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Polynomial Helices in the n-Dimensional Semi-Euclidean Space with Index Two

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Abstract. In this article, we investigate polynomial helices in the n-dimensional semi-Euclidean space with index two for $n \geq 4$. We obtain some families of spacelike and timelike polynomial helices. These helices have spacelike or timelike or null axes. After that, we give some examples of the spacelike and the timelike polynomial helices in the n-dimensional semi-Euclidean space with index two for $n = 4, 5$ and 6.

1. Introduction

Helices have been one of the most fruitful subject for the differential geometry since it has many applications in the other branches of science. For instance, in biology, in simulation of kinematic motion, in the design of highways, in engineering and so on [1, 2].

The notion of helix is stated in 3-dimensional Euclidean space by M. A. Lancret in 1802. Helix is a curve whose tangent vector field make a constant angle with a fixed direction called the axis of the helix. In 1845, B. de Saint Venant gave the necessary and sufficient condition of a curve to be a general helix. Namely, a curve is a general helix if and only if the ratio of the curvature to the torsion is constant [13]. In Literature, there are several characterizations for helices in the Euclidean 3-space [5, 12].

In [11], Özdamar and Hacısalihoğlu defined harmonic curvature functions in the n-dimensional Euclidean space and used them to extend the concept of the helix from 3-dimensional Euclidean space to n-dimensional Euclidean space for $n > 3$. Since then, the characterization of helices has been studied in many ambiant spaces. For example, in n-dimensional Euclidean space [2, 3], in 3-dimensional Lorentzian space forms, which are de Sitter and anti de Sitter space [8], in Galilean space [4, 9], in Lie group [17], in n-dimensional Minkowski space [1, 15, 16].

Semi-Euclidean geometry has been an active research area in general relativity and mathematics, after Einstein's formulation of general relativitiy as a theory of space, time and gravitation in the semi-Euclidean space [14]. As far as we know, there is little information available in literature about helices in the semi-Euclidean space with index two. The main goal of this article is to obtain families of non-null polynomial helices depend on its casual character in the semi-Euclidean space with index two. In addition, we consider casual character of the axis of the helix.

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The remainder of this article is organized as follows. First, we give basic information about a non-degenerate curve of local differential geometry in the n-dimensional semi-Euclidean space with index two. After, we give some families of the spacelike and the timelike polynomial helices in the n-dimensional semi-Euclidean space with index two. This part was adopted from Minkowski spacetime in [1]. Finally, give some examples in the n-dimensional semi-Euclidean space with index two for $n = 4, 5$ and 6.

2. Preliminary

In this section, we give the basic theory of non-degenerate curves of local differential geometry in the n-dimensional semi Euclidean space with index two. For more details and background about this space, see [10].

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis of real vector space $\mathbb{R}^n$ and the vector space $\mathbb{R}^n$ endowed with the scalar product,

$$g(x, y) = -x_1y_1 - x_2y_2 + \sum_{i=3}^{n} x_iy_i,$$

for all $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The couple $\{\mathbb{R}^n, g(,)\}$ is called n-dimensional semi-Euclidean space with index two, which is denoted by $\mathbb{E}^n_2$. Recall that a vector $v \in \mathbb{R}^n$ is called spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$. In addition, if the vector $v = 0$, then v is still called spacelike. The norm of a vector $v \in \mathbb{R}^n$ is defined by $\|v\| = \sqrt{|g(v, v)|}$. A curve in $\mathbb{E}^n_2$ is a smooth mapping $\alpha : I \to \mathbb{E}^n_2$, where I is an open interval in $\mathbb{R}$. A curve $\alpha : I \to \mathbb{E}^n_2$ is called regular provided $\alpha'(t) \neq 0$ for all t. The regular curve $\alpha : I \to \mathbb{E}^n_2$ is said to be spacelike or timelike if its velocity vector $\alpha'(t)$ is a spacelike or a timelike vector at any $t \in I$.

Let $\{V_1, V_2, \ldots, V_n\}$ be non-null Frenet frame along a non-null arbitrary curve α in $\mathbb{E}^n_2$. Since $\{V_1, V_2, \ldots, V_n\}$ is an orthonormal frame, then $g\left(V_i, V_j\right) = \delta_{ij}\varepsilon_i$ whereby $\varepsilon_i \in \{-1, 1\}$ for $(i, j = 1, 2, \ldots, n)$. Now, we can give Frenet-Serret formulas according to the causal character of the curve α. It means that if $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$, then $\alpha(t)$ is the spacelike and timelike curve in $\mathbb{E}^n_2$, respectively. Then, the Frenet equations are as follows,

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-1}' \\ V_n' \end{pmatrix} = \begin{pmatrix} 0 & \nu\varepsilon_2\kappa_1 & 0 & \cdots & 0 & 0 \\ -\nu\varepsilon_1\kappa_1 & 0 & \nu\varepsilon_3\kappa_2 & \cdots & 0 & 0 \\ 0 & -\nu\varepsilon_2\kappa_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\nu\varepsilon_{n-1}\kappa_{n-1} \\ 0 & 0 & 0 & \cdots & -\nu\varepsilon_n\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-1} \\ V_n \end{pmatrix}$$

where $V_1 = \frac{\alpha'}{\|\alpha'\|}$, $\nu = \|\alpha'(t)\|$ and κ_i the ith curvatures of the curve α for $1 \leq i \leq n-1$ [6]. In this work, we assume all curvatures κ_i of the curve α are nowhere vanish. Such curves are called non-degenerate curve.

3. Spacelike Polynomial Helices in $\mathbb{E}^n_2$

In this section, after giving the definition of a helix in $\mathbb{E}^n_2$, we give families of polynomial spacelike helices in $\mathbb{E}^n_2$. For doing this, we have two cases where n is even or odd. In the case of n is even, there are three subcases; $n = 4, n = 6$ and $n \geq 8$. If n is odd there are also two subcases; $n = 5$ and $n \geq 7$.

Definition 3.1. *A regular curve $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ parameterized by arc length is called a helix if and only if there exist a constant vector $U \in \mathbb{E}^3_1$ with $L(T(s), U)$ is a constant where $T(s)$ is tangent vector of the curve β and $L(,)$ is the Lorentzian metric. Any line parallel this direction U is called the axis of the curve β [7].*

Similar to the definition above, we define helix in $\mathbb{E}^n_2$ as follows,

Definition 3.2. *A curve $\beta : I \subset \mathbb{R} \to \mathbb{E}_2^n$ is called a helix if and only if there exist a nonzero constant vector $U \in \mathbb{E}_2^n$ with $g(V_1, U)$ is a nonzero constant where V_1 is the tangent vector field of the curve β. Any line parallel this direction U is called the axis of the curve β.*

3.1. Spacelike polynomial helices in $\mathbb{E}_2^n$ when n is even

In this subsection, we give families of spacelike polynomial helices with spacelike, timelike or null axis for n is even.

Theorem 3.3. *Let $n = 4$ and $\beta : (1, d) \subset R \to \mathbb{E}_2^4$, $d > 1$, be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2}t^2, \frac{a_2}{3}t^3, a_3 t, \frac{a_4}{5}t^5 + \frac{a_5}{3}t^3\right).$$

If

$$a_1^2 = 2b_1b_2, \quad a_2^2 = 2b_1b_3, \quad a_3 = b_1, \quad a_4 = b_3, \quad a_5 = b_2,$$

with $1 \le j \le 3$, $b_j \in \mathbb{R}^+$, $b_3 + b_2 > b_1$ then β is a spacelike polynomial helix with the spacelike axis $U = (0, 0, 1, -1)$.

Proof. From the straightforward calculations, we have

$$g(\beta'(t), \beta'(t)) = \left(b_3t^4 + b_2t^2 - b_1\right)^2,$$

$$V_1(t) = \frac{1}{b_3t^4 + b_2t^2 - b_1}\left(a_1t, a_2t^2, b_1, b_3t^4 + b_2t^2\right),$$

$$g(V_1(t), U) = -1.$$

Therefore, β is a spacelike polynomial helix. □

Example 3.4. *If we choose $b_1 = 1, b_2 = b_3 = 2$ in Theorem 3.3, then we have the spacelike polynomial helix*

$$\beta(t) = \left(t^2, \frac{2t^3}{3}, t, \frac{2t^5}{5} + \frac{2t^3}{3}\right)$$

with the spacelike axis

$$U = (0, 0, 1, -1)$$

and the tangent vector

$$V_1(t) = \frac{1}{2t^4 + 2t^2 - 1}\left(2t, 2t^2, 1, 2t^4 + 2t^2\right).$$

Also,

$$g(V_1(t), U) = -1.$$

Theorem 3.5. *Let $n = 6$ and $\beta : (1, d) \subset R \to \mathbb{E}_2^6$, $d > 1$, be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2}t^2, \frac{a_2}{3}t^3, a_3 t, \frac{a_4}{4}t^4, \frac{a_5}{5}t^5, \frac{a_6}{7}t^7 + \frac{a_7}{5}t^5\right).$$

If

$$a_1^2 = 2b_1b_2, \quad a_2^2 = 2b_1b_3 - b_2^2, \quad a_3 = b_1, \quad a_4^2 = 2b_2b_3 - 2b_1b_4, \quad a_5^2 = 2b_2b_4, \quad a_6 = b_4, \quad a_7 = b_3,$$

with $1 \le j \le 4$, $b_j \in \mathbb{R}^+$, $\sum_{j=2}^{4} b_j > b_1$; $2b_1b_3 > b_2^2$; $b_2b_3 > b_1b_4$, *then* β *is a spacelike polynomial helix with the axis*

$$U = \left(0, \frac{b_2}{a_2}, 1, 0, 0, -1\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{-b_1 + b_2t^2 + b_3t^4 + b_4t^6}\left(a_1t, a_2t^2, a_3, a_4t^3, a_5t^4, a_6t^6 + a_7t^4\right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 3.3. □

Theorem 3.6. *Assume* $n \ge 8$ *is an even number,*

$$a_1^2 = 2b_1b_2, \quad a_2^2 = 2b_1b_3 - b_2^2 > 0, \quad a_3 = b_1, \quad a_{n-1}^2 = 2b_{\frac{n-2}{2}}b_{\frac{n+2}{2}}, \quad a_n = b_{\frac{n+2}{2}}, \quad a_{n+1} = b_{\frac{n}{2}},$$

$$a_{2k+1}^2 = b_{k+1}^2 - 2b_1b_{2k+1} + 2\sum_{j=2}^{k} b_jb_{2k-j+2} > 0 \text{ for } 2 \le k \le \frac{n-4}{2}$$

and

$$a_{2l}^2 = -2b_1b_{2l} + 2\sum_{j=2}^{l} b_jb_{2l-j+1} > 0 \text{ for } 2 \le l \le \frac{n-2}{2}$$

such that $1 \le j \le \frac{n+2}{2}$, $b_j \in \mathbb{R}^+$, $b_{\frac{n+4}{2}} = b_{\frac{n+6}{2}} = \cdots = b_{n-2} = 0$ *and* $\sum_{j=2}^{\frac{n+2}{2}} b_j > b_1$. *Then, the curve* $\beta : I \to \mathbb{E}_2^n$ *defined by*

$$\beta(t) = \left(\frac{a_1}{2}t^2, \frac{a_2}{3}t^3, a_3t, \frac{a_4}{4}t^4, \frac{a_5}{5}t^5, \ldots, \frac{a_{n-1}}{n-1}t^{n-1}, \frac{a_n}{n+1}t^{n+1} + \frac{a_{n+1}}{n-1}t^{n-1}\right)$$

is a spacelike polynomial helix with axis U *where*

$$U = \frac{b_2}{a_2}e_2 + e_3 - \sum_{m=3}^{\frac{n-2}{2}} \frac{b_m}{a_{2m-1}}e_{2m-1} - e_n,$$

$I = (1, d) \subset \mathbb{R}$ *and* $d > 1$.

Proof. By making calculations, we have

$$\begin{aligned} V_1(t) &= \frac{1}{\left\|\beta'(t)\right\|}\left(a_1t, a_2t^2, a_3, a_4t^3, a_5t^4, \ldots, a_{n-1}t^{n-2}, a_nt^n + a_{n+1}t^{n-2}\right) \\ &= \frac{1}{\left\|\beta'(t)\right\|}\left(a_1t, a_2t^2, b_1, a_4t^3, a_5t^4, \ldots, a_{n-1}t^{n-2}, b_{\frac{n+2}{2}}t^n + b_{\frac{n}{2}}t^{n-2}\right). \end{aligned}$$

Morever, we have

$$g(\beta'(t), \beta'(t)) = \left(-b_1 + \sum_{j=2}^{\frac{n+2}{2}} b_jt^{2(j-1)}\right)^2.$$

So, $g(\beta'(t), \beta'(t)) > 0$. In that case β is a spacelike polynomial curve with

$$g(V_1(t), U) = -1.$$

Eventually, β is a spacelike polynomial helix. □

As a result of Theorem 3.6, we have the following corollary.

Corollary 3.7. *It is easily seen that,*

$$\text{If } \frac{2a_2^2 - b_2^2}{a_2^2} + \sum_{m=3}^{\frac{n-2}{2}} \frac{b_m^2}{a_{2m-1}^2} > 0, \text{ then the axis } U \text{ is a spacelike vector,}$$

$$\text{If } \frac{2a_2^2 - b_2^2}{a_2^2} + \sum_{m=3}^{\frac{n-2}{2}} \frac{b_m^2}{a_{2m-1}^2} < 0, \text{ then the axis } U \text{ is a timelike vector}$$

and

$$\text{If } \frac{2a_2^2 - b_2^2}{a_2^2} + \sum_{m=3}^{\frac{n-2}{2}} \frac{b_m^2}{a_{2m-1}^2} = 0, \text{ then the axis } U \text{ is a null vector.}$$

Similarly, from the Theorem 3.5, one easily see that the axis U is a spacelike, a timelike and a null vector if $2a_2^2 - b_2^2 > 0, 2a_2^2 - b_2^2 < 0$ and $2a_2^2 = b_2^2$, respectively.

Now, we give an example of a spacelike polynomial helix with the null axis for $n = 6$.

Example 3.8. *If we choose $b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 1$ in Theorem 3.5, then we have the spacelike polynomial helix*

$$\beta(t) = \left(t^2, \frac{\sqrt{2}}{3}t^3, t, \frac{\sqrt{5}}{2\sqrt{2}}t^4, \frac{2t^5}{5}, \frac{t^7}{7} + \frac{3t^5}{5}\right)$$

with the null axis

$$U = \left(0, \sqrt{2}, 1, 0, 0, -1\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^6 + 3t^4 + 2t^2 - 1}\left(2t, \sqrt{2}\,t^2, 1, \sqrt{10}\,t^3, 2t^4, t^6 + 3t^4\right).$$

Also,

$$g(V_1(t), U) = -1.$$

Now, we give an example of a spacelike polynomial helix with the timelike axis for $n = 6$

Example 3.9. *If we choose $b_1 = 2, b_2 = \sqrt{3}, b_3 = 2, b_4 = 1$ in Theorem 3.5, then we have the spacelike polynomial helix*

$$\beta(t) = \left(\frac{\sqrt[4]{3}}{\sqrt{2}}t^2, \frac{t^3}{3}, t, \sqrt{\frac{2\sqrt{3}-1}{8}}t^4, \frac{\sqrt[4]{12}}{5}t^5, \frac{t^7}{7} + \frac{2t^5}{5}\right)$$

with the timelike axis

$$U = \left(0, \sqrt{3}, 1, 0, 0, -1\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^6 + 2t^4 + \sqrt{3}t^2 - 1}\left(\sqrt[4]{12}\,t, t^2, 1, \sqrt{4\sqrt{3}-2}\,t^3, \sqrt[4]{12}\,t^4, t^6 + 2t^4\right).$$

Also,

$$g(V_1(t), U) = -1.$$

3.2. Spacelike polynomial helices in $\mathbb{E}_2^n$ when n is odd

In this subsection, we give families of spacelike polynomial helices with spacelike, timelike or null axis when n is odd.

Theorem 3.10. *Let $n = 5$ and $\beta : (1, d) \subset R \to \mathbb{E}_2^5$, $d > 1$, be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2}t^2, \frac{a_2}{3}t^3, a_3 t, \frac{a_4}{4}t^4, \frac{a_5}{5}t^5\right).$$

If

$$a_1^2 = 2b_1b_2, \quad a_2^2 = 2b_1b_3 - b_2^2, \quad a_3 = b_1, \quad a_4^2 = 2b_2b_3, \quad a_5 = b_3,$$

with $1 \leq j \leq 3$, $b_j \in \mathbb{R}^+$, $b_3 + b_2 > b_1$ and $b_2^2 < 2b_1b_3$ then β is a spacelike polynomial helix with the axis

$$U = \left(0, \frac{b_2}{a_2}, 1, 0, -1\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{-b_1 + b_2t^2 + b_3t^4}\left(a_1 t, a_2 t^2, a_3, a_4 t^3, a_5 t^4\right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 3.3. □

Theorem 3.11. *Assume $n \geq 7$ is an odd number,*

$$a_1^2 = 2b_1b_2, \quad a_2^2 = 2b_1b_3 - b_2^2 > 0, \quad a_3 = b_1, \quad a_{n-1}^2 = 2b_{\frac{n-1}{2}}b_{\frac{n+1}{2}}, \quad a_n = b_{\frac{n+1}{2}},$$

$$a_{2k+1}^2 = b_{k+1}^2 - 2b_1b_{2k+1} + 2\sum_{j=2}^{k} b_j b_{2k-j+2} > 0 \text{ for } 2 \leq k \leq \frac{n-3}{2}$$

and

$$a_{2l}^2 = -2b_1b_{2l} + 2\sum_{j=2}^{l} b_j b_{2l-j+1} > 0 \text{ for } 2 \leq l \leq \frac{n-3}{2}$$

such that $1 \leq j \leq \frac{n+1}{2}$, $b_j \in \mathbb{R}^+$, $b_{\frac{n+3}{2}} = b_{\frac{n+5}{2}} = \cdots = b_{n-1} = 0$ and $\sum_{j=2}^{\frac{n+1}{2}} b_j > b_1$. Then, the curve $\beta : I \to \mathbb{E}_2^n$ defined by

$$\beta(t) = \left(\frac{a_1}{2}t^2, \frac{a_2}{3}t^3, a_3 t, \frac{a_4}{4}t^4, \frac{a_5}{5}t^5, \frac{a_6}{6}t^6, \ldots, \frac{a_{n-1}}{n-1}t^{n-1}, \frac{a_n}{n}t^n\right)$$

is a spacelike polynomial helix with the axis

$$U = \frac{b_2}{a_2}e_2 + e_3 - \sum_{m=3}^{\frac{n-1}{2}} \frac{b_m}{a_{2m-1}} e_{2m-1} - e_n,$$

where $I = (1, d) \subset \mathbb{R}$, $d > 1$ and the tangent vector

$$V_1(t) = \frac{1}{-b_1 + \sum_{j=2}^{\frac{n+1}{2}} b_j t^{2(j-1)}}\left(a_1 t, a_2 t^2, a_3, a_4 t^3, a_5 t^4, \ldots, a_{n-1}t^{n-2}, a_n t^{n-1}\right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 3.6. $\square$

As a result of Theorem 3.11, we have the following corollary.

Corollary 3.12. *It is easily seen that,*

$$\text{If } \frac{2a_2^2-b_2^2}{a_2^2}+\sum_{m=3}^{\frac{n-1}{2}}\frac{b_m^2}{a_{2m-1}^2}>0, \text{ then the axis } U \text{ is a spacelike vector,}$$

$$\text{If } \frac{2a_2^2-b_2^2}{a_2^2}+\sum_{m=3}^{\frac{n-1}{2}}\frac{b_m^2}{a_{2m-1}^2}<0, \text{ then the axis } U \text{ is a timelike vector}$$

and

$$\text{If } \frac{2a_2^2-b_2^2}{a_2^2}+\sum_{m=3}^{\frac{n-1}{2}}\frac{b_m^2}{a_{2m-1}^2}=0, \text{ then the axis } U \text{ is a null vector.}$$

Similarly, from the Theorem 3.10, one easily see the axis U is a spacelike, a timelike and a null vector if $2a_2^2-b_2^2>0, 2a_2^2-b_2^2<0$ and $2a_2^2=b_2^2$, respectively.

Now, we give an example of a spacelike polynomial helix with the spacelike axis for $n=5$.

Example 3.13. *If we choose $b_2=1, b_1=b_3=2$ in Theorem 3.10, then we have the spacelike polynomial helix*

$$\beta(t)=\left(t^2,\frac{\sqrt{7}}{3}t^3,2t,\frac{t^4}{2},\frac{2t^5}{5}\right)$$

with the spacelike axis

$$U=\left(0,\frac{1}{\sqrt{7}},1,0,-1\right)$$

and the tangent vector

$$V_1(t)=\frac{1}{2t^4+t^2-2}\left(2t,\ \sqrt{7}\,t^2,2,2t^3,2t^4\right).$$

Also,

$$g(V_1(t),U)=-1.$$

Now, we give an example of a spacelike polynomial helix with the timelike axis for $n=5$.

Example 3.14. *If we choose $b_1=1, b_2=\sqrt{3}, b_3=2$ in Theorem 3.10, then we have the spacelike polynomial helix*

$$\beta(t)=\left(\frac{\sqrt[4]{3}}{\sqrt{2}}t^2,\frac{t^3}{3},t,\frac{\sqrt[4]{3}}{2}t^4,\frac{2t^5}{5}\right)$$

with the timelike axis

$$U=\left(0,\ \sqrt{3},1,0,-1\right).$$

and the tangent vector

$$V_1(t)=\frac{1}{2t^4+\sqrt{3}t^2-1}\left(\sqrt[4]{12}\,t,t^2,1,\sqrt[4]{48}\,t^3,2t^4\right).$$

Also,

$$g(V_1(t),U)=-1.$$

Now, we give an example of a spacelike polynomial helix with the null axis for $n = 5$.

Example 3.15. *If we choose $b_1 = 1, b_2 = 2, b_3 = 3$ in Theorem 3.10, then we have the spacelike polynomial helix*

$$\beta(t) = \left(t^2, \frac{\sqrt{2}}{3}t^3, t, \frac{\sqrt{3}}{2}t^4, \frac{3t^5}{5}\right)$$

with the null axis

$$U = \left(0, \sqrt{2}, 1, 0, -1\right).$$

and the tangent vector

$$V_1(t) = \frac{1}{3t^4 + 2t^2 - 1}\left(2t, \sqrt{2}\,t^2, 1, 2\sqrt{3}\,t^3, 3t^4\right).$$

Also,

$$g(V_1(t), U) = -1.$$

4. Timelike Polynomial Helices in $\mathbb{E}_2^n$

In this section we give families of timelike polynomial helices with spacelike, timelike or null axis.

Theorem 4.1. *Let $n = 4$ and $\beta : \mathrm{I} - \{0\} \subset \mathbb{R} \to \mathbb{E}_2^4$ be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{5}t^5 + a_2 t, \frac{a_3}{4}t^4, \frac{a_4}{3}t^3, -a_2 t\right).$$

If

$$a_1 = b_2, \quad a_2 = \frac{b_1^2}{b_2}, \quad a_3^2 = 2b_1 b_2, \quad a_4 = b_1$$

with $b_1, b_2 \in \mathbb{R}^+$, then β is a timelike polynomial helix with the spacelike axis

$$U = (-1, 0, 1, 1).$$

Proof. From the straightforward calculations, we have

$$g(\beta'(t), \beta'(t)) = -\left(b_1 t^2 + b_2 t^4\right)^2,$$

$$V_1(t) = \frac{1}{b_2 t^4 + b_1 t^2}\left(b_2 t^4 + a_2, a_3 t^3, b_1 t^2, -a_2\right),$$

$$g(V_1(t), U) = 1.$$

Therefore, β is a timelike polynomial helix. □

Example 4.2. *If we choose $b_1 = 2, b_2 = 1$ in Theorem 4.1, then we have the timelike polynomial helix*

$$\beta(t) = \left(\frac{t^5}{5} + 4t, \frac{t^4}{2}, \frac{2t^3}{3}, -4t\right)$$

with the spacelike axis

$$U = (-1, 0, 1, 1)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^4+2t^2}\left(t^4+4, 2t^3, 2t^2, -4\right).$$

Also,

$$g(V_1(t), U) = 1.$$

Theorem 4.3. *Let $n=5$ and $\beta : \mathrm{I} - \{0\} \subset \mathbb{R} \to \mathbb{E}_2^5$ be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{7}t^7 + \frac{a_2}{5}t^5 + a_3 t, \frac{a_4}{5}t^5, \frac{a_5}{4}t^4, \frac{a_6}{3}t^3, -a_3 t\right).$$

If

$$a_1 = b_3, \quad a_2 = b_2, \quad a_3 = \frac{b_1^2}{b_2}, \quad a_4^2 = 2b_1 b_3, \quad a_5^2 = \frac{2b_1^2 b_3 - 2b_1 b_2^2}{b_2} > 0, \quad a_6 = b_1$$

with $1 \leq j \leq 3$, $b_j \in \mathbb{R}^+$ then β is a timelike polynomial helix with the spacelike axis

$$U = (-1, 0, 0, 1, 1)$$

and the tangent vector

$$V_1(t) = \frac{1}{b_1 t^2 + b_2 t^4 + b_3 t^6}\left(a_1 t^6 + a_2 t^4 + a_3, a_4 t^4, a_5 t^3, a_6 t^2, -a_3\right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 4.1. □

Example 4.4. *If we choose $b_1 = 2, b_2 = 1, b_3 = 2$ in Theorem 4.3, then we have the timelike polynomial helix*

$$\beta(t) = \left(\frac{2t^7}{7} + \frac{t^5}{5} + 4t, \frac{2\sqrt{2}}{5}t^5, \frac{\sqrt{3}}{2}t^4, \frac{2t^3}{3}, -4t\right)$$

with the spacelike axis

$$U = (-1, 0, 0, 1, 1)$$

and the tangent vector

$$V_1(t) = \frac{1}{2t^6 + t^4 + 2t^2}\left(2t^6 + t^4 + 4, 2\sqrt{2}\,t^4, 2\sqrt{3}\,t^3, 2t^2, -4\right).$$

Also,

$$g(V_1(t), U) = 1.$$

Theorem 4.5. *Let $n \geq 6$ and $\beta : \mathrm{I} - \{0\} \subset \mathbb{R} \to \mathbb{E}_2^n$ be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2n-3}t^{2n-3} + \frac{a_2}{2n-5}t^{2n-5} + \cdots + \frac{a_{n-3}}{5}t^5 + a_{n-2}t, \frac{a_{n-1}}{n}t^n, \frac{a_n}{n-1}t^{n-1}, \frac{a_{n+1}}{n-2}t^{n-2}, \ldots, \frac{a_{2n-4}}{3}t^3, -a_{n-2}t\right).$$

If

$$a_{2n-4} = b_1, \quad a_{n-3} = b_2, \quad a_{n-4} = b_3, \ldots, a_2 = b_{n-3}, \quad a_1 = b_{n-2},$$

$$a_{n-2} = \frac{b_1^2}{b_2}, \quad a_{n-1}^2 = 2b_1 b_{n-2}, \quad a_{2n-5}^2 = \frac{2b_1^2 b_3 - 2b_1 b_2^2}{b_2} > 0$$

and

$$a_k^2 = 2a_{n-2}a_{k-n+1} - 2a_{2n-4}a_{k-n+2} > 0 \text{ for } n \leq k \leq 2n-6$$

with b_i is a positive constant for $1 \leq i \leq n-2$ then β is a timelike helix with the spacelike axis

$$U = -e_1 + e_{n-1} + e_n$$

and the tangent vector

$$V_1(t) = \frac{1}{\sum_{j=1}^{n-2} b_j t^{2j}} \left(a_1 t^{2n-4} + a_2 t^{2n-6} + \cdots + a_{n-3}t^4 + a_{n-2}, a_{n-1}t^{n-1}, a_n t^{n-2}, a_{n+1}t^{n-3}, \ldots, a_{2n-4}t^2, -a_{n-2} \right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 4.1. □

Theorem 4.6. *Let $n \geq 4$ and $\beta : \mathrm{I} - \{0\} \subset \mathbb{R} \to \mathbb{E}_2^n$ be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2n-3} t^{2n-3} + \frac{a_2}{2n-5} t^{2n-5} + \cdots + \frac{a_{n-2}}{3} t^3, a_{n-1}t, \frac{a_n}{2} t^2, \frac{a_{n+1}}{3} t^3, \ldots, \frac{a_{2n-3}}{n-1} t^{n-1} \right).$$

If

$$a_1 = b_{n-1}, \quad a_2 = b_{n-2}, \quad a_3 = b_{n-3}, \quad \ldots \quad a_{n-1} = b_1$$

and

$$a_n^2 = 2b_1 b_2, \quad a_{n+1}^2 = 2b_1 b_3, \quad \ldots, \quad a_{2n-3}^2 = 2b_1 b_{n-1},$$

with b_i is a positive constant for $1 \leq i \leq n-1$ then β is a timelike helix with the timelike axis

$$U = e_1 - e_2$$

and tangent vector

$$V_1(t) = \frac{1}{-b_1 + \sum_{j=2}^{n-1} b_j t^{2(j-1)}} \left(a_1 t^{2n-4} + a_2 t^{2n-6} + \cdots + a_{n-2}t^2, a_{n-1}, a_n t, a_{n+1}t^2, \ldots, a_{2n-3}t^{n-2} \right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 3.6. □

Example 4.7. *If we choose $n = 4; b_1 = 1, b_2 = 2, b_3 = 1$ in Theorem 4.6, then we have the timelike polynomial helix*

$$\beta(t) = \left(\frac{t^5}{5} + \frac{2t^3}{3}, t, t^2, \frac{\sqrt{2}t^3}{3} \right)$$

with the timelike axis

$$U = (1, -1, 0, 0)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^4 + 2t^2 - 1} \left(t^4 + 2t^2, 1, 2t, \sqrt{2}\, t^2 \right).$$

Also,

$$g(V_1(t), U) = -1.$$

Theorem 4.8. *Let $n \geq 4$ and $\beta : \mathrm{I} = (0,1) \subset \mathbb{R} \rightarrow \mathbb{E}_2^n$ be a curve defined by*

$$\beta(t) = \left(\frac{a_1}{2n-3} t^{2n-3} + \frac{a_2}{2n-5} t^{2n-5} + \cdots + \frac{a_{n-2}}{3} t^3 + t, \frac{a_{n-1}}{n-1} t^{n-1}, \frac{a_n}{n-2} t^{n-2}, \ldots, \frac{a_{2n-4}}{2} t^2, t \right).$$

If

$$a_1 = -b_{n-2}, \quad a_2 = b_{n-3}, \quad a_3 = b_{n-4}, \quad \ldots \quad a_{n-2} = b_1$$

and

$$a_{n-1}^2 = 2b_{n-2}, \quad a_n^2 = 2b_{n-3}, \quad a_{n+1}^2 = 2b_{n-4}, \quad \ldots \quad , a_{2n-4}^2 = 2b_1$$

with b_i is a positive constant for $1 \leq i \leq n-2$ and $b_1 \geq b_{n-2}$ then β is a timelike helix with the null axis

$$U = e_1 + e_n,$$

and tangent vector

$$V_1(t) = \frac{1}{t^2 \left(-b_{n-2} t^{2n-6} + \sum\limits_{j=2}^{n-2} b_{j-1} t^{2(j-2)} \right)} \left(a_1 t^{2n-4} + a_2 t^{2n-6} + \cdots + a_{n-2} t^2 + 1, a_{n-1} t^{n-2}, a_n t^{n-3}, \ldots, a_{2n-4} t, 1 \right).$$

Proof. We omit the proof since it is analogous to the proof of the Theorem 3.6. □

Example 4.9. *If we choose $n = 5; b_1 = b_2 = 2$ and $b_3 = 1$ in Theorem 4.8, then we have the timelike polynomial helix*

$$\beta(t) = \left(-\frac{2t^7}{7} + \frac{2t^5}{5} + \frac{2t^3}{3} + t, \frac{t^4}{2}, \frac{2t^3}{3}, t^2, t \right)$$

with the null axis

$$U = (1, 0, 0, 0, 1)$$

and the tangent vector

$$V_1(t) = \frac{1}{-2t^6 + 2t^4 + 2t^2} \left(-2t^6 + 2t^4 + 2t^2 + 1, 2t^3, 2t^2, 2t, 1 \right).$$

Also,

$$g(V_1(t), U) = -1.$$

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