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The p-Drazin Inverse for Operator Matrix over Banach Algebras

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Abstract. An element a in a Banach algebra $\mathcal{A}$ has p-Drazin inverse provided that there exists $b \in comm(a)$ such that $b = b^2a$, $a^k - a^{k+1}b \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. In this paper, we present new conditions for a block operator matrix to have p-Drazin inverse. As applications, we prove the p-Drazin invertibility of the block operator matrix under certain spectral conditions.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in A \mid xa = ax\}$. An element a in a Banach algebra $\mathcal{A}$ has p-Drazin inverse provided that there exists $b \in comm(a)$ such that $b = b^2a, a^k - a^{k+1}b \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. The preceding b is unique if exists, and we denote it by $a^{\ddagger}$. We refer the reader to [11, 13, 17, 18] and [20] for more properties of p-Drazin inverse in a Banach algebra.

Recall that $a \in \mathcal{A}$ has g-Drazin inverse provided that there exists $b \in comm(a)$ such that $b = b^2a, a - a^2b \in \mathcal{A}^{qnil}$ (see [6]). More results on g-Drazin inverse can be found in [1–5, 10, 19, 21]. As is well known, every p-Drazin inverse is just the g-Drazin inverse. The p-Drazin inverse should be expressed as that of the g-Drazin inverse if exists. We will suffice to investigate the existence for p-Drazin inverse. This motivates us to present new conditions for a block operator matrix to have p-Drazin inverse.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$. Let $a, d \in \mathcal{A}^{\ddagger}$. If

$$bd^{\ddagger} = 0 \text{ or } d^{\ddagger}c = 0, \text{ and } bd^ic = 0 \text{ for all } i \geq 0,$$

in Section 2, we prove that M has p-Drazin inverse.

In Section 3, we determine the p-Drazin invertibility of the block operator matrix M under certain spectral conditions. If $(bc)^{\pi}abc = 0$ or $bca(bc)^{\pi} = 0$, and $bd = 0$, then M has p-Drazin inverse.

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Throughout the paper, all Banach algebras are complex with an identity. We use $J(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. $\sqrt{J(\mathcal{A})}$ stands for the radical of $J(\mathcal{A})$, i.e., $\sqrt{J(\mathcal{A})} = \{x \mid x^m \in J(\mathcal{A}) \text{ for some } m \in \mathbb{N}\}$. $\mathcal{A}^{\ddagger}$ denotes the set of all elements having p-Drazin inverses in $\mathcal{A}$. For any $a \in \mathcal{A}^{\ddagger}$, we use a^{π} to stand for the spectral idempotent $1 - aa^{\ddagger}$.

2. 2 × 2 Operator matrices

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$. The aim of this section is to determine when M has p-Drazin inverse under certain conditions and generalize [9, Theorem 3.2] from g-Drazin inverse to p-Drazin inverse. The following lemmas are crucial.

Lemma 2.1. *Let $a, b \in \mathcal{A}$ and $a, b \in \sqrt{J(\mathcal{A})}$. If $ab^k a = 0$ for any $k \in \mathbb{N}$, then $a + b \in \sqrt{J(\mathcal{A})}$.*

Proof. Let $a^m, b^n \in J(\mathcal{A})$. Assume that $t = m + n$. Then $a^t, b^t \in J(\mathcal{A})$. Let $s = 3t + 1$. Then every term of $(a + b)^s$ should be

$$a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots a^{i_s} b^{j_s},$$

where $i_1, j_1, \cdots, i_s, j_s \geq 0, i_1 + j_1 + \cdots + i_s + j_s = s$. If the term contains $ab^k a (k \in \mathbb{N})$, then it is zero. So the nonzero terms should be written in the form $b^k a^l b^r (k + l + r = s)$. Then k or l or r is greater than t. Hence, $b^k a^l b^r \in J(\mathcal{A})$. Therefore $(a + b)^s \in J(\mathcal{A})$, as asserted. □

Lemma 2.2. *Suppose that $a, d \in \sqrt{J(\mathcal{A})}$. If $bd^i c = 0$ for all $i \geq 0$, then $M \in \sqrt{J\big(M_2(\mathcal{A})\big)}$.*

Proof. Let $a^m, d^n \in J(\mathcal{A})$ and $M = P + Q$, where $P = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, and $Q = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$.

Since $bc = 0$, we see that

$$\begin{aligned} \begin{pmatrix} a & 1 \\ bc & 0 \end{pmatrix}^{m+1} &= \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}^{m+1} \\ &\in J\big(M_2(\mathcal{A})\big), \end{aligned}$$

and so

$$\begin{aligned} P^{m+2} &= \left[\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \right]^{m+2} \\ &= \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} a & 1 \\ bc & 0 \end{pmatrix}^{m+1} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \\ &\in J\big(M_2(\mathcal{A})\big). \end{aligned}$$

Clearly, $Q^n \in J\big(M_2(\mathcal{A})\big)$. We easily check that

$$PQ^k P = 0$$

for any $k \in \mathbb{N}$. Hence, $M = P + Q \in \sqrt{J(\mathcal{A})}$ by Lemma 2.1. □

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra. If $a \in A^{\ddagger}$, $d \in \sqrt{J(\mathcal{A})}$ and $bd^i c = 0$ for all $i \geq 0$, then M has p-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} a^{\ddagger} & \gamma \\ \delta & \delta a \gamma \end{pmatrix}$, where

$$\gamma = \sum_{i=0}^{\infty} (a^{\ddagger})^{i+2} b d^i, \delta = \sum_{i=0}^{\infty} d^i c (a^{\ddagger})^{i+2}.$$

By hypothesis, we have

$$\gamma d^i c = 0, bd^i\delta = 0, \gamma d^i\delta = 0$$

for any $i \geq 0$. We shall prove that $N = M^{\ddagger}$.

Step 1. $MN = NM$. We compute that

$$MN = \begin{pmatrix} aa^{\ddagger} & a\gamma \\ ca^{\ddagger} + d\delta & c\gamma + d\delta a\gamma \end{pmatrix},$$
$$NM = \begin{pmatrix} a^{\ddagger}a & a^{\ddagger}b + \gamma d \\ \delta a & \delta b + \delta a\gamma d \end{pmatrix}.$$

As in the proof of [9, Lemma 3.1], we easily check that $MN = NM$.

Step 2. $N = MN^2$. We have

$$I - MN = \begin{pmatrix} a^{\pi} & -a\gamma \\ -ca^{\ddagger} - d\delta & 1 - c\gamma - d\delta a\gamma \end{pmatrix}.$$

As $\gamma c = \gamma\delta = 0$, we have

$$N(I - MN) = \begin{pmatrix} 0 & \gamma - a^{\ddagger}a\gamma \\ \delta a^{\pi} & 0 \end{pmatrix} = 0.$$

Hence, $N = MN^2$.

Step 3. $M - M^2N \in \sqrt{J(\mathcal{A})}$. Since $bd^ic = 0$ for all $i \geq 0$, we easily verify that

$$M(I - MN) = \begin{pmatrix} aa^{\pi} & b - a^2\gamma \\ ca^{\pi} - dca^{\ddagger} - d^2\delta & d - \sigma \end{pmatrix},$$

where $\sigma = ca\gamma + dc\gamma + d^2\delta a\gamma$.

Clearly, $aa^{\pi} \in \sqrt{J(\mathcal{A})}$. By hypothesis, we see that $\sigma d^i\sigma = 0$ for all $i \geq 0$. Hence, $\sigma^2 = 0$, and so $d, -\sigma \in \sqrt{J(\mathcal{A})}$. In view of Lemma 2.1, we see that $d - \sigma \in \sqrt{J(\mathcal{A})}$. Moreover, we have

$$(b - a^2\gamma)(d - \sigma)^m(ca^{\pi} - dca^{\ddagger} - d^2\delta) = 0$$

for all $m \geq 0$. Therefore $M - M^2N \in \sqrt{J(\mathcal{A})}$ by Lemma 2.2.

Therefore $N = M^{\ddagger}$, as asserted. □

We have accumulated all information necessary to prove the following.

Theorem 2.4. *Let $a, d \in \mathcal{A}^{\ddagger}$. If*

$$bd^{\ddagger} = 0, bd^ic = 0 \text{ for all } i \geq 0,$$

then M has p-Drazin inverse.

Proof. Clearly, $M = P + Q$, where

$$P = \begin{pmatrix} a & b \\ c & dd^{\pi} \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & d^2d^{\ddagger} \end{pmatrix}.$$

Obviously, Q has p-Drazin inverse. Clearly, $bc = 0$. For any $k \geq 0$, we have $b\big(dd^{\pi}\big)^k c = bd^kd^{\pi}c = bd^kc = 0$, and so $b\big(dd^{\pi}\big)^k c = 0$ for all $k \geq 0$. Clearly, $dd^{\pi} = d - d^2d^{\ddagger} \in \sqrt{J(\mathcal{A})}$. In light of Lemma 2.3, P has p-Drazin inverse. On the other hand,

$$PQ = \begin{pmatrix} a & b \\ c & dd^{\pi} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & d^2d^{\ddagger} \end{pmatrix} = 0.$$

Therefore M has p-Drazin inverse, by [13, Theorem 5.4]. □

Corollary 2.5. *Let $a, d \in \mathcal{A}^{\ddagger}$. If*

$$ca^{\ddagger} = 0, ca^{i}b = 0 \text{ for all } i \geq 0,$$

M has p-Drazin inverse.

Proof. In view of Theorem 2.4, the matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ has p-Drazin inverse. It is easy to check that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = I_2$, we easily obtain the result. □

Theorem 2.6. *Let $a, d \in \mathcal{A}^{\ddagger}$. If*

$$d^{\ddagger}c = 0, bd^{i}c = 0 \text{ for all } i \geq 0,$$

M has p-Drazin inverse.

Proof. Clearly, $M = P + Q$, where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & d^2d^{\ddagger} \end{pmatrix}, \; Q = \begin{pmatrix} a & b \\ c & dd^{\pi} \end{pmatrix}.$$

Obviously, P has p-Drazin inverse. As in the proof of Theorem 2.4 we easily check that $b\big(dd^{\pi}\big)^k c = bd^k d^{\pi} c = bd^k c = 0$ for any $k \geq 0$. In view of Lemma 2.3, Q has p-Drazin inverse. By hypothesis, we see that

$$PQ = \begin{pmatrix} 0 & 0 \\ 0 & d^2d^{\ddagger} \end{pmatrix}\begin{pmatrix} a & b \\ c & dd^{\pi} \end{pmatrix} = 0.$$

According to [13, Theorem 5.4], M has p-Drazin inverse, as asserted. □

Corollary 2.7. *Let $a, d \in \mathcal{A}^{\ddagger}$. If*

$$a^{\ddagger}b = 0, ca^{i}b = 0 \text{ for all } i \geq 0,$$

M has p-Drazin inverse.

Proof. In view of Theorem 2.6, the matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ has p-Drazin inverse. As in the proof Corollary 2.5, we easily obtain the result. □

3. Spectral conditions

In this section we apply the preceding results and demonstrate the p-Drazin invertibility of the block matrix M under certain spectral conditions. We now derive

Lemma 3.1. *Let $a, d \in \mathcal{A}^{\ddagger}$. If $abc = 0, bd = 0$ and $bc \in \sqrt{J(\mathcal{A})}$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$.*

Proof. Clearly, we have

$$M^2 = \begin{pmatrix} a^2 + bc & ab \\ ca + dc & cb + d^2 \end{pmatrix}.$$

Clearly, a^2 has p-Drazin inverse. By hypothesis, bc has p-Drazin inverse. Since $a^2(bc) = 0$, it follows from [13, Theorem 5.4] that $a^2 + bc$ has p-Drazin inverse. In light of [13, Theorem 3.6], cb has p-Drazin inverse. Then we easily see that $cb + d^2$ has p-Drazin inverse as $(cb)d^2 = 0$. It is easy to verify that

$$\begin{gathered} ab(cb+d^2)^{\ddagger} = ab(cb+d^2)\big((cb+d^2)^{\ddagger}\big)^2 = 0; \\ ab(cb+d^2)^i(ca+dc) = 0. \end{gathered}$$

In view of Theorem 2.4, M^2 has p-Drazin inverse. Therefore M has p-Drazin inverse by [20, Lemma 2.8]. □

We come now to generalize [16, Theorem 3.1 and Corollary 3.3] from the generalized Drazin inverse to the p-Drazin inverse.

Theorem 3.2. *Let $a, d, bc, (bc)^{\pi}a \in \mathcal{A}^{\ddagger}$. If $(bc)^{\pi}abc = 0$ and $bd = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$.*

Proof. Step 1. Let $h = bc, N = \begin{pmatrix} a & 1 \\ h & 0 \end{pmatrix}$ and $e = \begin{pmatrix} h^{\pi} & 0 \\ 0 & 0 \end{pmatrix}$. Then $N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}_e$, where

$$a' = eNe, b' = eN(I-e), c' = (I-e)Ne, d' = (I-e)N(I-e).$$

Since $(bc)^{\pi}abc = 0$, we have $h^{\pi}ah = 0$, we easily check that

$$\begin{gathered} a' = \begin{pmatrix} h^{\pi}a & 0 \\ 0 & 0 \end{pmatrix}, b' = \begin{pmatrix} 0 & h^{\pi} \\ 0 & 0 \end{pmatrix}, \\ c' = \begin{pmatrix} hh^{\ddagger}ah^{\pi} & 0 \\ hh^{\pi} & 0 \end{pmatrix}, d' = \begin{pmatrix} ahh^{\ddagger} & hh^{\ddagger} \\ h^2h^{\ddagger} & 0 \end{pmatrix}. \end{gathered}$$

Since $h^{\pi}ahh^{\ddagger} = (bc)^{\pi}abc(bc)^{\ddagger} = 0$, it follows by [16, Lemma 2.2] that $(h^{\pi}a)^d = h^{\pi}a^d$. This shows that

$$\begin{gathered} (h^{\pi}a)(h^{\pi}a^d) = (h^{\pi}a^d)(h^{\pi}a), \\ h^{\pi}a^d = (h^{\pi}a^d)(h^{\pi}a)(h^{\pi}a^d). \end{gathered}$$

Since $a \in \mathcal{A}^{\ddagger}$, we have $a^k - a^{k+1}a^d \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. Then we verify that

$$\begin{array}{cl} & \big[h^{\pi}a - (h^{\pi}a)^2(h^{\pi}a^d)\big]^k \\ = & (h^{\pi}a)^k\big[1 - (h^{\pi}a)(h^{\pi}a^d)\big] \\ = & (h^{\pi}a)^k - (h^{\pi}a)^{k+1}a^d \\ = & h^{\pi}a^k - h^{\pi}a^{k+1}a^d \\ = & h^{\pi}(a^k - a^{k+1}a^d) \\ \in & J(\mathcal{A}), \end{array}$$

and so $(h^{\pi}a)^{\ddagger} = h^{\pi}a^{\ddagger}$. Hence, we easily verify that

$$(a')^{\ddagger} = \begin{pmatrix} h^{\pi}a^{\ddagger} & 0 \\ 0 & 0 \end{pmatrix}, (d')^{\ddagger} = \begin{pmatrix} 0 & h^{\ddagger} \\ hh^{\ddagger} & -ahh^{\ddagger} \end{pmatrix}.$$

Hence, $a', d' \in \mathcal{A}^{\ddagger}$. Moreover, we have

$$\begin{gathered} a'b'c' = \begin{pmatrix} h^{\pi}ahh^{\pi} & 0 \\ 0 & 0 \end{pmatrix}; \\ b'd' = \begin{pmatrix} h^{\pi}h^2h^{\ddagger} & 0 \\ 0 & 0 \end{pmatrix}; \\ b'c' = \begin{pmatrix} hh^{\pi} & 0 \\ 0 & 0 \end{pmatrix}. \end{gathered}$$

Therefore $a'b'c' = 0, b'd' = 0$ and $b'c' \in \sqrt{J(\mathcal{A})}$. In light of Lemma 3.1, N has p-Drazin inverse.

Step 2. It is easy to check that

$$N = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix},$$

it follows by [13, Theorem 3.6] that $\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ has p-Drazin inverse. Therefore $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ has p-Drazin inverse.

Step 3. Write $M = P + Q$, where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}.$$

Then $QP = 0$. Clearly, P has p-Drazin inverse. By the preceding discussion, we have Q has p-Drazin inverse. In light of [13, Theorem 5.4], M has p-Drazin inverse, as asserted. □

Corollary 3.3. *Let $a, d, cb, (cb)^{\pi}d \in \mathcal{A}^{\ddagger}$. If $(cb)^{\pi}dcb = 0$ and $ca = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$.*

Proof. In view of [13, Theorem 3.6], $cb \in \mathcal{A}^{\ddagger}$. By virtue of Theorem 3.2, we prove that $\begin{pmatrix} d & c \\ b & a \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$. We easily check that

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This completes the proof. □

Corollary 3.4. *Let $a, d, bc, (bc)^{\pi}a \in \mathcal{A}^{\ddagger}$. If $(bc)^{\pi}abc = 0, d^{\pi}dc = 0$ and $bd^{\ddagger} = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$.*

Proof. Obviously, we have $M = P + Q$, where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & d^{\pi}d \end{pmatrix}, Q = \begin{pmatrix} a & b \\ c & d^2d^{\ddagger} \end{pmatrix}.$$

In light of Theorem 3.2, Q has p-Drazin inverse. Since $d^{\pi}dc = 0$, we have $PQ = 0$, and therefore we complete the proof by [13, Theorem 5.4]. □

The following is the symmetric version of Theorem 3.2.

Theorem 3.5. *Let $a, d, bc, (bc)^{\pi}a \in \mathcal{A}^{\ddagger}$. If $bca(bc)^{\pi} = 0$ and $bd = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^{\ddagger}$.*

Proof. Step 1. Let $h = bc$ and let $N = \begin{pmatrix} a & 1 \\ h & 0 \end{pmatrix}$. Let $e = \begin{pmatrix} hh^{\ddagger} & 0 \\ 0 & 1 \end{pmatrix}$. Then $N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}_e$, where

$$a' = eNe, b' = eN(1-e), c' = (1-e)Ne, d' = (1-e)N(1-e).$$

By hypothesis, we have

$$a' = \begin{pmatrix} hh^{\ddagger}a & hh^{\ddagger} \\ h^2h^{\ddagger} & 0 \end{pmatrix}, b' = \begin{pmatrix} 0 & 0 \\ hh^{\pi} & 0 \end{pmatrix},$$
$$c' = \begin{pmatrix} h^{\pi}ahh^{\ddagger} & h^{\pi} \\ 0 & 0 \end{pmatrix}, d' = \begin{pmatrix} ah^{\pi} & 0 \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we have

$$a'b'c' = 0;$$
$$b'd' = \begin{pmatrix} 0 & 0 \\ h^\pi hah^\pi & 0 \end{pmatrix} = 0;$$
$$b'c' = \begin{pmatrix} 0 & 0 \\ hh^\pi ahh^\ddagger & hh^\pi \end{pmatrix} \in \sqrt{J\big(M_2(\mathcal{A})\big)}.$$

As in the proof of Theorem 3.2, $a', d' \in \mathcal{A}^\ddagger$. In light of Lemma 3.1, N has p-Drazin inverse.

Step 2. Since

$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix}^{-1} \begin{pmatrix} a & 1 \\ bc & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix},$$

we prove that $\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}$ has p-Drazin inverse.

Step 3. Obviously, we have

$$\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}.$$

In light of [13, Theorem 3.6], $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has p-Drazin inverse. Therefore $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ has p-Drazin inverse.

Step 4. Write $M = P + Q$, where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}.$$

Then $QP = 0$, and therefore we complete the proof by the discussion above. □

As in the proof of Corollary 3.3, we now derive

Corollary 3.6. *Let $a, d, cb, (cb)^\pi d \in \mathcal{A}^\ddagger$. If $cbd(cb)^\pi = 0$ and $ca = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^\ddagger$.*

Corollary 3.7. *Let $a, d, bc, (bc)^\pi a \in \mathcal{A}^\ddagger$. If $bca(bc)^\pi = 0, d^\pi dc = 0$ and $bd^\ddagger = 0$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})^\ddagger$.*

Example 3.8. *Let $\mathbb{C}$ be the field of complex number, and let*

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\}.$$

Let $M = \begin{pmatrix} E & I_4 \\ F & 0 \end{pmatrix}$, where

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_2(\mathcal{A}).$$

Then M has p-Drazin inverse.

Proof. We see that

$$J(\mathcal{A}) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| b \in \mathbb{C} \right\}.$$

Clearly, E and F have p-Drazin inverses. In fact, we have

$$E^{\ddagger} = E = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, F^{\ddagger} = F = \begin{pmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$(F^{\ddagger}E)^{\ddagger} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we have

$$F^{\pi}EF = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

and we are done by Theorem 3.2. □

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