \frac{1}{L}$, combining (34) and (37), we deduce

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{38}$$

Note that the sequence $\{z_n\}$ is bounded. Selecting any $x^\ddagger \in \omega_w(z_n)$, there exists a subsequence $\{z_{n_i}\} \subset \{z_n\}$ such that

$$z_{n_i} \rightharpoonup x^\ddagger \in C. \tag{39}$$

From (8), we obtain

$$f(z_{n_i}, z^\ddagger) \ge f(z_{n_i}, y_{n_i}) + \frac{1}{\tau_{n_i}} \langle z_{n_i} - y_{n_i}, z^\ddagger - y_{n_i} \rangle, \ \forall z^\ddagger \in C. \tag{40}$$

Thanks to (35), (A1) and (A3), we get

$$\lim_{i\to\infty} f(z_{n_i}, y_{n_i}) = 0.$$

This together with (40) implies that

$$f(x^\ddagger, z^\ddagger) \ge 0, \ \forall z^\ddagger \in C.$$

Therefore, $x^\ddagger \in EP(f, C)$.

By (36) and (39), we have $x_{n_i} \rightharpoonup x^\ddagger \in C$. Combining with (38), we deduce

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0.$$

Applying Lemma 2.5, we conclude that $x^\ddagger \in Fix(T)$. Therefore, $x^\ddagger \in Fix(T) \cap EP(f, C)$.

Setting $u^\ddagger = P_{Fix(T)\cap EP(f,C)}[x_0]$, from (31), we obtain

$$\|x_{n+1} - x_0\| \le \|x_0 - u^\ddagger\|, \quad \forall n \ge 0. \tag{41}$$

Applying Lemma 2.6 to (41), we conclude that $x_n \to x^\ddagger$. □

Setting $T = I$, the identity operator, we obtain the following iterative algorithm for finding a solution in $EP(f, C)$.

Algorithm 3.3. *Step 0. (Initialization) Fix* $x_0 \in C$.

Step 1.(Extragradient technique) For given $\{x_n\}$*, solve the successively strong convex programs*

$$\min\left\{f(x_n, x^\ddagger) + \frac{1}{2\tau_n}\|x_n - x^\ddagger\|^2 : x^\ddagger \in C\right\}$$

and

$$\min\left\{f(y_n, x^\ddagger) + \frac{1}{2\tau_n}\|x_n - x^\ddagger\|^2 : x^\ddagger \in C\right\},$$

to achieve their unique solutions y_n *and* u_n*, respectively.*

Step 2. (CQ technique) Construct the following two half-spaces to cut C:

$$C_n = \{q^\ddagger \in C : \|u_n - q^\ddagger\| \le \|x_n - q^\ddagger\|\}$$

and

$$Q_n = \{q^\ddagger \in C : \langle x_n - q^\ddagger, x_0 - x_n\rangle \ge 0\}.$$

Step 3. (Projection technique) Compute the sequence $\{x_{n+1}\}$ *by the following projection method*

$$x_{n+1} = P_{C_n \cap Q_n}[x_0].$$

Step 4. Set $n := n + 1$ *and return to Step 1.*

Corollary 3.4. *Suppose that* $EP(f, C) \ne \emptyset$*. Then, the sequence* $\{x_n\}$ *generated by Algorithm 3.3 converges strongly to* $u^\ddagger = P_{EP(f,C)}[x_0]$.

Setting $f = 0$, we obtain the following iterative algorithm for finding a point in $Fix(T)$.

Algorithm 3.5. *Step 0. (Initialization) Fix* $x_0 \in C$.

Step 1. (Fixed point step) For given $\{x_n\}$*, compute the sequence* $\{z_n\}$ *by*

$$z_n = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n].$$

Step 2. (CQ technique) Construct the following two half-spaces to cut C:

$$C_n = \{q^\ddagger \in C : \|z_n - q^\ddagger\| \le \|x_n - q^\ddagger\|\}$$

and

$$Q_n = \{q^\ddagger \in C : \langle x_n - q^\ddagger, x_0 - x_n\rangle \ge 0\}.$$

Step 3. (Projection technique) Compute the sequence $\{x_{n+1}\}$ *by the following projection method*

$$x_{n+1} = P_{C_n \cap Q_n}[x_0].$$

Step 4. Set $n := n + 1$ *and return to Step 1.*

Corollary 3.6. *Suppose that* $Fix(T) \ne \emptyset$*. Then, the sequence* $\{x_n\}$ *generated by Algorithm 3.5 converges strongly to* $u^\ddagger = P_{Fix(T)}[x_0]$.

Remark 3.7. *If T is nonexpansive, then the above conclusions hold.*

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