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A Fixed Point Theorem of Kannan Type That Characterizes Fuzzy Metric Completeness

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Abstract. We obtain a fixed point theorem for complete fuzzy metric spaces, in the sense of Kramosil and Michalek, that extends the classical Kannan fixed point theorem. We also show that, in fact, our theorem allows to characterize the fuzzy metric completeness, extending in this way the well-known Reich-Subrahmanyam theorem that a metric space is complete if and only if every Kannan contraction on it has a fixed point.

1. Introduction and preliminaries

The first characterization of metric completeness by means of fixed point theorems was obtained by Hu in [7]. He proved that a metric space is complete if and only if every Banach contraction on any of its closed subsets has a fixed point. The antecedent of Hu's theorem is in an article of Connell [4, Second part of Example 4], where an example of a non complete metric space for which every Banach contraction has fixed point was given. Later, Reich [13] and Subrahmanyam [16], independently proved that the renowned Kannan fixed point theorem [8] characterizes complete metric spaces, while Kirk established in [9] that the celebrated Caristi fixed point theorem [2] also characterizes the metric completeness. Further solutions to the problem of obtaining necessary and sufficient conditions to a metric space be complete in terms of fixed point results may be found in [14, 17, 18].

Kirk's theorem has been partially generalized to fuzzy metric spaces, in the sense of Kramosil and Michalek, in [1, 3], while Hu's theorem was recently extended in [15] to this setting. In this paper, and continuing with this aproach, we propose a notion of Kannan contraction for fuzzy metric spaces that allows us, by one hand, to extend Kannan's fixed point theorem to the fuzzy metric framework and, on the other hand, to characterize fuzzy metric completeness by means of a suitable version of the Reich-Subrahmanyam characterization of metric completeness. In fact, Reich-Subrahmanyam's theorem will be get back from our fuzzy approach.

In order to help the reader, we next recall several notions and properties which will be useful in the rest of the paper.

The sets of positive integer numbers will be denoted by $\mathbb{N}$. Our basic reference for general topology is [6] and for (continuous) t-norms it is [10].

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A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $u * 1 = u$ for every $u \in [0,1]$; (iv) $u * w \leq v * w$ if $u \leq v$, where $u, v, w \in [0,1]$.

Three distinguished examples of continuous t-norms are the minimum $\wedge$, the product $*_P$ and the Łukasiewicz t-norm $*_L$, which are defined, respectively, by $u \wedge v = \min\{u, v\}$, $u *_P v = uv$, and $u *_L v = \max\{u + v - 1, 0\}$.

It is well known that $*_L \leq *_P \leq \wedge$. In fact, $* \leq \wedge$ for any continuous t-norm $*$.

The following well-established concepts and facts may be found, for instance, in [1, Section 1].

Definition 1 (Kramosil and Michalek [11]). A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function from $X \times X \times [0, \infty)$ to $[0,1]$ such that for all $x, y, z \in X$:
(i) $M(x, y, 0) = 0$;
(ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
(v) $M(x, y, \cdot) : [0, \infty) \to [0,1]$ is left continuous.

If $(M, *)$ is a fuzzy metric on a set X, the triple $(X, M, *)$ is said to be a fuzzy metric space.

Remark 1. It is well known, and easy to see, that for each $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function on $[0, \infty)$.

Remark 2. It easily follows from Remark 1 that, given $x, y \in X$, if $M(x, y, t) > 1 - t$ for all $t > 0$, then $x = y$.

Every fuzzy metric $(M, *)$ on a set X induces a metrizable topology τ_M on X which has as a base the family of open sets $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0,1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $\varepsilon \in (0,1)$, $t > 0$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if for any $\varepsilon \in (0,1)$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges with respect to the topology τ_M, i.e., if there exists $y \in X$ such that for each $t > 0$, $\lim_n M(y, x_n, t) = 1$.

We conclude this section by recalling a basic but important example of a fuzzy metric space which will be very useful, mainly, in the final part of our study.

Example 1. Let (X, d) be a metric space. Define a function $M_{01}^d : X \times X \times [0, \infty) \to [0,1]$ as $M_{01}^d(x, y, t) = 1$ if $d(x, y) < t$, and $M_{01}^d(x, y, t) = 0$ if $d(x, y) \geq t$. Then, $(M_{01}^d, *)$ is a fuzzy metric on X for any continuous t-norm $*$, such that the topologies induced by d and $(M_{01}^d, *)$ coincide. Furthermore $(X, M_{01}^d, *)$ is complete if and only if (X, d) is complete.

2. Kannan contractions on fuzzy metric spaces

We start this section by recalling Kannan fixed point theorem and Reich-Subrahmanyam theorem.

Theorem 1 (Kannan [8]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists a constant $c \in (0, 1/2)$ satisfying, for any $x, y \in X$,*

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)]. \qquad \text{(cK)}$$

Then T has a unique fixed point.

A mapping T satisfying condition (cK) is said to be a Kannan contraction on X.

Theorem 2 (Reich [13], Subrahmanyam [16]). *A metric space (X, d) is complete if and only if every Kannan contraction on X has a fixed point.*

We propose the following notion of Kannan contraction for fuzzy metric spaces.

Definition 2. Let $(X, M, *)$ be a fuzzy metric space. We say that a mapping $T : X \to X$ is a (1)-Kannan contraction on X if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct. \qquad (1K)$$

Similarly, and by analogy to the classical metric case, we say that a mapping $T : X \to X$ is a (1/2)-Kannan contraction on X if there is a constant $c \in (0, 1/2)$ such that for any $x, y \in X$ and $t > 0$,

$$\min\{M(x, Tx, t), M(y, Ty, t)\} > 1 - t \Rightarrow M(Tx, Ty, ct) > 1 - ct.$$

Obviously, every (1/2)-Kannan contraction is a (1)-Kannan contraction. However (see Example 2 below) there exist (1)-Kannan contractions, with constant $c = 1/2$, that are not (1/2)-Kannan contractions. In our first result we prove a fixed point theorem valid for (1)-Kannan contractions which contrasts with the classical metric case where is a crucial requirement that the constant c belongs to $(0, 1/2)$.

Theorem 3. *Every (1)-Kannan contraction on a complete fuzzy metric space has a unique fixed point.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a (1)-Kannan contraction on X. Then, there exists a constant $c \in (0, 1)$ for which condition (1K) in Definition 2 is satisfied.

Fix $t_0 > 1$. For any $x, y \in X$ we have

$$M(x, Tx, t_0) > 1 - t_0 \quad \text{and} \quad M(y, Ty, t_0) > 1 - t_0.$$

So, by condition (1K),

$$M(Tx, Ty, ct_0) > 1 - ct_0.$$

From

$$M(x, Tx, t_0) > 1 - t_0 \quad \text{and} \quad M(Tx, T^2x, t_0) > 1 - t_0,$$

we deduce that

$$M(Tx, T^2x, ct_0) > 1 - ct_0,$$

and, similarly,

$$M(Ty, T^2y, ct_0) > 1 - ct_0.$$

Therefore, by condition (1K),

$$M(T^2x, T^2y, c^2t_0) > 1 - c^2t_0.$$

Following this process we obtain, for each $n \in \mathbb{N}$,

$$M(T^nx, T^ny, c^nt_0) > 1 - c^nt_0.$$

Now fix an $x_0 \in X$. Define $x_n := T^nx_0$ for all $n \in \mathbb{N}$. We show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$.

Indeed, given $t > 0$ and $\varepsilon \in (0,1)$, there is $n_\varepsilon \in \mathbb{N}$ such that $c^n t_0 < \min\{\varepsilon, t\}$ for all $n \geq n_\varepsilon$. Let $m, n \geq n_\varepsilon$. Suppose $m > n$. Then $m = n + k$ for some $k \in \mathbb{N}$, and hence

$$\begin{aligned} M(x_n, x_m, t) &= M(T^n x_0, T^n T^k x_0, t) \geq M(T^n x_0, T^n T^k x_0, c^n t_0) \\ &> 1 - c^n t_0 > 1 - \varepsilon. \end{aligned}$$

Consequently $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. So there is $z \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to z in τ_M.

Next we shall prove that z is a fixed point of T. (The proof of this claim is not immediate, so several details must be checked.)

Fix $r, s > 0$ such that $c < s < r < 1$.

Our first step is to show, by applying induction, that for each $k \in \mathbb{N}$,

$$M(z, Tz, r^k t_0) \geq 1 - r^k t_0. \qquad (\diamond)$$

(We can suppose, without loss of generality that $r^k t_0 \leq 1$ for all $k \in \mathbb{N}$: Otherwise, the inequality $(\diamond)$ is trivially satisfied.)

For each $k \in \mathbb{N}$ we define

$$A_{k,r,s} := \{\varepsilon \in (0,1) : \varepsilon + sr^{k-1} t_0 < r^k t_0\}.$$

Let $k = 1$. Then, we have

$$M(z, Tz, t_0) > 1 - t_0 \quad \text{and} \quad M(x_n, x_{n+1}, t_0) > 1 - t_0.$$

So, by Remark 1 and condition (1K), we obtain

$$M(Tz, x_{n+1}, st_0) \geq M(Tz, x_{n+1}, ct_0) > 1 - ct_0 > 1 - st_0.$$

for all $n \in \mathbb{N} \cup \{0\}$.

Since $(x_n)_{n\in\mathbb{N}}$ converges to z, for any $\varepsilon \in A_{1,r,s}$ there is $n_\varepsilon \in \mathbb{N}$ for which $M(z, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Consequently

$$\begin{aligned} M(z, Tz, rt_0) &\geq M(z, x_{n_\varepsilon}, \varepsilon) * M(x_{n_\varepsilon}, Tz, st_0) \geq (1 - \varepsilon) * (1 - st_0) \geq \\ &(1 - \varepsilon) * (1 - rt_0). \end{aligned}$$

Taking limit when $\varepsilon \to 0$ we deduce, by continuity of $*$, that

$$M(z, Tz, rt_0) \geq 1 - rt_0.$$

Suppose now that the inequality $(\diamond)$ holds for $k = j$, $j \in \mathbb{N}$. We are going to show that $M(z, Tz, r^{j+1} t_0) \geq 1 - r^{j+1} t_0$.

Indeed, we get

$$M(z, Tz, r^j t_0) > 1 - r^j t_0.$$

Moreover, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there is $n_j \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, r^j t_0) > 1 - r^j t_0,$$

for all $n \geq n_j$. Then, it follows from condition (1K) that

$$M(Tz, x_{n+1}, cr^j t_0) > 1 - cr^j t_0,$$

for all $n \geq n_j$. Since $s > c$, we deduce from the preceding inequality that

$$M(Tz, x_{n+1}, sr^j t_0) > 1 - sr^j t_0,$$

for all $n \geq n_j$.

Now let $\varepsilon \in A_{j+1,r,s}$. Then $\varepsilon + sr^j t_0 < r^{j+1} t_0$, and there exists $n_\varepsilon > n_j$ such that $M(z, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Therefore

$$\begin{aligned} M(z, Tz, r^{j+1} t_0) &\geq M(z, x_{n_\varepsilon}, \varepsilon) * M(x_{n_\varepsilon}, Tz, sr^j t_0) \\ &\geq (1 - \varepsilon) * (1 - sr^j t_0) \geq (1 - \varepsilon) * (1 - r^{j+1} t_0). \end{aligned}$$

Taking limit when $\varepsilon \to 0$ we deduce, by continuity of $*$, that

$$M(z, Tz, r^{j+1} t_0) \geq 1 - r^{j+1} t_0.$$

We have shown that inequality ($\diamond$) holds.

Now, given $t > 0$ there is $k \in \mathbb{N}$ such that $r^k t_0 < t$, so $M(z, Tz, t) \geq M(z, Tz, r^k t_0) > 1 - r^k t_0 > 1 - t$. By Remark 2 we conclude that $z = Tz$.

It remains to show that z is the unique fixed point of T. Let $u \in X$ such that $u = Tu$. Then

$$\min\{M(z, Tz, t), M(u, Tu, t)\} = 1,$$

for all $t > 0$. Hence, by condition (1K), $M(Tz, Tu, ct) > 1 - ct$, for all $t > 0$, which implies that $Tz = Tu$, i.e., $z = u$. This concludes the proof.

Corollary 1. *Every (1/2)-Kannan contraction on a complete fuzzy metric space has a unique fixed point.*

We finish this section with the promised example of a (1)-Kannan contraction, for $c = 1/2$, that is not (1/2)-Kannan contraction. In fact, it also provides an instance where we can apply Theorem 3 but not Corollary 1.

Example 2. Let $X = [0, 3/2]$ and let d be the usual metric on X. Then $(X, M_{01}^d, \wedge)$ is a complete fuzzy metric space. Define $T : X \to X$ as $Tx = 0$ for all $x \in [0, 1]$, and $Tx = x/3$ for all $x \in (1, 3/2]$.

We first show that T is not a (1/2)-Kannan contraction.

Choose $c \in (0, 1/2)$. Take $x = 0$, $y = 3/2$ and $t = 1/2c$. Then $t > 1$, so

$$\min\{M_{01}^d(x, Tx, t), M_{01}^d(y, Ty, t)\} > 1 - t.$$

However

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(0, 1/2, 1/2) = 0 < 1 - ct.$$

Therefore T is not a (1/2)-Kannan contraction on X.

Now we shall prove that T is a (1)-Kannan contraction for $c = 1/2$.

To this end, we shall distinguish three cases for any $x, y \in X$.

Case 1. $x, y \in [0, 1]$. Then, for any $t > 0$,

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(0, 0, t/2) = 1 > 1 - ct,$$

and, hence, condition (1K) is obviously satisfied.

Case 2. $x \in [0, 1]$, $y \in (1, 3/2]$.

If $t > 1$ we have $y/3 \leq 1/2 < t/2$, so

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(0, y/3, t/2) = 1 > 1 - ct,$$

and, hence, condition (1K) is obviously satisfied.

If $t \leq 1$ and assuming that

$$\min\{M_{01}^d(x, Tx, t), M_{01}^d(y, Ty, t)\} > 1 - t,$$

we deduce that $M_{01}^d(x, 0, t) = M_{01}^d(y, y/3, t) = 1$. Consequently ($x < t$ and) $2y/3 < t$, which implies that

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(0, y/3, t/2) = 1 > 1 - ct.$$

Case 3. $x, y \in (1, 3/2]$. We shall assume $x > y$, without loss of generality.

If $t > 1$ we have $(x - y)/3 < x/3 \leq 1/2 < t/2$, so

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(x/3, y/3, t/2) = 1 > 1 - ct,$$

and, hence, condition (1K) is obviously satisfied.

If $t \leq 1$ and assuming that

$$\min\{M_{01}^d(x, Tx, t), M_{01}^d(y, Ty, t)\} > 1 - t,$$

we deduce that $M_{01}^d(x, x/3, t) = M_{01}^d(y, y/3, t) = 1$. Consequently $2x/3 < t$ (and $2y/3 < t$); thus $(x - y)/3 < x/3 < t/2$, which implies that

$$M_{01}^d(Tx, Ty, ct) = M_{01}^d(x/3, y/3, t/2) = 1 > 1 - ct.$$

We have proved that T is a (1)-Kannan contraction on X.

3. A characterization of fuzzy metric completeness

In this section we prove the main result of the paper (Theorem 5 below). The notion of a semi-metric and an important result of Radu will be crucial ingredients in obtaining that result.

A semi-metric (compare e.g. [5]) for a topological space (X, τ) is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions for every $x, y \in X$ and a subset A of X :

(i) $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $x \in \overline{A} \Leftrightarrow d(x, A) = 0$. (As usual, $\overline{A}$ denotes the closure of A in (X, τ).)

As in the metric case, if d is a semi-metric for a topological space (X, τ), a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

Theorem 4. (Radu [12] Proposition 2.1.1) *Let $(X, M, *)$ be a fuzzy metric space. For each $x, y \in X$ put*

$$d_M(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}.$$

Then d_M is a semi-metric for (X, τ_M), satisfying $d_M \leq 1$ and

$$d_M(x, y) < t \Longleftrightarrow M(x, y, t) > 1 - t,$$

for all $t > 0$.Therefore, a sequence in X is a Cauchy sequence for d_M if and only if it is a Cauchy sequence in $(X, M, *)$.
Moreover, if $*_L \leq *$, then d_M is a metric on X whose induced topology coincides with τ_M.

Theorem 5. *For a fuzzy metric space $(X, M, *)$ the following are equivalent.*
(1) *$(X, M, *)$ is complete.*
(2) *Every (1)-Kannan contraction on X has a fixed point.*

(3) *Every (1/2)-Kannan contraction on X has a fixed point.*

Proof. (1) ⇒ (2). Theorem 3.
(2) ⇒ (3). Obvious.
(3) ⇒ (1). Suppose that $(X, M, *)$ is not complete. Then there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, M, *)$ that does not converge in τ_M. We assume, without loss of generality, that $x_n \neq x_m$ whenever $n \neq m$.

Set $C := \{x_n : n \in \mathbb{N}\}$. Then, both C and all sets of the form $C\backslash\{y\}$, $y \in X$, are closed for τ_M.

Then, by condition (iii) in the definition of a semi-metric, for any $y \in X\backslash C$ we have $d_M(y, C\backslash\{y\}) > 0$ (note that, in this case, the sets C and $C\backslash\{y\}$ coincide). Since, by Theorem 4, $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence for the semi-metric d_M, there is $n(y) \in \mathbb{N}$ such that

$$d_M(x_j, x_k) < \frac{1}{4} d_M(y, C\backslash\{y\}),$$

for all $j, k \geq n(y)$.

On the other hand, for any $y \in C$ there is $n \in \mathbb{N}$ such that $y := x_n$. Since $d_M(y, C\backslash\{y\}) > 0$, there is $n(y) > n$ such that

$$d_M(x_j, x_k) < \frac{1}{4} d_M(y, C\backslash\{y\}),$$

for all $j, k \geq n(y)$.

Define a mapping $T : X \to X$ as follows:

$$Ty = x_{n(y)},$$

for all $y \in X$. Obviously T has no fixed points (observe, in particular, that $x_n \neq x_{n(y)}$ because we have taken $n(y) > n$).

We shall show that, however, T is a (1/2)-Kannan contraction on $(X, M, *)$.

Indeed, let $y, z \in X$ such that

$$\min\{M(y, Ty, t), M(z, Tz, t)\} > 1 - t,$$

for $t > 0$. Therefore, by Theorem 4,

$$d_M(y, Ty) < t \quad \text{and} \quad d_M(z, Tz) < t.$$

If $n(y) < n(z)$, we obtain

$$d_M(Ty, Tz) = d_M(x_{n(y)}, x_{n(z)}) < \frac{1}{4} d_M(y, C\backslash\{y\}) \leq \frac{1}{4} d_M(y, Ty) < \frac{1}{4} t.$$

Consequently, by Theorem 4,

$$M(Ty, Tz, t/4) > 1 - t/4.$$

If $n(z) < n(y)$, we similarly obtain

$$d_M(Ty, Tz) = d_M(x_{n(y)}, x_{n(z)}) < \frac{1}{4} d_M(z, C\backslash\{z\}) \leq \frac{1}{4} d_M(z, Tz) < \frac{1}{4} t,$$

and thus

$$M(Ty, Tz, t/4) > 1 - t/4.$$

We have shown that T is a (1/2)-Kannan contraction. This finishes the proof.

We will conclude the paper by showing that we can get back Reich-Subrahmanyam's theorem from our fuzzy approach in a similar way to the one given in [15] for the case of Hu's theorem.

If (X, d) is a metric space, we denote by $d_{1/2}$ the metric defined on X by $d_{1/2}(x, y) = \min\{1/2, d(x, y)\}$ for all $x, y \in X$. It is clear that the topologies induced by d and $d_{1/2}$ coincide, and that (X, d) is complete if and only if $(X, d_{1/2})$ is complete. Hence $(X, M_{01}^d, *)$ is complete if and only if $(X, M_{01}^{d_{1/2}}, *)$ is complete, for any continuous t-norm $*$ (compare Example 1).

Proposition 2. *Let (X, d) be a metric space. Then, every Kannan contraction on $(X, d_{1/2})$ is a Kannan contraction on (X, d).*

Proof. Let T be a Kannan contraction on $(X, d_{1/2})$. Then, there exists $c \in (0, 1/2)$ such that $d_{1/2}(Tx, Ty) \leq c[d_{1/2}(x, Tx) + d_{1/2}(y, Ty)]$, for all $x, y \in X$.

If $d_{1/2}(Tx, Ty) = 1/2$ for some $x, y \in X$, we deduce that $1/2 \leq c[1/2 + 1/2] = c$, a contradiction. Hence $d_{1/2}(Tx, Ty) = d(Tx, Ty) < 1/2$ for all $x, y \in X$, and thus

$$\begin{aligned} d(Tx, Ty) &= d_{1/2}(Tx, Ty) \leq c[d_{1/2}(x, Tx) + d_{1/2}(y, Ty)] \\ &\leq c[d(x, Tx) + d(y, Ty)]. \end{aligned}$$

Consequently T is a Kannan contraction on (X, d).

Proposition 3. *Let (X, d) be a metric space with $d \leq 1$. Then, for any continuous t-norm $*$, every (1/2)-Kannan contraction on $(X, M_{01}^d, *)$ is a Kannan contraction on (X, d).*

Proof. Let $T : X \to X$ be a (1/2)-Kannan contraction on $(X, M_{01}^d, *)$, with constant $c \in (0, 1/2)$. We want to show that $d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)]$, for all $x, y \in X$.

Indeed, assume the contrary. Then, there exist $x, y \in X$ such that $d(Tx, Ty) > c[d(x, Tx) + d(y, Ty)]$. Put $L = d(x, Tx) + d(y, Ty)$. Then $L \in [0, 2)$. Choose an $\varepsilon > 0$ for which $c(L + \varepsilon) < 1$ and $d(Tx, Ty) > c(L + \varepsilon)$. Since $M_{01}^d(x, Tx, L + \varepsilon) = 1$ and $M_{01}^d(y, Ty, L + \varepsilon) = 1$, and T is a (1/2)-Kannan contraction we deduce that $M_{01}^d(Tx, Ty, c(L + \varepsilon)) > 1 - c(L + \varepsilon) > 0$. But, on the other hand, from $d(Tx, Ty) > c(L + \varepsilon)$, it follows that $M_{01}^d(Tx, Ty, c(L + \varepsilon)) = 0$. This contradiction concludes the proof.

Proof of Theorem 2: We only show the "if" part. Suppose that every Kannan contraction on (X, d) has a fixed point. Let T be a (1/2)-Kannan contraction on the fuzzy metric space $(X, M_{01}^{d_{1/2}}, *)$. Since $d_{1/2} \leq 1$ it follows from Proposition 3 that T is a Kannan contraction on $(X, d_{1/2})$, and thus it is a Kannan contraction on (X, d) by Proposition 2. Hence T has a fixed point. Consequently $(X, M_{01}^{d_{1/2}}, *)$ is complete by Theorem 5. Thus $(X, M_{01}^d, *)$ is complete and, hence, (X, d) is complete.

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