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Some Results About Spectral Continuity and Compact Perturbations

S. Sánchez-Perales[a], S. V. Djordjević[b], S. Palafox[a]

[a]*Universidad Tecnológica de la Mixteca, Instituto de Física y Matemáticas, Km. 2.5 Carretera a Acatlima, 69000 Oaxaca, Mexico.*
[b]*Facultad de Ciencias Físico-Matemáticas, BUAP Río Verde y Av. San Claudio San Manuel, Puebla Pue. 72570, Mexico.*

Abstract. In this paper, we are interested in the continuity of the spectrum and some of its parts in the setting of Hilbert spaces. We study the continuity of the spectrum in the class of operators $\{T\} + K(H)$, where $K(H)$ denote the ideal of compact operators. Also, we give conditions in order to transfer the continuity of spectrum from T to $T + K$, where $K \in K(H)$. Then, we characterize those operators for which the continuity of spectrum is stable under compact perturbations.

1. Introduction

Let H be a Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators defined on H. For $T \in B(H)$ denote with $\sigma(T)$, $\sigma_a(T)$ and $\sigma_s(T)$ the *spectrum*, the *approximate point spectrum*, and the *surjective spectrum* of T. Let $N(T)$ and $R(T)$ be denote the null space and the range of the mapping $T \in B(H)$. We set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim H/R(T)$, if theses spaces are finite dimensional, otherwise let $\alpha(T) = \infty$ and $\beta(T) = \infty$. If the range $R(T)$ of $T \in B(H)$ is closed and $\alpha(T) < \infty$ then T is said to be an *upper semi-Fredholm* operator ($T \in \Phi_+(H)$). Similarly, if $\beta(T) < \infty$ then T is said to be a *lower semi-Fredholm* operator ($T \in \Phi_-(H)$). If $T \in \Phi_-(H) \cup \Phi_+(H)$ then T is called a *semi-Fredholm* operator ($T \in \Phi_\pm(H)$) and for $T \in \Phi_-(H) \cap \Phi_+(H)$ we say that T is a *Fredholm* operator ($T \in \Phi(H)$). For $T \in \Phi_\pm(H)$, the *index* of T is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

We set $\Phi_+^-(H) = \{T \in B(H) \mid T \in \Phi_+(H) \text{ and } \operatorname{ind}(T) \leq 0\}$, $\Phi_-^+(H) = \{T \in B(H) \mid T \in \Phi_-(H) \text{ and } \operatorname{ind}(T) \geq 0\}$ and $\Phi_0(H) = \{T \in \Phi(H) \mid \operatorname{ind}(T) = 0\}$. The *Weyl spectrum*, the *Weyl approximate point spectrum*, and the *Weyl surjectivity spectrum* of $T \in B(H)$ are given by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_0(H)\}, \quad \sigma_{aw}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_+^-(H)\} \text{ and } \sigma_{sw} = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_-^+(H)\}.$$

Put $\rho_w(T) = \mathbb{C} \setminus \sigma_w(T)$ and $\rho_{aw}(T) = \mathbb{C} \setminus \sigma_{aw}(T)$. The *ascent* and the *descent* of an operator $T \in B(H)$ are defined by $\operatorname{asc}(T) = \inf\{n \in \mathbb{N} \mid N(T^n) = N(T^{n+1})\}$ and $\operatorname{dsc}(T) = \inf\{n \in \mathbb{N} \mid R(T^n) = R(T^{n+1})\}$; the infimum

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Email addresses: es21254@yahoo.com.mx (S. Sánchez-Perales), slavdj@fcfm.buap.mx (S. V. Djordjević), sergiopalafoxd@gmail.com (S. Palafox)

over the empty set is taken to be ∞. The *Browder spectrum*, the *Browder approximative point spectrum* and the *Browder surjectivity spectrum* of $T \in B(H)$ are given by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_0(H) \text{ or } \operatorname{asc}(T - \lambda) = \infty\},$$
$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_+^-(H) \text{ or } \operatorname{asc}(T - \lambda) = \infty\},$$
$$\sigma_{sb}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin \Phi_-^+(H) \text{ or } \operatorname{dsc}(T - \lambda) = \infty\}.$$

It is well known that

$$\sigma_w(T) = \bigcap_{K \in K(H)} \sigma(T + K), \qquad \sigma_b(T) = \bigcap_{\substack{K \in K(H), \\ TK = KT}} \sigma(T + K); \tag{1}$$

$$\sigma_{aw}(T) = \bigcap_{K \in K(H)} \sigma_a(T + K), \qquad \sigma_{ab}(T) = \bigcap_{\substack{K \in K(H), \\ TK = KT}} \sigma_a(T + K). \tag{2}$$

For $T \in B(H)$, let $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } T\}$, $\pi_0(T) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } T \text{ of finite algebraic multiplicity}\}$ and for $k \in \mathbb{Z} \cup \{-\infty, \infty\}$, let $\rho_{sf}^k(T)$ be denote the set of $\lambda \in \mathbb{C}$ for which $T - \lambda \in \Phi_\pm(H)$ and $\operatorname{ind}(T - \lambda) = k$. Put

$$\rho_{sf}(T) = \bigcup_{-\infty \le k \le \infty} \rho_{sf}^k(T), \quad \rho_{sf}^-(T) = \bigcup_{-\infty \le k \le -1} \rho_{sf}^k(T), \quad \rho_{sf}^+(T) = \bigcup_{1 \le k \le \infty} \rho_{sf}^k(T), \quad \rho_{sf}^\pm(T) = \rho_{sf}^-(T) \cup \rho_{sf}^+(T).$$

The semi-Fredholm spectrum of $T \in B(H)$ is given by $\sigma_{sf}(T) = \mathbb{C} \setminus \rho_{sf}(T)$ i.e $\sigma_{sf}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_\pm(H)\}$.

The next concepts are part of classical point set topology. Let $\{E_n\}$ be a sequence of arbitrary subsets of $\mathbb{C}$ and define the limits inferior and superior of $\{E_n\}$ as follows:

$$\liminf E_n = \{\lambda \in \mathbb{C} \mid \text{ for every } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } B(\lambda, \epsilon) \cap E_n \neq \emptyset \text{ for all } n \geq N\}.$$

$$\limsup E_n = \{\lambda \in \mathbb{C} \mid \text{ for every } \epsilon > 0, \text{ there exists } J \subseteq \mathbb{N} \text{ infinite such that } B(\lambda, \epsilon) \cap E_n \neq \emptyset \text{ for all } n \in J\}.$$

Remark 1.1. *Let $\{E_n\}$ be a sequence of non-empty subsets of $\mathbb{C}$. The following properties are hold:*

(a) $\liminf E_n$ *and* $\limsup E_n$ *are closed subsets of* $\mathbb{C}$.
(b) $\lambda \in \limsup E_n$ *if and only if there exists an increasing sequence of natural numbers* $n_1 < n_2 < n_3 < \cdots$ *and points* $\lambda_{n_k} \in E_{n_k}$*, for all* $k \in \mathbb{N}$*, such that* $\lim \lambda_{n_k} = \lambda$.
(c) $\lambda \in \liminf E_n$ *if and only if there exists a sequence* $\{\lambda_n\}$ *such that* $\lambda_n \in E_n$ *for all* $n \in \mathbb{N}$*, and* $\lim \lambda_n = \lambda$.

Proposition 1.2. *[20, Remark 2 (d)] Let K, E, E_n be non-empty compact subsets of $\mathbb{C}$ such that $E_n \subseteq K$, for all $n \in \mathbb{N}$. Then $E_n \to E$ in the Hausdorff metric if and only if* $\limsup E_n \subseteq E$ *and* $E \subseteq \liminf E_n$.

2. Spectral continuity

We say that a sequence (T_n) in $B(H)$ converge in norm to $T \in B(H)$, and is denoted by $T_n \to T$, if $\lim_{n\to\infty} \|T_n - T\| = 0$. A function τ, defined on $B(H)$ whose values are non-empty compact subsets of $\mathbb{C}$, is said to be continuous at T, if $\tau(T_n) \to \tau(T)$ for all $T_n \to T$. Also, τ is said to be upper (lower) semi-continuous at T, when if $T_n \to T$ then $\limsup \tau(T_n) \subseteq \tau(T)$ ($\tau(T) \subseteq \liminf \tau(T_n)$).

Proposition 2.1. *Let $\tau \in \{\sigma, \sigma_a, \sigma_s, \sigma_w, \sigma_{aw}, \sigma_{sw}, \sigma_b, \sigma_{ab}, \sigma_{sb}\}$. Then τ is continuous at $T \in B(H)$ if and only if τ is upper and lower semi-continuous at T.*

Proof. It results by Proposition 1.2, because

$$\tau(T_n) \subseteq \sigma(T_n) \subseteq B(0, \|T_n\|) \subseteq B[0, M]$$

for all $n \in \mathbb{N}$ and some $M > 0$. $\square$

The upper semi-continuity of the spectrum and its parts is proved using the condition that the class of all invertible (Fredholm, Weyl, Browser, etc.) operators are open in algebra $B(H)$ (for more details see [15, Proposition 9, p. 55]:

Theorem 2.2. *Let $\tau \in \{\sigma, \sigma_a, \sigma_s, \sigma_w, \sigma_{aw}, \sigma_{sw}, \sigma_b, \sigma_{ab}, \sigma_{sb}\}$. Then for every $T \in B(H)$, τ is upper semi-continuous at T.*

One of the first discussions about the continuity of the spectrum is found in the work of Newburgh ([16]) and Apostol et al. ([4]). More complete results in the case of a Hilbert space H are given in the series of papers by Conway and Morrel (see [7], [8] and [9]). Those results are moved by Burlando ([6]) for the case of Banach space operators. Some of these results can be summarized in the following:

Remark 2.3. *Let $T \in B(H)$ and (T_n) be a sequence in $B(H)$ such that $T_n \to T$. The following inclusions are well known:*

1. $iso\sigma(T) \subseteq \liminf \sigma(T_n)$
2. $\pi_0(T) \subseteq \liminf \pi_0(T_n) \subseteq \liminf \sigma_a(T_n) \subseteq \liminf \sigma(T_n)$
3. $\overline{\rho^+_{sf}(T)} \subseteq \liminf \sigma_a(T_n)$
4. $\overline{\rho^-_{sf}(T)} \subseteq \liminf \sigma_s(T_n)$
5. $\overline{\rho^\pm_{sf}(T)} \subseteq \liminf \sigma_w(T_n) \subseteq \liminf \sigma(T_n)$.

Another way to observe the spectral continuity is giving relations between the continuity of different parts of the spectrum. For example, we have that the continuity of approximate point spectrum implies the continuity of spectrum (see next theorem or [8]). The opposite implication is not true in general, for this we need more.

Theorem 2.4. *Let $T \in B(H)$ be such that $dsc(T - \lambda) < \infty$, for any $\lambda \in \rho_{aw}(T)$. Then, σ is continuous at T if and only if σ_a is continuous at T.*

Proof. Suppose that σ_a is continuous at T. Then, for all $\lambda \in \sigma_a(T)$, we have

$$\lambda \in \sigma_a(T) \subseteq \liminf \sigma_a(T_n) \subset \liminf \sigma(T_n).$$

By [18, Proposition 2.3], we have $\sigma(T) \setminus \sigma_a(T) \subseteq \liminf \sigma(T_n)$. Therefore, σ is continuous at T.

Now, suppose that σ is continuous at T. By Remark 2.3 (2) and (3), $\overline{\rho^+_{sf}(T)} \cup \overline{\pi_0(T)} \subseteq \liminf \sigma_a(T_n)$. Let $\lambda \in \sigma_a(T) \setminus (\overline{\rho^+_{sf}(T)} \cup \overline{\pi_0(T)})$. Suppose that $\rho^-_{sf}(T) \neq \emptyset$, take $\mu \in \rho^-_{sf}(T)$ then $\mu \in \rho_{aw}(T)$ and so $\mathrm{dsc}(T - \mu) < \infty$, thus by [1, Theorems 3.8 and 3.19], it follows that $\mathrm{ind}(T - \mu) > 0$. By contradiction we have that $\rho^-_{sf}(T) = \emptyset$.

Therefore, $\lambda \in \sigma(T) \setminus (\overline{\rho^\pm_{sf}(T)} \cup \overline{\pi_0(T)})$. Let $\epsilon > 0$ be such that $B(\lambda, \epsilon) \cap (\overline{\rho^\pm_{sf}(T)} \cup \overline{\pi_0(T)}) = \emptyset$. Then, from [7, Theorem 3.1], $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$. Thus, by [8, Lemma 3.1], $\lambda \in \liminf \sigma_{sf}(T)(\subseteq \liminf \sigma_a(T_n))$. Consequently, σ_a is continuous at T. □

An operator $T \in B(H)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to H$ which satisfies the equation $(T - \lambda_0)f(\mu) = 0$, for all $\mu \in U$, is the function $f \equiv 0$. An operator $T \in B(H)$ is said to have SVEP if T has SVEP at every point in the complex plane.

The descent and the ascent of an operator is connected with SVEP property. For example, for $\lambda \in \rho_{aw}(T)$, $T - \lambda$ has SVEP if and only if $\mathrm{asc}(T - \lambda) < \infty$. On the other hand, any of those properties determines the sign of the index of a semi-Fredholm operator, the SVEP or the finite ascent property for $T \in \Phi_\pm(H)$ implies $\mathrm{ind}(T) \leq 0$ and the SVEP for T^* or the finite descent property for $T \in \Phi_\pm(H)$ implies that $\mathrm{ind}(T) \geq 0$ (see [1]). In this way we have next:

Corollary 2.5. *Let $T \in B(H)$ be such that T^* has SVEP at λ, for any $\lambda \in \rho_{aw}(T)$. Then, σ is continuous at T if and only if σ_a is continuous at T.*

The conditions given in the previous paragraph to Corollary 2.5 are also connected to one of the Browder-type theorems. We say that

$T \in B(H)$ satisfies the Browder's theorem if $\sigma_w(T) = \sigma_b(T)$;

$T \in B(H)$ satisfies the a-Browder's theorem if $\sigma_{aw}(T) = \sigma_{ab}(T)$;

$T \in B(H)$ satisfies the s-Browder's theorem if $\sigma_{sw}(T) = \sigma_{sb}(T)$,

(for more details see [12]). The a-Browder and s-Browder theorems are not equivalent to each other, but each implies the Browder's theorem ([12, Corollary 8.3.4]). On the other hand, the Browder's theorem is strongly connected with the continuity of the spectrum, Weyl spectrum and Browder spectrum. It is known that the continuity of σ or σ_b at T implies the Browder's theorem for T, and the continuity of σ_{ab} or σ_a implies the a-Browder's theorem (see, [11, Theorem 2.1 and Corollary 3.4] and [18, Theorems 2.2 and 2.4]). From Example 2.9 we have that the Browder's theorem does not imply the continuity of spectrum. In [11] was put out that if σ_w is continuous at T and it satisfies Browder's theorem then σ is continuous at T, however the converse of this implication is in general not true, see [18, Remark 4.2].

Theorem 2.6. *If σ is continuous at $T \in B(H)$ and $\overline{\pi_0(T)} \cap \sigma_{sf}(T) = \emptyset$, then σ_w is continuous at T.*

Proof. Let $\lambda \in \sigma_w(T)$. If $\lambda \notin \sigma_{sf}(T)$ then $\lambda \in \overline{\rho_{sf}^{\pm}(T)}$ and so by Remark 2.3 (5), $\lambda \in \liminf \sigma_w(T_n)$. Now, suppose that $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$. From hypothesis, there exists $r > 0$ such that $B(\lambda, r) \cap \overline{\pi_0(T)} = \emptyset$. Let $0 < \epsilon < r$, by [7, Theorem 3.1], there exists C a component of $\sigma_{sf}(T) \cup \pi_0(T)$ such that $C \subseteq B(\lambda, \epsilon)$. Then C is a component of $\sigma_{sf}(T)$ such that $C \subseteq B(\lambda, \epsilon)$. Consequently, by [8, Lemma 3.1], $\lambda \in \liminf \sigma_{sf}(T_n) \subseteq \liminf \sigma_w(T_n)$. □

From duality of a-Browder and s-Browder theorems of T and T^*, we have the following theorem.

Theorem 2.7. *Let $T \in B(H)$. Then σ_{sb} is continuous at T if and only if T satisfies the s-Browder theorem and σ_{sw} is continuous at T.*

Let C be the set of opertors $T \in B(H)$ for which

$$T = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} \quad \text{on} \quad N(T-\lambda) \oplus N(T-\lambda)^{\perp} \tag{3}$$

for all $\lambda \in \sigma_p(T) \setminus \{0\}$. For instant, (3) holds for operators T which are of class $\mathcal{A}(k)$ and of class (n,k)-quasi-$*$-paranormal ([21]):

$T \in B(H)$ is said to be class $\mathcal{A}(k)$ if $|T|^2 \leq |T^{k+1}|^{\frac{2}{k+1}}$;

$T \in B(H)$ is said to be class (n,k)-quasi-$*$-paranormal if $\|T^*(T^k x)\| \leq \|T^{n+1}(T^k x)\|^{\frac{1}{n+1}} \|T^k x\|^{\frac{n}{n+1}}$.

For $T \in B(H)$ we consider the set

$$\Delta(T) = \{\lambda \in \mathbb{C} : R(T-\lambda) \text{ is not closed}\}.$$

Theorem 2.8. *If Δ is lower semi-continuous at $T \in C$ then σ is continuous at T.*

Proof. Let $\lambda \in \sigma(T) \setminus \{0\}$. If $\lambda \in \Delta(T)$ then $\lambda \in \liminf \Delta(T_n) \subseteq \liminf \sigma(T_n)$. Now, suppose that $\lambda \notin \Delta(T)$, this implies that $R(T-\lambda)$ is closed.

Case I. $\alpha(T-\lambda) = 0$. In this situation, $\lambda \in \rho_{sf}^{-}(T)$, so by Remark 2.3 (4), $\lambda \in \liminf \sigma_{su}(T_n) \subseteq \liminf \sigma(T_n)$.

Case II. $\alpha(T-\lambda) \neq 0$. By Remark 2.3 (1), we may assume without loss of generality that $\lambda \in \operatorname{acc}\sigma(T)$. Since $T \in C$, it follows that $T = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}$ on $N(T-\lambda) \oplus N(T-\lambda)^{\perp}$. Observe that $R(B-\lambda) = R(T-\lambda)$ and $\alpha(B-\lambda) = 0$. Therefore $B-\lambda$ is a semi-Fredholm operator and by [10, Theorem 4.2.1], there exists $\epsilon > 0$ such that if $|\gamma - \lambda| < \epsilon$ then $B - \gamma \in \Phi_+(H)$ and $\alpha(B-\gamma) = 0$. This implies that $R(T-\gamma) = (\lambda-\gamma)N(T-\lambda) \oplus R(B-\gamma)$ is closed and $\alpha(T-\gamma) = \alpha((\lambda-\gamma)I) + \alpha(B-\gamma) = 0$ for all $\gamma \in B(\lambda, \epsilon)$ with $\gamma \neq \lambda$. Since λ is an accumulation point of $\sigma(T)$, there exists a sequence (γ_k) in $\sigma(T)$ such that $\gamma_k \to \lambda$, and for each $k \in \mathbb{N}$, $\gamma_k \neq \lambda$, $T - \gamma_k \in \Phi_+(H)$ and $\alpha(T - \gamma_k) = 0$. If there exists $k_0 \in \mathbb{N}$ such that $\gamma_{k_0} \notin \liminf \sigma(T_n)$, then there exists a subsequence (T_{n_l}) of (T_n) such that $T_{n_l} - \gamma_{k_0}$ is invertible for all $l \in \mathbb{N}$. By continuity of the index, it follows that $\operatorname{ind}(T_{n_l} - \gamma_{k_0}) \to \operatorname{ind}(T - \gamma_{k_0})$, thus $\operatorname{ind}(T - \gamma_{k_0}) = 0$, but $\alpha(T - \gamma_{k_0}) = 0$. Therefore $T - \gamma_{k_0}$ is invertible, which is a contradiction. Consequently, $\gamma_k \in \liminf \sigma(T_n)$ for all $k \in \mathbb{N}$. Thus, since $\liminf \sigma(T_n)$ is a closed set, it follows that $\lambda \in \liminf \sigma(T_n)$. $\square$

The fact that $T \in \mathcal{C}$ does not necessarily imply that σ is continuous at T, see Example 2.9. The same example shows that the Browder's theorem does not imply the continuity of the spectrum.

Example 2.9. *Let $H = L^2([0,1])$ and define the multiplication operator $M : H \to H$ by $M(f)(x) = xf(x)$. It is not difficult to prove that M is a bounded operator such that $\sigma_p(M) = \emptyset$ and $\sigma(M) = [0,1]$. Observe that $M^* = M$, then M is a normal operator and so $M \in \mathcal{C}$. It is clear that M satisfies Browder's theorem. Therefore $\sigma(M) \setminus \sigma_w(M) = \emptyset$ which implies that $\sigma(M) = \sigma_w(M)$. Consequently, by continuity of the index, $\rho_{sf}^{\pm}(M) = \emptyset$ and $\sigma_{sf}(M) = [0,1]$. Let $D = B(0, 1/4)$, since every component of $\sigma_{sf}(M) \cup \pi_0(M)$ meets $\overline{D}$, it follows from [5, Theorem 3.1], that there exists a sequence (A_n) of operators in $B(H)$ such that $\|A_n - M\| \to 0$ and $\sigma(A_n) \subseteq D$ for all $n \in \mathbb{N}$. Thus, $\sigma(A_n) \not\to \sigma(M)$.*

3. Compact perturbations and continuity of spectra

In [18] the continuity of spectrum is studied on the class $\{T\} + K(H)$ where the commutativity of T and K is essential. In this way we have next:

Proposition 3.1. *Let $T \in B(H)$ and $K, K_n \in K(H)$ be such that $K_n \to K$. If $K_n T = T K_n$ for all $n \in \mathbb{N}$, then $\sigma(T + K_n) \to \sigma(T + K)$. If additionally $\sigma_p(T + K) \cap \rho_{sf}^-(T) = \emptyset$ then $\sigma_a(T + K_n) \to \sigma_a(T + K)$.*

Proof. Since $K_n T \to KT$, $TK_n \to TK$ and $TK_n = K_n T$ for all $n \in \mathbb{N}$, it follows that $TK = KT$. Let $\lambda \in \sigma(T+K)$. If $\lambda \in \sigma_b(T)$ then by (1), $\lambda \in \sigma(T+K_n)$ for all $n \in \mathbb{N}$, thus $\lambda \in \liminf \sigma(T+K_n)$. If $\lambda \notin \sigma_b(T)$ then $T - \lambda \in \Phi_0(H)$ and $\operatorname{asc}(T - \lambda) < \infty$. Consequently, by [1, Theorem 3.43], $\alpha(T + K - \lambda) = \beta(T + K - \lambda) < \infty$ and $\operatorname{asc}(T + K - \lambda) < \infty$, hence by [1, Theorem 3.4], $\alpha(T + K - \lambda) = \beta(T + K - \lambda) < \infty$ and $\operatorname{asc}(T + K - \lambda) = \operatorname{dsc}(T + K - \lambda) < \infty$. Thus $\lambda \in \pi_0(T + K)$, which implies by Remark 2.3 (2), that $\lambda \in \liminf \sigma(T + K_n)$.

Now, let $\lambda \in \sigma_a(T + K)$. If $\lambda \in \sigma_{ab}(T)$ then by (2), $\lambda \in \sigma_a(T + K_n)$ for all $n \in \mathbb{N}$, so $\lambda \in \liminf \sigma_a(T + K_n)$. If $\lambda \notin \sigma_{ab}(T)$ then $\lambda \in \rho_{sf}^-(T) \cup \rho_{sf}^0(T)$ and $\operatorname{asc}(T - \lambda) < \infty$. From hypothesis $\sigma_p(T + K) \cap \rho_{sf}^-(T) = \emptyset$, we have that $\lambda \in \rho_{sf}^0(T)$ and $\operatorname{asc}(T - \lambda) < \infty$ which implies by above that $\lambda \in \rho_{sf}^0(T + K)$. Thus, $\lambda \in \pi_0(T + K)$ and hence by Remark 2.3 (2), $\lambda \in \liminf \sigma_a(T + K_n)$. $\square$

Remark 3.2. *Proposition 3.1 is a generalization of [17, Corollary 3.2]. Thus in [17, Proposition 3.4 and Proposition 3.5], the condition "T has finite ascent at every $\lambda \in \sigma_p(T)$" can be lifted.*

Next we will study the continuity of the spectrum at $T + K$ without of assuming that $TK = KT$. In previous section we discussed the connection between the continuity of spectrum as well as some of its parts with different versions of the Browder's theorem. The Browder's theorem is always a good tool when dealing with the continuity of the spectrum and different parts of it. Observe, from (1) it is clear that

$$\sigma_w(T + K) = \sigma_w(T) \subseteq \liminf \sigma(T + K_n)$$

for all sequence (K_n) in $K(H)$ with $K_n \to K$. On the other hand, if $T + K$ satisfies Browder's theorem, then by Remark 2.3,

$$\sigma(T + K) \setminus \sigma_w(T + K) = \pi_0(T + K) \subseteq \liminf \sigma(T + K_n).$$

Consequently, we have the following proposition.

Proposition 3.3. *Let $T \in B(H)$ and $K \in K(H)$ be such that $T + K$ satisfies Browders' theorem. Then for every sequence(K_n) in $K(H)$ that in norm converge to K, we have*

1. $\sigma(T + K_n) \to \sigma(T + K)$,
2. *additionally, if* $\sigma_p(T + K) \cap \rho_{sf}^{-}(T) = \emptyset$ *then* $\sigma_a(T + K_n) \to \sigma_a(T + K)$.

Remark 3.4. By [2, Theorem 3.6] and [14, Theorem 1.4], we have that if $\rho_w(T)$ is connected then $T + K$ satisfies Browder's theorem, for all compact operator K. On the other hand, the connectedness of $\rho_w(T)$ is not enough for the continuity of the spectrum at $T + K$ for all $K \in K(H)$, see Example 2.9.

Corollary 3.5. *Let* $T \in B(H)$, *if*

(1) $\rho_w(T)$ *is connected*

or

(2) $\rho(T)$ *is connected and Browder's theorem holds for* T,

then $\sigma(T + K_n) \to \sigma(T + K)$ *for all sequence of compact operators* $\{K_n\}$ *that in norm converges to* K.

Proof. This is an immediate consequence of [2, Theorem 3.4], Remark 3.4 and Proposition 3.3. □

Lemma 3.6. *Let* $T \in B(H)$ *be such that it obeys the Browder's theorem. If* C *is a component of* $\sigma_{sf}(T) \cup \pi_0(T)$ *and* $C \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$ *then* C *is a component of* $\sigma(T)$.

Proof. Since T obeys Browder's theorem it follows that

$$\sigma(T) = [\sigma_{sf}(T) \cup \pi_0(T)] \cup \rho_{sf}^{\pm}(T). \tag{4}$$

Observe that $[\sigma_{sf}(T) \cup \pi_0(T)] \cap \rho_{sf}^{\pm}(T) = \emptyset$, which implies that $\sigma_{sf}(T) \cup \pi_0(T)$ is a closed set. Let D be a component of $\sigma(T)$ such that $C \subseteq D$. Then by (4),

$$D = [D \cap (\sigma_{sf}(T) \cup \pi_0(T))] \cup [D \cap \overline{\rho_{sf}^{\pm}(T)}]. \tag{5}$$

We set $E = D \cap (\sigma_{sf}(T) \cup \pi_0(T))$. Since $C \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$ it follows that $C \subseteq E \subseteq \sigma_{sf}(T) \cup \pi_0(T)$ and there exists $\epsilon > 0$ such that $(C)_\epsilon \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$. Suppose that E is disconnected. We claim that there exist E_1, E_2 compact sets such that

$$E = E_1 \cup E_2,\ E_1 \cap E_2 = \emptyset,\ E_1 \neq \emptyset,\ E_2 \neq \emptyset \tag{6}$$

and

$$C \subseteq E_1 \subseteq (C)_\epsilon. \tag{7}$$

Let E_1^* y E_2^* be compact sets such that satisfy conditions in (6). Without loss of generality we may assume that $C \subseteq E_1^* \nsubseteq (C)_\epsilon$. We set $P = E_1^* \setminus (C)_\epsilon$, and for each $x \in P$, let C_x be a component in $E_1^* \setminus (C)_\epsilon$ of x. Since $E_1^* \setminus (C)_\epsilon \subseteq \bigcup\limits_{x \in P} C_x$ and $E_1^* \setminus (C)_\epsilon$ is compact, it follows that there exists $\{x_i\}_{i=1}^n \subseteq P$ such that $E_1^* \setminus (C)_\epsilon \subseteq \bigcup\limits_{i=1}^{n} C_{x_i}$. Let $E_1 = E_1^* \setminus \bigcup\limits_{i=1}^{n} C_{x_i}$ and $E_2 = E_2^* \cup \left(\bigcup\limits_{i=1}^{n} C_{x_i}\right)$, then E_1, E_2 are compact sets such that $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $E_1 \neq \emptyset$, $E_2 \neq \emptyset$ and $C \subseteq E_1 \subseteq (C)_\epsilon$. So our claim is true.

Therefore by (5), (6) and (7), $D = E_1 \cup [E_2 \cup (D \cap \overline{\rho_{sf}^{\pm}(T)})]$ and $E_1 \cap [E_2 \cup (D \cap \overline{\rho_{sf}^{\pm}(T)})] = \emptyset$. Thus D is disconnected, which is a contradiction. Hence E is connected and so $C = E$. Consequently,

$$D = C \cup [D \cap \overline{\rho_{sf}^{\pm}(T)}].$$

Finally, since D is connected, $C \neq \emptyset$ and C, $D \cap \overline{\rho_{sf}^{\pm}(T)}$ are closed sets such that $C \cap [D \cap \overline{\rho_{sf}^{\pm}(T)}] = \emptyset$, we have that $D \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$, therefore $D = C$. □

Theorem 3.7. *Let $T \in B(H)$ and $K \in K(H)$ be such that $\sigma(T) = \sigma(T+K)$. Then, σ is continuous at T if and only if σ is continuous at $T+K$.*

Proof. $\Rightarrow$] Suppose that σ is continuous at T. Let (T_n) be a sequence in $B(H)$ such that $T_n \to T+K$ and let $\lambda \in \sigma(T+K)$.
Case I. $\lambda \in \overline{\rho_{sf}^{\pm}(T+K)}$. By Remark 2.3 (5), we have that $\lambda \in \liminf \sigma(T_n)$.
Case II. $\lambda \in \rho_{sf}^{0}(T+K)$. Since $\sigma(T) = \sigma(T+K)$, it follows that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. By continuity of σ at T and [18, Theorem 2.2], the operator T satisfies Browder' theorem, therefore $\lambda \in \pi_0(T)$. Thus $\lambda \in \mathrm{iso}\sigma(T+K)$ and so by Remark 2.3 (1), $\lambda \in \liminf \sigma(T_n)$.
Case III. $\lambda \in \sigma_{sf}(T+K) \setminus \overline{\rho_{sf}^{\pm}(T+K)}$. In this situation, $\lambda \in \sigma_{s-T}(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$. There exists $r > 0$ such that $B(\lambda, r) \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset$. Let $\epsilon > 0$, without loss of generality we may consider that $\epsilon < r$. Due continuity of σ at T, we have by [7, Theorem 3.1] that $B(\lambda, \epsilon)$ contains a component C of $\sigma_{sf}(T) \cup \pi_0(T)$. Then by Lemma 3.6, C is a component of $\sigma(T) = \sigma(T+K)$. Consequently, from [7, Lemma 1.5], there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $B(\lambda, \epsilon)$ contains a component of $\sigma(T_n)$. Therefore, for every $n \geq n_0$, $B(\lambda, \epsilon) \cap \sigma(T_n) \neq \emptyset$. Thus $\lambda \in \liminf \sigma(T_n)$.
$\Leftarrow$] Applying the first part of the proof for the operators $S = T+K$ and $-K$ we obtain the desired. □

Corollary 3.8. *Let $T \in B(H)$ and $K \in K(H)$ be such that $TK = KT$ and K^n is a finite rank operator for some $n \in \mathbb{N}$. If $\sigma(T) = \mathrm{acc}\,\sigma(T)$ and σ is continuous at T then σ is continuous at $T+K$.*

Proof. From [3, Theorem 3.20], $\sigma(T) = \mathrm{acc}\,\sigma(T) = \mathrm{acc}\,\sigma(T+K)$. Therefore, it is sufficient to observe that in the proof of Case III of Theorem 3.7, C is a component of $\sigma(T)(= \mathrm{acc}\,\sigma(T+K))$ and so C is a component of $\sigma(T+K)$. □

Example 3.9. *Consider the right shift $R : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by $R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$ and let $U \in B(^2(\mathbb{N}))$ be a finite rank operator. Let T, K be defined on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by*

$$T = \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}.$$

Then K is a finite rank operator and $TK = KT$. Since $\sigma(T) = B[0,1] = acc\sigma(T)$ it follows by Corollary 3.8 that σ is continuous at $T+K$.

In [8, p. 462] it is observed that σ is continuous at $T \in B(H)$ if and only if $\mathrm{int}\,(\sigma(T) \setminus \sigma_w(T)) = \emptyset$ and, for each $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$. But this proposition is false as shown in [19, Example 4.13]. Now, we use this condition to give a characterization for the continuity of the spectrum at compact perturbations.

Theorem 3.10. *The spectrum σ is continuous at $T+K$ for all $K \in K(H)$ if and only if $\rho_w(T)$ is connected and for every $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$.*

Proof. $\Rightarrow$] Let $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho_{sf}^{\pm}(T)}$ and $\epsilon > 0$. There exists $r > 0$ such that

$$B(\lambda, r) \cap \overline{\rho_{sf}^{\pm}(T)} = \emptyset.$$

By [13, Proposition 3.4], there exists $K \in K(H)$ such that

$$\sigma_p(T+K) = \rho_{sf}^{+}(T+K).$$

Since $\sigma_{sf}(T) = \sigma_{sf}(T+K)$ and $\rho^{\pm}_{sf}(T) = \rho^{\pm}_{sf}(T+K)$ it follows that $\lambda \in \sigma_{sf}(T+K) \setminus \overline{\rho^{\pm}_{sf}(T+K)}$. Suppose that $\epsilon < r$, from continuity of σ at $T+K$, we have by [7, Theorem 3.1] that the ball $B(\lambda, \epsilon)$ contains a component C of $\pi_0(T+K) \cup \sigma_{sf}(T+K)$. Observe that

$$\begin{aligned} C \cap \pi_0(T+K) &\subseteq C \cap \sigma_p(T+K) \\ &= C \cap \rho^{+}_{sf}(T+K) \\ &\subseteq B(\lambda, r) \cap \rho^{+}_{sf}(T) = \emptyset. \end{aligned}$$

Therefore, C is a component of $\sigma_{sf}(T+K)(= \sigma_{sf}(T))$. On the other hand, by [18, Theorem 2.4], $T+S$ satisfies Browder's theorem for all $S \in K(H)$, consequently by [14, Theorem 1.4], $\rho_w(T)$ is connected.
$\Leftarrow$] It follows by [14, Theorem 1.4] and [18, Theorem 4.4]. □

Corollary 3.11. *If σ is continuous at $T \in B(H)$, $\sigma_{sf}(T) \cap \overline{\pi_0(T)} = \emptyset$, and $\rho_w(T)$ is connected, then σ is continuous at $T+K$ for all $K \in K(H)$.*

Example 3.12. *Let $T : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ be defined by*

$$T = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix},$$

where $S : \ell^2 \to \ell^2$ is an injective quasinilpotent operator. Then $\sigma(T) = \sigma_w(T) = \sigma_{sf}(T) = \{0, 1\}$. This implies that $\rho_w(T)$ is connected and $\pi_0(T) = \emptyset$. Moreover, by Remark 2.3 (1), σ is continuous at T. Therefore, by Corollary 3.11, σ is continuous at $T+K$ for all $K \in K(\ell^2 \oplus \ell^2)$.

Theorem 3.13. *Let $T \in B(H)$ be such that T^* has SVEP at every $\lambda \in \rho_{aw}(T)$. Then the following statements are equivalent:*

1. *The approximate point spectrum σ_a is continuous at $T+K$ for all $K \in K(H)$.*
2. *$\rho_{aw}(T)$ is connected and for every $\lambda \in \sigma_{sf}(T) \setminus (\overline{\rho^{+}_{sf}(T)} \cup \overline{\pi_0(T)})$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$.*

Proof. Since T^* has SVEP at every $\lambda \in \rho_{aw}(T)$ if follows that $\rho^{-}_{sf}(T) = \emptyset$. Suppose that (1) holds, then by Theorem 3.10, for every $\lambda \in \sigma_{sf}(T) \setminus (\overline{\rho^{+}_{sf}(T)} \cup \overline{\pi_0(T)})$ and $\epsilon > 0$, the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)$. Moreover, from [18, Theorem 2.4], $T+K$ satisfies a–Browder's theorem for all $k \in K(H)$. Thus by [14, Theorem 1.5], $\rho_{aw}(T)$ is connected.

Now, suppose that (2) holds. Since $\rho_{aw}(T)$ is connected it follows by [14, Theorem 1.5] that $T+K$ satisfies a-Browder's theorem for all $K \in K(H)$. Let (T_n) be a sequence in $B(H)$ such that $T_n \to T+K$ and let $\lambda \in \sigma_a(T+K)$. If $\lambda \in \sigma(T+K) \setminus \sigma_{sf}(T+K)$ then $\lambda \in \overline{\rho^{+}_{sf}(T+K)} \cup \overline{\pi_0(T+K)}$. From Remark 2.3 we have that $\overline{\pi_0(T+K)} \subseteq \liminf \sigma_a(T_n)$ and $\overline{\rho^{+}_{sf}(T+K)} \subseteq \liminf \sigma_a(T_n)$, therefore $\lambda \in \liminf \sigma_a(T_n)$. Now, let $\lambda \in \sigma_{sf}(T+K) \setminus (\overline{\rho^{+}_{sf}(T+K)} \cup \overline{\pi_0(T+K)})$ this implies that $\lambda \in \sigma_{sf}(T) \setminus (\overline{\rho^{+}_{sf}(T)} \cup \overline{\pi_0(T)})$. From hypothesis we have that for each $\epsilon > 0$ the ball $B(\lambda, \epsilon)$ contains a component of $\sigma_{sf}(T)(= \sigma_{sf}(T+K))$, therefore by [8, Lemma 3.1], $\lambda \in \liminf \sigma_{sf}(T_n)(\subseteq \liminf \sigma_a(T_n))$. □

Corollary 3.14. *If σ is continuous at $T \in B(H)$ and $\rho_{sf}(T)$ is connected then σ_a is continuous at $T+K$ for all $K \in K(H)$.*

Proof. Suppose that $\mathrm{int}\, \sigma_{sf}(T) \neq \emptyset$. Take $\lambda \in \mathrm{int}\sigma_{sf}(T)$ then there exists $r > 0$ such that $B(\lambda, r) \subseteq \sigma_{sf}(T)$. This implies that $\lambda \in \sigma_{sf}(T) \setminus \overline{\rho^{\pm}_{sf}(T)}$ and so by continuity of σ at T, it follows from [7, Theorem 3.1] that there exists a component C of $\sigma_{sf}(T) \cup \pi_0(T)$ such that $C \subseteq B(\lambda, \frac{r}{2})$. The ball $B(\lambda, r)$ is a connected set in $\sigma_{sf}(T) \cup \pi_0(T)$ such that $C \subseteq B(\lambda, r)$, therefore $C = B(\lambda, r)$ which is a contradiction. Thus $\mathrm{int}\sigma_{sf}(T) = \emptyset$. Then by [22, Theorem 1.3], T has SVEP and so $\rho^+_{sf}(T) = \emptyset$. Observe that $\rho_{sf}(T) = \rho_{aw}(T) \cup \rho^+_{sf}(T)$, hence $\rho_{sf}(T) = \rho_{aw}(T)$ which implies that $\rho_{aw}(T)$ is connected. Now, since $\sigma_{sf}(T^*) = \sigma_{sf}(T)^*$ and $\rho_{sf}(T^*) = \rho_{sf}(T)^*$ it follows that $\operatorname{int}\sigma_{sf}(T^*) = \emptyset$ and $\rho_{sf}(T^*)$ is connected. Then by [22, Theorem 1.3], T^* has SVEP which implies that T^* has SVEP at every $\lambda \in \rho_{aw}(T)$. Finally, if $\lambda \in \sigma_{sf}(T) \setminus (\overline{\rho^+_{sf}(T)} \cup \overline{\pi_0(T)})$ and $\epsilon > 0$ then by [7, Theorem 3.1], the ball $B(\lambda, \epsilon)$ contains a component C of $\sigma_{sf}(T) \cup \pi_0(T)$. For ϵ small enough we have that $B(\lambda, \epsilon) \cap \overline{\pi_0(T)} = \emptyset$ and so C is a component of $\sigma_{sf}(T)$. Consequently by Theorem 3.13, σ_a is continuous at $T + K$ for all $K \in K(H)$. □

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