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Improved Brauer-Type Eigenvalue Localization Sets for Tensors with Their Applications

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Abstract. In this paper, by excluding some sets from the Brauer-type eigenvalue inclusion sets for tensors developed by Bu et al. (Linear Algebra Appl. 512 (2017) 234-248) and Li et al. (Linear and Multilinear Algebra 64 (2016) 727-736), some improved Brauer-type eigenvalue localization sets for tensors are given, which are proved to be much tighter than those put forward by Bu et al. and Li et al. As applications, some new criteria for identifying the nonsingularity of tensors are developed, which are better than some previous results. This fact is illustrated by some numerical examples.

1. Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, n be a positive integer with $n \geq 2$, and $N = \{1, 2, \ldots, n\}$. The tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is called a complex (real) order m dimension n tensor, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if $a_{i_1 \ldots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j \in N$ for $j = 1, 2, \ldots, m$ [17].

The tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called the unit tensor [14], denoted by $\mathcal{I}$, if its entries $\delta_{i_1 \ldots i_m} (i_1, \ldots, i_m \in N)$ satisfy the following conditions:

$$\delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \ldots = i_m, \\ 0, & \text{otherwise}, \end{cases}$$

and for $x \in \mathbb{C}^n$. $\mathcal{A}x^{m-1}$ is a column vector of dimension n and its i-th entry is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \ i \in N.$$

Some notations used in this paper are given. For $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$, $i, j \in N$, $j \neq i$, we denote

$$\Delta_i = \{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j = 2, 3, \ldots, m\},$$
$$\overline{\Delta}_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j = 2, 3, \ldots, m\}$$

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and

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} |a_{ii_2\ldots i_m}|,\ r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0,\\ \delta_{ji_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| = r_i(\mathcal{A}) - |a_{ij\ldots j}|,\ \bar{r}_i^j(\mathcal{A}) = r_i(\mathcal{A}) - |a_{i\ldots ij}|.$$

In 2005, Qi [21] and Lim [19] independently gave the definition of the eigenvalues of tensors.

Definition 1.1. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenpair of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T$. Here x^T denotes the transpose of x. Furthermore, we call (λ, x) an H-eigenpair, if λ is a real number and x is a real vector.

Recently, Che et al. [3] consider the homogeneous dynamical system related to the tensor $\mathcal{A}$ and derived the definition of ϵ-pseudospectrum of $\mathcal{A}$.

Definition 1.2. [3] Let $\epsilon \geq 0$. The ϵ-pseudospectrum of $A = (a_{i_1 i_2 \ldots i_m}) \in C^{[m,n]}$ is defined as

$$\Lambda_\epsilon(\mathcal{A}) = \{\lambda \in \mathbb{C} : (\mathcal{A} + \varepsilon)x^{m-1} = \lambda x^{[m-1]} \text{ for } \varepsilon \in \mathbb{C}^{[m,n]} \text{ with } \|\varepsilon\|_F \leq \epsilon \text{ and some } x \in \mathbb{C}^n \backslash \{0\}\},$$

where $\|\varepsilon\|_F$ is the Frobenius norm of $\varepsilon = (\varepsilon_{i_1 i_2 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, i.e., $\|\varepsilon\|_F = \sqrt{\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} |\varepsilon_{i_1 \ldots i_m}|^2}$.

Next we exhibit the definition of symmetry of tensor, which was put forward firstly by Qi [21].

Definition 1.3. [11, 12, 15, 16, 21, 24] A real tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is called symmetric if its entries satisfy

$$a_{i_1 \ldots i_m} = a_{\pi(i_1 \ldots i_m)},\ \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices.

Eigenvalue problems of tensors play significant roles in many fields, and they have wide practical applications, such as magnetic resonance imaging [22], higher order Markov chains [20], spectral hypergraph theory [4] and so forth. Due to this fact and the difficulty of computing eigenvalues of tensors directly, it is vital to study the eigenvalue inclusion sets for tensors. As observed in [12, 14, 17], we can utilize the smallest H-eigenvalue of an even-order real symmetric tensor to determine its positive (semi-)definiteness, but getting the smallest H-eigenvalue of tensors is a task work for us on many occasions. In addition, [13] posed that the determinant of the tensor $\mathcal{A}$, denoted by $\det(\mathcal{A})$, is the resultant of the ordered system of homogeneous equations $\mathcal{A}x^{m-1} = 0$ and is closely related to the eigenvalues of $\mathcal{A}$. If $\det(\mathcal{A}) \neq 0$, i.e., 0 is not an eigenvalue of $\mathcal{A}$, then $\mathcal{A}$ is nonsingular. While the nonsingularity of tensors is hard to be identified by computing their eigenvalues directly. Considering above situations, a set containing all eigenvalues of tensors should be derived. Much literature have been devoted to this topic recently, refer to [1, 2, 6–17, 21] for more details. A great eigenvalue localization set is conducive to judge the positive definiteness and the nonsingularity of tensors, so we establish the new eigenvalue localization sets called improved Brauer-type eigenvalue localization sets for tensors in this paper, which are proved to be tighter than those in [1, 13, 18, 23].

Before establishing the new eigenvalue inclusion sets for tensors in this paper, we first review some related results. For the real supersymmetric tensors, Qi in [21] gave the Geršgorin eigenvalue localization sets as follows.

Lemma 1.1. [21] *Let* $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ *is the set of all the eigenvalues of* $\mathcal{A}$ *and*

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\ldots i}| \leq r_i(\mathcal{A})\}.$$

This result is also valid for general tensors [15, 24]. To improve the accuracy of $\Gamma(\mathcal{A})$, Bu et al. [1] derived the following eigenvalue localization set recently for tensors.

Lemma 1.2. [1] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} \mathcal{B}_{i,j}(\mathcal{A}), \tag{1}$$

where

$$\mathcal{B}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i\ldots i}|^{m-1}|z - a_{j\ldots j}| \leq (r_i(\mathcal{A}))^{m-1} r_j(\mathcal{A}) \right\}.$$

$\mathcal{B}(\mathcal{A})$ is called the Brauer-type eigenvalue localization set. Besides, another Brauer-type eigenvalue localization set is also proposed by the authors in [1] as follows.

Lemma 1.3. [1] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$ *and* $r_i(\mathcal{A}) \neq 0$ $(i \in N)$. *Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) = \bigcup_{\substack{a_{i_1 i_2\ldots i_m}\neq 0,\\ \delta_{i_1 i_2\ldots i_m}=0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j\ldots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}) \right\}.$$

The set in Lemma 1.3 was confirmed to be better than that in Lemma 1.1. In addition, by dividing the set N into two disjoint parts, Li et al. in [17] constructed the new Brauer-type eigenvalue localization set for tensors in the following lemma.

Lemma 1.4. [17] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \left(\bigcup_{i\in N} \hat{\Omega}_i(\mathcal{A}) \right) \bigcup \left(\bigcup_{i,j\in N, i\neq j} \left(\tilde{\Omega}^1_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A}) \right) \right), \tag{2}$$

where

$$\hat{\Omega}_i(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{i\ldots i}| \leq r_i^{\Delta_i}(\mathcal{A}) \},$$
$$\tilde{\Omega}_{i,j}(\mathcal{A}) = \{ z \in \mathbb{C} : (|z - a_{i\ldots i}| - r_i^{\Delta_i}(\mathcal{A}))(|z - a_{j\ldots j}| - r_j^{\overline{\Delta}_i}(\mathcal{A})) \leq r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j^{\Delta_i}(\mathcal{A}) \}.$$

In [13], the authors excluded some proper subsets, which do not include any eigenvalues of tensors, from eigenvalue localization set in Lemma 1.1. And they skillfully constructed a tighter eigenvalue localization set as follows.

Lemma 1.5. [13] *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \bigcup_{i\in N} \Upsilon_i(\mathcal{A}),$$

where $\Upsilon_i(\mathcal{A}) = \Gamma_i(\mathcal{A}) \backslash \Delta_i(\mathcal{A})$,

$$\Delta_i(\mathcal{A}) = \bigcup_{j\neq i} \Delta_{ij}(\mathcal{A}).$$

and

$$\Delta_{ij}(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{j\ldots j}| \leq 2|a_{ji\ldots i}| - r_j(\mathcal{A}) \}.$$

Note that in recent published literature [5], He et al. made use of the idea of excluding the subsets, and constructed the exclusion set for the pseudospectrum of tensors, which is significant in practical applications. Moreover, when the tensor $\epsilon = 0$, the exclusion set for the pseudospectrum of tensors in [5] reduces to an eigenvalue inclusion set for tensors, whose form is similar to that in Lemma 1.5.

In this work, motivated by the idea of [13, 18], several improved Brauer-type eigenvalue localization sets are established, which are sharper than those in Lemmas 1.2-1.4. And their forms are different from those of the exclusion sets in Lemma 1.5 and [5] as $\epsilon = 0$. As applications of the new sets, some new criteria for identifying the nonsingularity of tensors are given, which have advantages over some existing ones.

2. Improved Brauer-type eigenvalue localization sets for tensors

In this section, we construct the improved Brauer-type eigenvalue localization sets, and the comparisons between the new sets and those in Lemmas 1.2-1.4 are given.

Theorem 2.1. *Let* $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} (\mathcal{B}_{i,j}(\mathcal{A}) \setminus \Omega_{i,j}(\mathcal{A})), \tag{3}$$

where

$$\mathcal{B}_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i\dots i}|^{m-1} |z - a_{j\dots j}| \leq (r_i(\mathcal{A}))^{m-1} r_j(\mathcal{A}) \right\},$$
$$\Omega_{i,j}(\mathcal{A}) = \left\{ z \in \mathbb{C} : (|z - a_{i\dots i}| + \bar{r}_i^j(\mathcal{A}))^{m-1} |z - a_{j\dots j}| < |a_{i\dots ij}|^{m-1} (2|a_{ji\dots i}| - r_j(\mathcal{A})) \right\}.$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{4}$$

In view of the proof of Theorem 3.1 in [1], let $|x_p| \geq |x_q| \geq \max\{|x_i|, i \in N, i \neq p, i \neq q\}$. Then, $|x_p| > 0$. It follows from the pth equation of (4) that

$$(\lambda - a_{p\dots p}) x_p^{m-1} = \sum_{\delta_{pi_2 \dots i_m = 0}} a_{pi_2 \dots i_m} x_{i_2} \cdots x_{i_m}. \tag{5}$$

Taking absolute values in Equation (5) and applying the triangle inequality yield

$$\begin{aligned} |\lambda - a_{p\dots p}||x_p|^{m-1} &\leq \sum_{\delta_{pi_2 \dots i_m = 0}} |a_{pi_2 \dots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{\delta_{pi_2 \dots i_m = 0}} |a_{pi_2 \dots i_m}||x_p|^{m-2}|x_q| \\ &= r_p(\mathcal{A})|x_p|^{m-2}|x_q|, \end{aligned}$$

which leads to

$$|\lambda - a_{p\dots p}||x_p|^m \leq r_p(\mathcal{A})|x_p|^{m-1}|x_q|. \tag{6}$$

If $|x_q| = 0$, then it follows from (6) that $|\lambda - a_{p\dots p}| \leq 0$ by $|x_p| > 0$, which implies that $\lambda = a_{p\dots p}$. Evidently, $\lambda \in \mathcal{B}_{p,q}(\mathcal{A})$. Otherwise, $|x_q| > 0$. Then qth equation of (4) gives

$$(\lambda - a_{q\dots q}) x_q^{m-1} = \sum_{\delta_{qi_2 \dots i_m = 0}} a_{qi_2 \dots i_m} x_{i_2} \cdots x_{i_m}, \tag{7}$$

and it follows that

$$\begin{aligned} |\lambda - a_{q\dots q}||x_q|^{m-1} &\leq \sum_{\delta_{qi_2 \dots i_m = 0}} |a_{qi_2 \dots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ &\leq \sum_{\delta_{qi_2 \dots i_m = 0}} |a_{qi_2 \dots i_m}||x_p|^{m-1} \\ &= r_q(\mathcal{A})|x_p|^{m-1}. \end{aligned} \tag{8}$$

Combining (6) with (8) results in

$$|z - a_{p\dots p}|^{m-1} |z - a_{q\dots q}| \leq (r_p(\mathcal{A}))^{m-1} r_q(\mathcal{A})$$

by $|x_p| \geq |x_q| > 0$, which means that $\lambda \in \mathcal{B}_{p,q}(\mathcal{A})$ holds true. It follows from (5) that

$$a_{p\ldots pq}x_p^{m-2}x_q = (\lambda - a_{p\ldots p})x_p^{m-1} - \left(\sum_{\delta_{pi_2\ldots i_m=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} - a_{p\ldots pq}x_p^{m-2}x_q\right). \tag{9}$$

By taking modulus in both sides of (9) and utilizing the triangle inequality, it has

$$\begin{aligned}|a_{p\ldots pq}||x_p|^{m-2}|x_q| &\leq |\lambda - a_{p\ldots p}||x_p|^{m-1} + \left(\sum_{\delta_{pi_2\ldots i_m=0}} |a_{pi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}| - |a_{p\ldots pq}||x_p|^{m-2}|x_q|\right)\\ &\leq (|\lambda - a_{p\ldots p}| + \bar{r}_p^q(\mathcal{A}))|x_p|^{m-1},\end{aligned}$$

which results in

$$|a_{p\ldots pq}||x_p|^{m-1}|x_q| \leq (|\lambda - a_{p\ldots p}| + \bar{r}_p^q(\mathcal{A}))|x_p|^m. \tag{10}$$

Furthermore, from (7), it has

$$a_{qp\ldots p}x_p^{m-1} = (\lambda - a_{q\ldots q})x_q^{m-1} - \sum_{\substack{\delta_{qi_2\ldots i_m=0},\\ \delta_{pi_2\ldots i_m=0}}} a_{qi_2\ldots i_m}x_{i_2}\cdots x_{i_m}. \tag{11}$$

Applying the same operations utilized in (10) to (11) results in

$$\begin{aligned}|a_{qp\ldots p}||x_p|^{m-1} &\leq |\lambda - a_{q\ldots q}||x_q|^{m-1} + \sum_{\substack{\delta_{qi_2\ldots i_m=0},\\ \delta_{pi_2\ldots i_m=0}}} |a_{qi_2\ldots i_m}||x_{i_2}|\cdots|x_{i_m}|\\ &\leq |\lambda - a_{q\ldots q}||x_q|^{m-1} + \sum_{\substack{\delta_{qi_2\ldots i_m=0},\\ \delta_{pi_2\ldots i_m=0}}} |a_{qi_2\ldots i_m}||x_p|^{m-1}\\ &= |\lambda - a_{q\ldots q}||x_q|^{m-1} + (r_q(\mathcal{A}) - |a_{qp\ldots p}|)|x_p|^{m-1},\end{aligned}$$

which yields that

$$(2|a_{qp\ldots p}| - r_q(\mathcal{A}))|x_p|^{m-1} \leq |\lambda - a_{q\ldots q}||x_q|^{m-1}. \tag{12}$$

If $|x_q| > 0$, then combining (10) with (12) leads to

$$|a_{p\ldots pq}|^{m-1}(2|a_{qp\ldots p}| - r_q(\mathcal{A}))|x_p|^{m(m-1)}|x_q|^{m-1} \leq (|\lambda - a_{p\ldots p}| + \bar{r}_p^q(\mathcal{A}))|\lambda - a_{q\ldots q}||x_p|^{m(m-1)}|x_q|^{m-1},$$

and hence

$$|a_{p\ldots pq}|^{m-1}(2|a_{qp\ldots p}| - r_q(\mathcal{A})) \leq (|\lambda - a_{p\ldots p}| + \bar{r}_p^q(\mathcal{A}))|\lambda - a_{q\ldots q}| \tag{13}$$

as $|x_p| \geq |x_q| > 0$. If $|x_q| = 0$, then (12) implies that $2|a_{qp\ldots p}| - r_q(\mathcal{A}) \leq 0$, and (13) is also valid. (13) means that $\lambda \notin \Omega_{p,q}(\mathcal{A})$. Therefore, $\lambda \in \left(\mathcal{B}_{p,q}(\mathcal{A})\backslash\Omega_{p,q}(\mathcal{A})\right)$.

It is uncertain which p and q are appropriate to each eigenvalue λ, we conclude that

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} (\mathcal{B}_{i,j}(\mathcal{A}) \setminus \Omega_{i,j}(\mathcal{A})),$$

which completes the proof of Theorem 2.1. □

Next, we prove that $\Theta(\mathcal{A})$ is better than $\mathcal{B}(\mathcal{A})$ in Lemma 1.2.

Theorem 2.2. *Let* $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$, *then*

$$\Theta(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. By Theorem 3.1 in [1], we see that $\mathcal{B}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ holds. Thus, we only need to prove $\Theta(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A})$. For any $i, j \in N$ and $i \neq j$, if $|a_{i\dots ij}|^{m-1}(2|a_{ji\dots i}| - r_j(\mathcal{A})) \leq 0$, then $\Omega_{i,j}(\mathcal{A}) = \emptyset$, and therefore $\Omega_{i,j}(\mathcal{A}) \subseteq \mathcal{B}_{i,j}(\mathcal{A})$. Now we consider the case that $|a_{i\dots ij}|^{m-1}(2|a_{ji\dots i}| - r_j(\mathcal{A})) > 0$. By the definition of $\bar{r}_i^j(\mathcal{A})$, we see that $\bar{r}_i^j(\mathcal{A}) \geq 0$ and therefore

$$|z - a_{i\dots i}|^{m-1}|z - a_{j\dots j}| \leq (|z - a_{i\dots i}| + \bar{r}_i^j(\mathcal{A}))^{m-1}|z - a_{j\dots j}|. \tag{14}$$

Since $|a_{ji\dots i}| \leq r_j(\mathcal{A})$ and $|a_{i\dots ij}| \leq r_i(\mathcal{A})$, it has

$$|a_{i\dots ij}|^{m-1}(2|a_{ji\dots i}| - r_j(\mathcal{A})) \leq |a_{i\dots ij}|^{m-1}(2r_j(\mathcal{A}) - r_j(\mathcal{A})) = (r_i(\mathcal{A}))^{m-1}r_j(\mathcal{A}), \tag{15}$$

which together with (14) shows that

$$\begin{aligned} |z - a_{i\dots i}|^{m-1}|z - a_{j\dots j}| &\leq (|z - a_{i\dots i}| + \bar{r}_i^j(\mathcal{A}))^{m-1}|z - a_{j\dots j}| \\ &< |a_{i\dots ij}|^{m-1}(2|a_{ji\dots i}| - r_j(\mathcal{A})) \leq (r_i(\mathcal{A}))^{m-1}r_j(\mathcal{A}). \end{aligned}$$

Thus $\Omega_{i,j}(\mathcal{A}) \subseteq \mathcal{B}_{i,j}(\mathcal{A})$ is valid for this case. Thus we conclude that $\Theta(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A})$. □

Remark 2.3. For a tensor $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$, obtaining the set $\Theta(\mathcal{A})$ needs to compute $n(n+1)$ sets, which includes $\frac{n(n+1)}{2}$ sets $\mathcal{B}_{i,j}(\mathcal{A})$ and $\frac{n(n+1)}{2}$ sets $\Omega_{i,j}(\mathcal{A})$. And the set $\mathcal{B}(\mathcal{A})$ consists of $\frac{n(n+1)}{2}$ sets $\mathcal{B}_{i,j}(\mathcal{A})$. This implies that there are more computations to determine $\Theta(\mathcal{A})$ than $\mathcal{B}(\mathcal{A})$, while $\Theta(\mathcal{A})$ is tighter than $\mathcal{B}(\mathcal{A})$ as showed in Theorem 2.2.

The following example is given to compare the sets in Theorem 2.1 and Theorem 3.1 of [1], and we depict them in Figure 1.

Example 2.4. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ with elements defined as follows:*

$$\begin{aligned} &a_{111} = 60,\ a_{222} = 5,\ a_{333} = 90 + 30\mathrm{i},\ a_{444} = 15,\ a_{114} = 1,\ a_{122} = 30 + \mathrm{i},\ a_{133} = 1 - \mathrm{i},\ a_{144} = 1 + \mathrm{i}, \\ &a_{211} = 2,\ a_{221} = 120,\ a_{223} = 1,\ a_{233} = 1,\ a_{311} = 1,\ a_{322} = 1,\ a_{332} = 1,\ a_{334} = 2,\ a_{441} = 2,\ a_{442} = 1 \end{aligned}$$

and other elements of $\mathcal{A}$ are zeros.

The localization sets $\Theta(\mathcal{A})$ and $\mathcal{B}(\mathcal{A})$ are plotted in Figure 1. Besides, all eigenvalues of the tensor $\mathcal{A}$ computed by the Matlab code *teig*, are depicted in Figure 1 with the black plus. It is clear that $\Theta(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{A})$ and all eigenvalues of the tensor $\mathcal{A}$ are included in $\Theta(\mathcal{A})$, which are in accordance with the results of Theorem 2.1 and Theorem 2.2 (see Figure 1).

Theorem 2.5. *Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$ and $r_i(\mathcal{A}) \neq 0$ $(i \in N)$. Then*

$$\sigma(\mathcal{A}) \subseteq \Psi(\mathcal{A}) = (\Psi_1(\mathcal{A})) \bigcup (\Psi_2(\mathcal{A})), \tag{16}$$

where

$$\begin{aligned} \Psi_1(\mathcal{A}) &= \bigcup_{i\in N}\{a_{i\dots i}\}, \\ \Psi_2(\mathcal{A}) &= \bigcup_{\substack{a_{i_1i_2\dots i_m}\neq 0,\\ \delta_{i_1i_2\dots i_m}=0}} \left(\left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j\dots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}) \right\} \right. \\ &\quad \left. \setminus \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j\dots i_j}| < \prod_{j=1}^{m} (2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})); 2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0, j = 1, \dots, m \right\} \right), \end{aligned}$$

and $|\bar{a}_{i_j}| = \max_{\pi\in\Pi_{m-1}} \{|a_{i_j\pi(i_1\dots i_{j-1}i_{j+1}\dots i_m)}|\}$ with Π_{m-1} being the permutation group of $m-1$ indices.

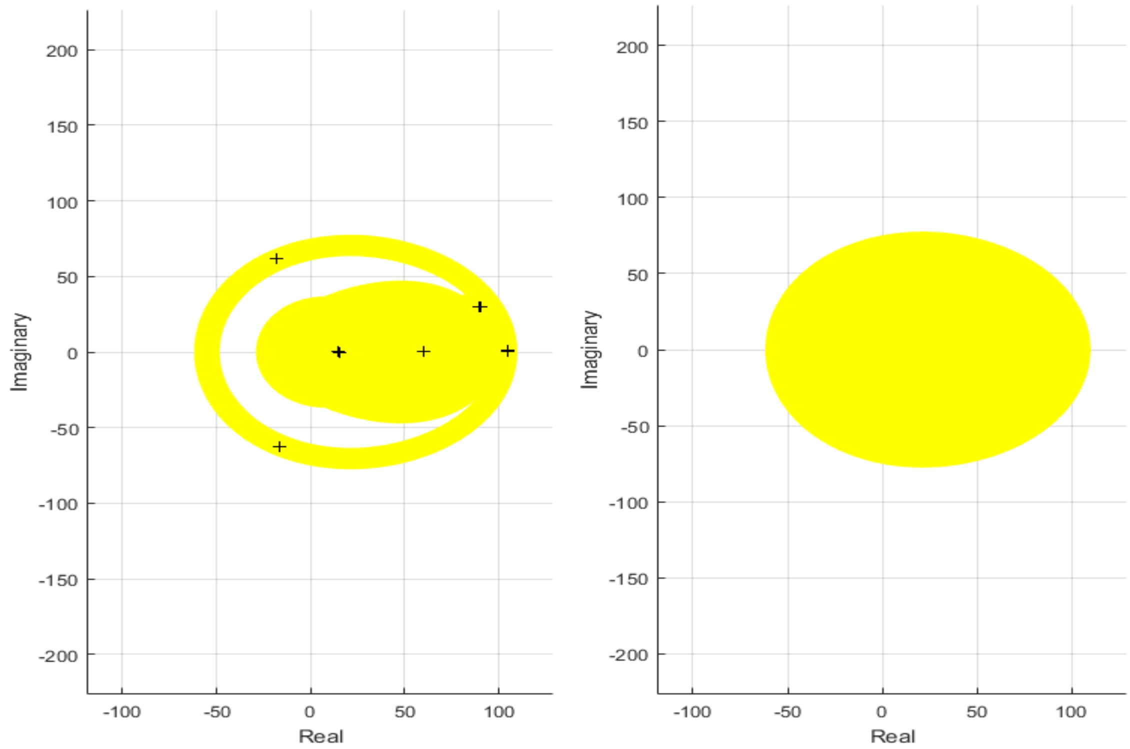

Figure 1: Eigenvalue localization sets $\Theta(\mathcal{A})$ (left) and $\mathcal{B}(\mathcal{A})$ (right).

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{17}$$

By making use of the technique of Theorem 3.3 in [1], let $|x_\beta| = \max\{|x_{i_1}||x_{i_2}|\cdots|x_{i_m}| : a_{i_1 i_2 \dots i_m} \neq 0, \delta_{i_1 i_2 \dots i_m} = 0, i_1, \dots, i_m \in N\}$. Then for all $i \in N$, it has

$$(\lambda - a_{i\dots i})x_i^{m-1} = \sum_{\delta_{ii_2\dots i_m}=0} a_{ii_2\dots i_m} x_{i_2} \cdots x_{i_m}. \tag{18}$$

Taking absolute values in Equation (18) and applying the triangle inequality give

$$\begin{aligned} |\lambda - a_{i\dots i}||x_i|^m &\leq \sum_{\delta_{ii_2\dots i_m}=0} |a_{ii_2\dots i_m}||x_i||x_{i_2}|\cdots|x_{i_m}| \\ &= \sum_{\substack{a_{ii_2\dots i_m}\neq 0,\\ \delta_{i_1 i_2\dots i_m}=0}} |a_{ii_2\dots i_m}||x_i||x_{i_2}|\cdots|x_{i_m}| \leq r_i(\mathcal{A})|x_\beta|. \end{aligned} \tag{19}$$

Since $x \neq 0$, there exists one index k such that $x_k \neq 0$. Taking $i = k$ in (19) leads to

$$|\lambda - a_{k\dots k}||x_k|^m \leq r_k(\mathcal{A})|x_\beta|. \tag{20}$$

If $|x_\beta| = 0$, then it follows from (20) that $\lambda = a_{k\dots k}$ and therefore $\lambda \in \Psi_1(\mathcal{A})$.

For the case that $|x_\beta| \neq 0$, without loss of generality, we assume that $|x_\beta| = |x_{j_1}||x_{j_2}|\cdots|x_{j_m}|$. Then from (20), it holds that

$$\begin{aligned} |\lambda - a_{j_1\ldots j_1}||x_{j_1}|^m &\leq r_{j_1}(\mathcal{A})|x_\beta|, \\ |\lambda - a_{j_2\ldots j_2}||x_{j_2}|^m &\leq r_{j_2}(\mathcal{A})|x_\beta|, \\ &\vdots \\ |\lambda - a_{j_m\ldots j_m}||x_{j_m}|^m &\leq r_{j_m}(\mathcal{A})|x_\beta|, \end{aligned}$$

which yields that

$$\prod_{i=1}^{m} |\lambda - a_{j_i\ldots j_i}||x_{j_i}|^m \leq |x_\beta|^m \prod_{i=1}^{m} r_{j_i}(\mathcal{A}),$$

and hence

$$\prod_{i=1}^{m} |\lambda - a_{j_i\ldots j_i}| \leq \prod_{i=1}^{m} r_{j_i}(\mathcal{A}), \tag{21}$$

which also implies that

$$\lambda \in \bigcup_{\substack{a_{i_1 i_2\ldots i_m}\neq 0,\\ \delta_{i_1 i_2\ldots i_m}=0}} \left\{ z \in \mathbb{C} : \prod_{i=1}^{m} |z - a_{j_i\ldots j_i}| \leq \prod_{i=1}^{m} r_{j_i}(\mathcal{A}) \right\}. \tag{22}$$

Considering $i = j_1$ in (18):

$$(\lambda - a_{j_1\ldots j_1})x_{j_1}^{m-1} = \sum_{\delta_{j_1 i_2\ldots i_m=0}} a_{j_1 i_2\ldots i_m} x_{i_2}\cdots x_{i_m}. \tag{23}$$

Without loss of generality, we assume $|a_{j_1 j_2\ldots j_m}| = \max_{\pi\in\Pi_{m-1}} \{|a_{j_1\pi(j_2 j_3\ldots j_m)}|\}$ with Π_{m-1} being the permutation group of $m-1$ indices. Then it follows from (23) that

$$a_{j_1 j_2\ldots j_m} x_{j_1}\ldots x_{j_m} = (\lambda - a_{j_1\ldots j_1})x_{j_1}^m - \left(\sum_{\delta_{j_1 i_2\ldots i_m=0}} a_{j_1 i_2\ldots i_m} x_{j_1} x_{i_2}\cdots x_{i_m} - a_{j_1 j_2\ldots j_m} x_{j_1}\ldots x_{j_m} \right).$$

Taking modulus in the above equation and applying the triangle inequality lead to

$$|a_{j_1 j_2\ldots j_m}||x_\beta| \leq |\lambda - a_{j_1\ldots j_1}||x_{j_1}|^m + \left(r_{j_1}(\mathcal{A}) - |a_{j_1 j_2\ldots j_m}| \right)|x_\beta|,$$

which is equivalent to

$$(2|\bar{a}_{j_1}| - r_{j_1}(\mathcal{A}))|x_\beta| \leq |\lambda - a_{j_1\ldots j_1}||x_{j_1}|^m. \tag{24}$$

Similarly, for $i = j_2, i = j_3, \ldots, i = j_m$ in (18), we have

$$\begin{aligned} (2|\bar{a}_{j_2}| - r_{j_2}(\mathcal{A}))|x_\beta| &\leq |\lambda - a_{j_2\ldots j_2}||x_{j_2}|^m, \\ (2|\bar{a}_{j_3}| - r_{j_3}(\mathcal{A}))|x_\beta| &\leq |\lambda - a_{j_3\ldots j_3}||x_{j_3}|^m, \\ &\vdots \\ (2|\bar{a}_{j_m}| - r_{j_m}(\mathcal{A}))|x_\beta| &\leq |\lambda - a_{j_m\ldots j_m}||x_{j_m}|^m, \end{aligned} \tag{25}$$

which together with (24) gives

$$\prod_{i=1}^{m} |\lambda - a_{j_i\ldots j_i}||x_{j_i}|^m = |x_\beta|^m \prod_{i=1}^{m} |\lambda - a_{j_i\ldots j_i}| \geq |x_\beta|^m \prod_{i=1}^{m} (2|\bar{a}_{j_i}| - r_{j_i}(\mathcal{A})) \tag{26}$$

under the condition $2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0$ $(j = 1, \ldots, m)$. Then it follows that

$$\prod_{i=1}^{m} |\lambda - a_{j_i \ldots j_i}| \geq \prod_{i=1}^{m} (2|\bar{a}_{j_i}| - r_{j_i}(\mathcal{A}))$$

in terms of $|x_\beta| > 0$. This implies that

$$\lambda \notin \left\{ z \in \mathbb{C} : \prod_{i=1}^{m} |z - a_{j_i \ldots j_i}| < \prod_{i=1}^{m} (2|\bar{a}_{j_i}| - r_{j_i}(\mathcal{A})); 2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0, j = 1, \ldots, m \right\}. \tag{27}$$

By combining (22) with (27), we have $\lambda \in \Psi_2(\mathcal{A})$. This proof is completed. □

Remark 2.6. In the proof of Theorem 2.5, Inequality (26) is valid under the assumptions $2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0$ $(j = 1, \ldots, m)$. Actually, (26) also holds true in other cases. For example, existing even number of $2|\bar{a}_{j_i}| - r_{j_i}(\mathcal{A}) \leq 0$ in Inequalities (24)-(25) or satisfying other proper restrictions, which is not easy to be described, may result in (26). Hence it is convenience for us to prove our theorem under the condition $2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0$ $(j = 1, \ldots, m)$.

The following theorem illustrates that $\Psi(\mathcal{A})$ in Theorem 2.2 is sharper than $\mathcal{Z}(\mathcal{A})$ in Lemma 1.3.

Theorem 2.7. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$ and $r_i(\mathcal{A}) \neq 0$ $(i \in N)$. Then*

$$\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. Theorem 3.3 of [1] has proven $\mathcal{Z}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. Thus, we only need to prove $\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$. First, we show that

$$\mathcal{Z}(\mathcal{A}) = \mathcal{Z}_1(\mathcal{A}) := \left(\bigcup_{i \in N} \{a_{i \ldots i}\} \right) \bigcup \left(\bigcup_{\substack{a_{i_1 i_2 \ldots i_m} \neq 0, \\ \delta_{i_1 i_2 \ldots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}) \right\} \right).$$

Obviously, $\mathcal{Z}(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A})$. So it is remain to prove $\mathcal{Z}_1(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$.

Let $z \in \mathcal{Z}_1(\mathcal{A})$. For the case that $z \in \bigcup_{i \in N} \{a_{i \ldots i}\}$, then there exists $p \in N$ such that $z = a_{p \ldots p}$. Since $r_p(\mathcal{A}) > 0$, there exists $a_{p p_2 \ldots p_m} \neq 0$ and

$$|z - a_{p \ldots p}| \prod_{i=2}^{m} |z - a_{p_i \ldots p_i}| = 0 < r_p(\mathcal{A}) \prod_{i=2}^{m} r_{p_i}(\mathcal{A}),$$

which means that

$$z \in \bigcup_{\substack{a_{i_1 i_2 \ldots i_m} \neq 0, \\ \delta_{i_1 i_2 \ldots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}) \right\} = \mathcal{Z}(\mathcal{A}).$$

Moreover, if

$$z \in \bigcup_{\substack{a_{i_1 i_2 \ldots i_m} \neq 0, \\ \delta_{i_1 i_2 \ldots i_m} = 0}} \left\{ z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}) \right\},$$

then it is easy to see that $z \in \mathcal{Z}(\mathcal{A})$. Therefore, $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}_1(\mathcal{A})$.

In the sequel, we prove that

$$\Psi(\mathcal{A}) \subseteq \mathcal{Z}_1(\mathcal{A}) = \mathcal{Z}(\mathcal{A}).$$

For any $a_{i_1 i_2 \ldots i_m} \neq 0$ and $\delta_{i_1 i_2 \ldots i_m} = 0$, if $\prod_{j=1}^{m}(2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})) \leq 0$, then

$$\left\{z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| < \prod_{j=1}^{m}(2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}))\right\} = \emptyset,$$

and then

$$\left\{z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| < \prod_{j=1}^{m}(2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}))\right\} \subseteq \left\{z \in \mathbb{C} : \prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A})\right\}. \tag{28}$$

Now we consider the case that $\prod_{j=1}^{m}(2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})) > 0$. By the definition of $|\bar{a}_{i_j}|$ in Theorem 2.5, it can be seen that $0 \leq |\bar{a}_{i_j}| \leq r_{i_j}(\mathcal{A})$ and therefore

$$2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \leq 2r_{i_j}(\mathcal{A}) - r_{i_j}(\mathcal{A}) = r_{i_j}(\mathcal{A}).$$

In addition, we see that $-r_{i_j}(\mathcal{A}) \leq 2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})$ and hence

$$|2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})| \leq r_{i_j}(\mathcal{A}),$$

which leads to

$$\prod_{j=1}^{m} |z - a_{i_j \ldots i_j}| < \prod_{j=1}^{m}(2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A})) \leq \prod_{j=1}^{m} r_{i_j}(\mathcal{A}).$$

Thus (28) also holds true and $\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$. □

Example 2.8. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ with elements defined as follows:*

$$a_{111} = 2,\ a_{222} = 2,\ a_{333} = 50,\ a_{444} = 50,\ a_{122} = 30 + \mathrm{i},\ a_{133} = 3 - \mathrm{i},$$
$$a_{221} = 30,\ a_{233} = 1,\ a_{311} = 1,\ a_{344} = 20,\ a_{443} = 50$$

and other elements of $\mathcal{A}$ are zeros.

The localization sets $\Psi(\mathcal{A})$ and $\mathcal{Z}(\mathcal{A})$ are plotted in Figure 2 where all eigenvalues of $\mathcal{A}$ are indicated by the plus. It can be seen that $\Psi(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{A})$ and the new set $\Psi(\mathcal{A})$ contains all eigenvalues of the tensor $\mathcal{A}$ (see Figure 2), which confirms the correctness of Theorem 2.7 and the feasibility of the new set $\Psi(\mathcal{A})$.

In the sequel, we establish another new Brauer-type eigenvalue localization set for tensors in the following theorem, which is better than that in Lemma 1.4.

Theorem 2.9. *Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \bar{\Omega}(\mathcal{A}) = \left(\bigcup_{i \in N} \hat{\Omega}_i(\mathcal{A})\right) \bigcup \left(\bigcup_{i,j \in N, i \neq j} \left(\left(\tilde{\Omega}_{i,j}(\mathcal{A}) \backslash \bar{\Omega}_{i,j}(\mathcal{A})\right) \bigcap \Gamma_i(\mathcal{A})\right)\right), \tag{29}$$

where

$$\hat{\Omega}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \ldots i}| \leq r_i^{\Delta_i}(\mathcal{A})\},$$
$$\tilde{\Omega}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}| - r_i^{\Delta_i}(\mathcal{A}))(|z - a_{j \ldots j}| - r_j^{\overline{\Delta}_i}(\mathcal{A})) \leq r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j^{\Delta_i}(\mathcal{A})\},$$
$$\bar{\Omega}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \ldots i}| + r_i^{j}(\mathcal{A}))(|z - a_{j \ldots j}| + r_j^{\overline{\Delta}_i}(\mathcal{A})) < |a_{ij \ldots j}|(2|a_{ji \ldots i}| - r_j^{\Delta_i}(\mathcal{A}))\}.$$

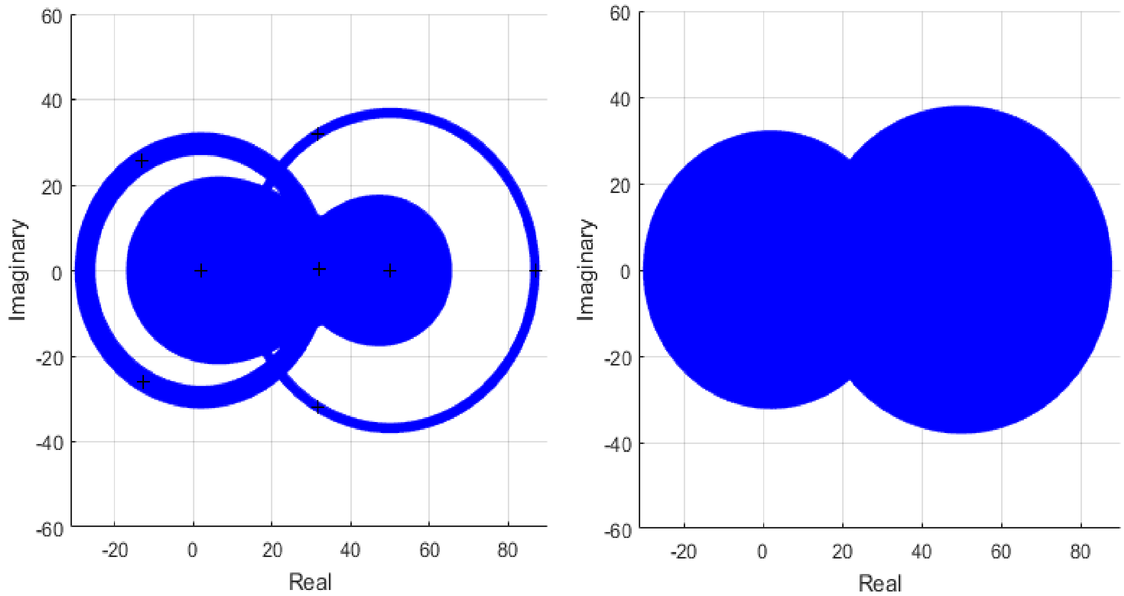

Figure 2: Eigenvalue localization sets $\Psi(\mathcal{A})$ (left) and $\mathcal{Z}(\mathcal{A})$ (right).

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \backslash \{0\}$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{30}$$

Let $|x_p| \geq |x_q| \geq \max\{|x_i| : i \in N, i \neq p, i \neq q\}$. Then $|x_p| > 0$. According to the proof of Theorem 2.1 in [17], pth and qth equations of (30) give

$$(|\lambda - a_{p\dots p}| - r_p^{\Delta_p}(\mathcal{A}))|x_p|^{m-1} \leq r_p^{\overline{\Delta}_p}(\mathcal{A})|x_q|^{m-1} \tag{31}$$

and

$$(|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta}_p}(\mathcal{A}))|x_q|^{m-1} \leq r_q^{\Delta_p}(\mathcal{A})|x_p|^{m-1}. \tag{32}$$

If $|x_q| = 0$, then Equation (31) is equivalent to $|\lambda - a_{p\dots p}| \leq r_p^{\Delta_p}(\mathcal{A})$ and hence $\lambda \in \hat{\Omega}_p(\mathcal{A}) \subseteq \bigcup_{i \in N} \hat{\Omega}_i(\mathcal{A}) \subseteq \bar{\Omega}(\mathcal{A})$. If $|x_q| > 0$, then it follows from (31) and (32) that

$$(|\lambda - a_{p\dots p}| - r_p^{\Delta_p}(\mathcal{A}))(|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta}_p}(\mathcal{A})) \leq r_p^{\overline{\Delta}_p}(\mathcal{A}) r_q^{\Delta_p}(\mathcal{A}) \tag{33}$$

in view of $|x_p| \geq |x_q| > 0$, which means that $\lambda \in \tilde{\Omega}_{p,q}(\mathcal{A})$. Moreover, it follows from (31) that $|\lambda - a_{p\dots p}| \leq r_p(\mathcal{A})$, which together with $\lambda \in \tilde{\Omega}_{p,q}(\mathcal{A})$ results in $\lambda \in (\tilde{\Omega}_{p,q}(\mathcal{A}) \bigcap \Gamma_p(\mathcal{A}))$.

It follows from (30) that

$$a_{pq\dots q} x_q^{m-1} = (\lambda - a_{p\dots p}) x_p^{m-1} - \sum_{\substack{(i_2,\dots,i_m) \in \Delta_p, \\ \delta_{pi_2\dots i_m} = 0}} a_{pi_2\dots i_m} x_{i_2} \cdots x_{i_m} - \sum_{\substack{(i_2,\dots,i_m) \in \overline{\Delta}_p, \\ \delta_{qi_2\dots i_m} = 0}} a_{pi_2\dots i_m} x_{i_2} \cdots x_{i_m}. \tag{34}$$

By taking modulus in both sides of (34) and utilizing the triangle inequality, it has

$$\begin{aligned}|a_{pq\dots q}||x_q|^{m-1} &\le |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta_p,\\ \delta_{pi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\le |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{\delta_{pi_2\dots i_m}=0,\\ \delta_{qi_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_p|^{m-1} = (|\lambda - a_{p\dots p}| + r_p^q(\mathcal{A}))|x_p|^{m-1}. \end{aligned} \tag{35}$$

Furthermore, we consider the qth equation of (30) which can be written as

$$a_{qp\dots p}x_p^{m-1} = (\lambda - a_{q\dots q})x_q^{m-1} - \sum_{\substack{(i_2,\dots,i_m)\in\Delta_p,\\ \delta_{pi_2\dots i_m}=0}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\dots i_m}=0}} a_{qi_2\dots i_m}x_{i_2}\cdots x_{i_m}. \tag{36}$$

Applying the same operations used in (35) to (36) results in

$$\begin{aligned}|a_{qp\dots p}||x_p|^{m-1} &\le |\lambda - a_{q\dots q}||x_q|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta_p,\\ \delta_{pi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_{i_2}|\cdots|x_{i_m}| \\ &\le |\lambda - a_{q\dots q}||x_q|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\Delta_p,\\ \delta_{pi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\dots i_m}=0}} |a_{qi_2\dots i_m}||x_q|^{m-1} \\ &= |\lambda - a_{q\dots q}||x_q|^{m-1} + (r_q^{\Delta_p}(\mathcal{A}) - |a_{qp\dots p}|)|x_p|^{m-1} + r_q^{\overline{\Delta}_p}(\mathcal{A})|x_q|^{m-1}, \end{aligned}$$

which yields that

$$(2|a_{qp\dots p}| - r_q^{\Delta_p}(\mathcal{A}))|x_p|^{m-1} \le (|\lambda - a_{q\dots q}| + r_q^{\overline{\Delta}_p}(\mathcal{A}))|x_q|^{m-1}. \tag{37}$$

If $|x_q| > 0$, then multiplying (35) with (37) leads to

$$|a_{pq\dots q}|(2|a_{qp\dots p}| - r_q^{\Delta_p}(\mathcal{A}))|x_p|^{m-1}|x_q|^{m-1} \le (|\lambda - a_{p\dots p}| + r_p^q(\mathcal{A}))(|\lambda - a_{q\dots q}| + r_q^{\overline{\Delta}_p}(\mathcal{A}))|x_p|^{m-1}|x_q|^{m-1},$$

and therefore

$$|a_{pq\dots q}|(2|a_{qp\dots p}| - r_q^{\Delta_p}(\mathcal{A})) \le (|\lambda - a_{p\dots p}| + r_p^q(\mathcal{A}))(|\lambda - a_{q\dots q}| + r_q^{\overline{\Delta}_p}(\mathcal{A})) \tag{38}$$

as $|x_p| \ge |x_q| > 0$. If $|x_q| = 0$, then (37) implies that $2|a_{qp\dots p}| - r_q^{\Delta_p}(\mathcal{A}) \le 0$, and (38) is also valid. (38) means that $\lambda \notin \bar{\Omega}_{p,q}(\mathcal{A})$. Therefore, $\lambda \in \left(\left(\tilde{\Omega}_{p,q}(\mathcal{A})\backslash\bar{\Omega}_{p,q}(\mathcal{A})\right)\bigcap \Gamma_p(\mathcal{A})\right) \subseteq \bar{\Omega}(\mathcal{A})$. □

Next theorem shows that $\bar{\Omega}(\mathcal{A})$ is sharper than $\Omega(\mathcal{A})$ in Lemma 1.4.

Theorem 2.10. *Let* $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{C}^{[m,n]}$, *then*

$$\bar{\Omega}(\mathcal{A}) \subseteq \Omega(\mathcal{A}).$$

Proof. For any $i, j \in N$ and $j \ne i$, if $|a_{ij\dots j}|(2|a_{ji\dots i}| - r_j^{\Delta_i}(\mathcal{A})) \le 0$, then $\bar{\Omega}_{i,j}(\mathcal{A}) = \emptyset$, and therefore $\bar{\Omega}_{i,j}(\mathcal{A}) \subseteq \tilde{\Omega}_{i,j}(\mathcal{A})$. Now we consider the case that $|a_{ij\dots j}|(2|a_{ji\dots i}| - r_j^{\Delta_i}(\mathcal{A})) > 0$. Since $r_i^j(\mathcal{A}) \ge r_i^{\Delta_i}(\mathcal{A})$, it has

$$\begin{aligned}&(|z - a_{i\dots i}| + r_i^j(\mathcal{A}))(|z - a_{j\dots j}| + r_j^{\overline{\Delta}_i}(\mathcal{A})) - (|z - a_{i\dots i}| - r_i^{\Delta_i}(\mathcal{A}))(|z - a_{j\dots j}| - r_j^{\overline{\Delta}_i}(\mathcal{A})) \\ =\ &2|z - a_{i\dots i}|r_j^{\overline{\Delta}_i}(\mathcal{A}) + |z - a_{j\dots j}|(r_i^j(\mathcal{A}) + r_i^{\Delta_i}(\mathcal{A})) + r_j^{\overline{\Delta}_i}(\mathcal{A})(r_i^j(\mathcal{A}) - r_i^{\Delta_i}(\mathcal{A})) \ge 0. \end{aligned} \tag{39}$$

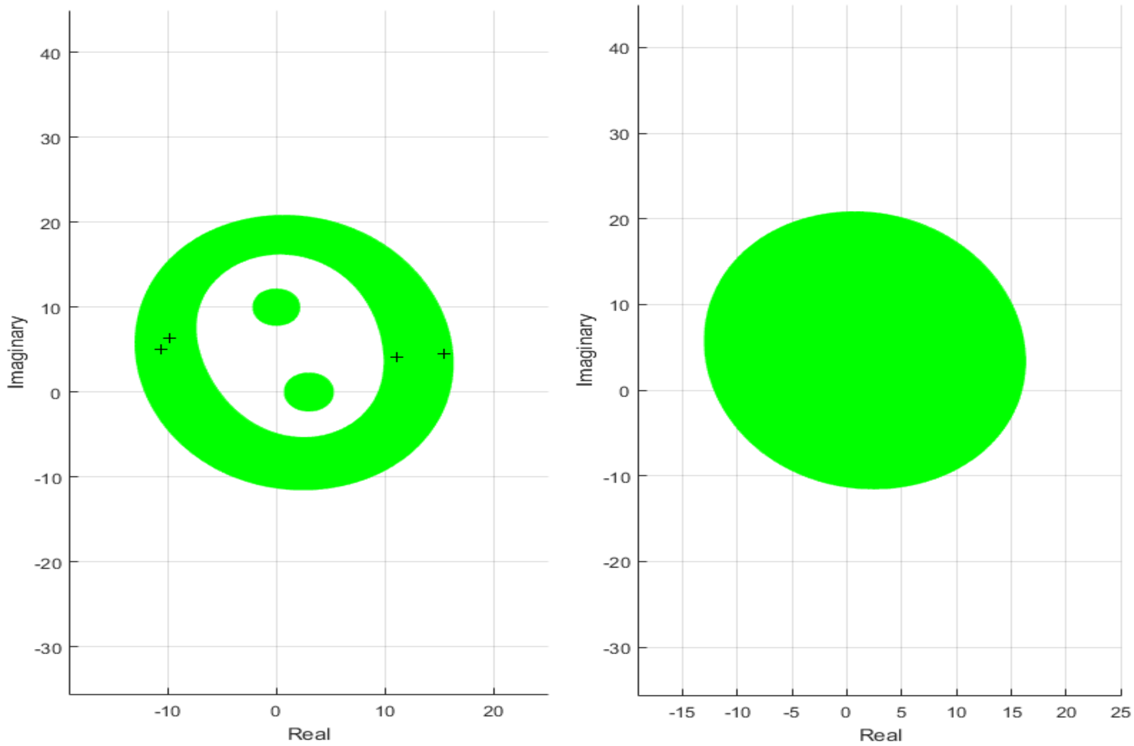

Figure 3: Eigenvalue localization sets $\bar{\Omega}(\mathcal{A})$ (left) and $\Omega(\mathcal{A})$ (right).

Moreover, it is not difficult to verify that $|a_{ij\ldots j}| \leq r_i^{\bar{\Delta}_i}(\mathcal{A})$ and $2|a_{ji\ldots i}| \leq 2r_j^{\Delta_i}(\mathcal{A})$, that is $0 < 2|a_{ji\ldots i}| - r_j^{\Delta_i}(\mathcal{A}) \leq r_j^{\Delta_i}(\mathcal{A})$, which implies that

$$|a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\Delta_i}(\mathcal{A})) \leq r_i^{\bar{\Delta}_i}(\mathcal{A}) r_j^{\Delta_i}(\mathcal{A}),$$

which together with (39) shows that $\bar{\Omega}_{i,j}(\mathcal{A}) \subseteq \tilde{\Omega}_{i,j}(\mathcal{A})$ and $\left(\tilde{\Omega}_{i,j}(\mathcal{A}) \backslash \bar{\Omega}_{i,j}(\mathcal{A})\right) \subseteq \tilde{\Omega}_{i,j}(\mathcal{A})$. Thus

$$\left(\left(\tilde{\Omega}_{i,j}(\mathcal{A}) \backslash \bar{\Omega}_{i,j}(\mathcal{A})\right) \bigcap \Gamma_i(\mathcal{A})\right) \subseteq \left(\tilde{\Omega}_{i,j}(\mathcal{A}) \bigcap \Gamma_i(\mathcal{A})\right),$$

which leads to $\bar{\Omega}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$. This proof is completed. □

Remark 2.11. For a tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, the set $\bar{\Omega}(\mathcal{A})$ in Theorem 2.9 contains n sets $\hat{\Omega}_i(\mathcal{A})$, $\frac{n(n+1)}{2}$ sets $\tilde{\Omega}_{i,j}(\mathcal{A})$, $\frac{n(n+1)}{2}$ sets $\bar{\Omega}_{i,j}(\mathcal{A})$ and n sets $\Gamma_i(\mathcal{A})$. Hence there are $n^2 + 3n$ sets in $\bar{\Omega}(\mathcal{A})$. By Lemma 1.4, it can be seen that $\Omega(\mathcal{A})$ contains $\frac{n(n+1)}{2} + 2n$ sets. Thus computing $\bar{\Omega}(\mathcal{A})$ requires more computations than $\Omega(\mathcal{A})$. However, Theorem 2.10 reveals $\bar{\Omega}(\mathcal{A})$ is sharper than $\Omega(\mathcal{A})$.

Example 2.12. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,2]}$ with elements defined as follows:*

$$a_{111} = 10\mathrm{i},\ a_{222} = 3,\ a_{112} = 1,\ a_{121} = 1,$$
$$a_{122} = 8,\ a_{211} = 20,\ a_{212} = 2,\ a_{221} = 0.1$$

and other elements of $\mathcal{A}$ are zeros.

The localization sets $\bar{\Omega}(\mathcal{A})$, $\Omega(\mathcal{A})$ and the exact eigenvalues of the tensor $\mathcal{A}$ are plotted in Figure 3. Here, the exact eigenvalues of the tensor $\mathcal{A}$ are denoted by the black plus. It can be seen that $\bar{\Omega}(\mathcal{A})$ can capture all eigenvalues of $\mathcal{A}$ and $\bar{\Omega}(\mathcal{A}) \subseteq \Omega(\mathcal{A})$ (see Figure 3), which shows that the results of Theorem 2.9 and Theorem 2.10 are valid.

3. Some new sufficient criterias for nonsingularity of tensors

As applications of the sets proposed in Section 2, we develop new sufficient criterias for the nonsingularity of tensors in this section. Additionally, we use several examples to show the advantages of the proposed criterias over the existing ones in [1, 13, 18, 23].

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$. If for all $i, j \in N$ and $i \neq j$, one of the following two conditions holds:*
(i) $|a_{i\dots i}|^{m-1}|a_{j\dots j}| > (r_i(\mathcal{A}))^{m-1} r_j(\mathcal{A})$;
(ii) $(|a_{i\dots i}| + \bar{r}_i^j(\mathcal{A}))^{m-1}|a_{j\dots j}| < |a_{i\dots ij}|^{m-1}(2|a_{ji\dots i}| - r_j(\mathcal{A}))$,
then $\mathcal{A}$ is nonsingular.

Proof. Assume that λ is the eigenvalue of $\mathcal{A}$. From Theorem 2.1, it has $\lambda \in \Theta(\mathcal{A})$, which implies that there are $k, h \in N$ such that

$$|\lambda - a_{k\dots k}|^{m-1}|\lambda - a_{h\dots h}| \leq (r_k(\mathcal{A}))^{m-1} r_h(\mathcal{A}),$$
$$(|\lambda - a_{k\dots k}| + \bar{r}_k^h(\mathcal{A}))^{m-1}|\lambda - a_{h\dots h}| \geq |a_{k\dots kh}|^{m-1}(2|a_{hk\dots k}| - r_h(\mathcal{A})).$$

If $\lambda = 0$, then it follows that

$$|\lambda - a_{k\dots k}|^{m-1}|\lambda - a_{h\dots h}| = |a_{k\dots k}|^{m-1}|a_{h\dots h}| \leq (r_k(\mathcal{A}))^{m-1} r_h(\mathcal{A})$$

and

$$(|\lambda - a_{k\dots k}| + \bar{r}_k^h(\mathcal{A}))^{m-1}|\lambda - a_{h\dots h}| = (|a_{k\dots k}| + \bar{r}_k^h(\mathcal{A}))^{m-1}|a_{h\dots h}| \geq |a_{k\dots kh}|^{m-1}(2|a_{hk\dots k}| - r_h(\mathcal{A})),$$

which contradict with the conditions of this theorem. Hence, $\lambda \neq 0$ and $\mathcal{A}$ is nonsingular. □

Example 3.2. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,3]}$ with elements defined as follows:*

$$a_{111} = 80,\ a_{222} = 30,\ a_{333} = 90,\ a_{122} = 30,\ a_{133} = 1,$$
$$a_{221} = 120,\ a_{211} = 2,\ a_{223} = 1,\ a_{233} = 1,\ a_{332} = 1,\ a_{311} = 1,\ a_{322} = 1$$

and other elements of $\mathcal{A}$ are zeros.

By some calculations, we have

$$|a_{222}|^2|a_{111}| = 72000 < 476656 = (r_2(\mathcal{A}))^2 r_1(\mathcal{A}),$$

which implies that Corollary 3.2 in [1] can not be applied to identify the nonsingularity of $\mathcal{A}$ in this example. And we can obtain

$$(|a_{111}| - r_1^{\Delta_1}(\mathcal{A}))|a_{222}| = 2400 < 3844 = (r_1^{\bar{\Delta}_1}(\mathcal{A}))^2 r_2(\mathcal{A}),$$
$$(|a_{111}| + r_1^2(\mathcal{A}))(|a_{222}| + r_2^{\bar{\Delta}_1}(\mathcal{A})) = 2592 > -3540 = |a_{122}|(2|a_{211}| - r_2^{\Delta_1}(\mathcal{A})).$$

Thus Corollary 1 of [18] can not be used to determine the nonsingularity of $\mathcal{A}$. Besides,

$$|a_{222}| = 30 < 124 = r_2(\mathcal{A}),\ |a_{222}| = 30 > -120 = 2|a_{211}| - r_2(\mathcal{A}),\ |a_{222}| = 30 > -122 = 2|a_{233}| - r_2(\mathcal{A}).$$

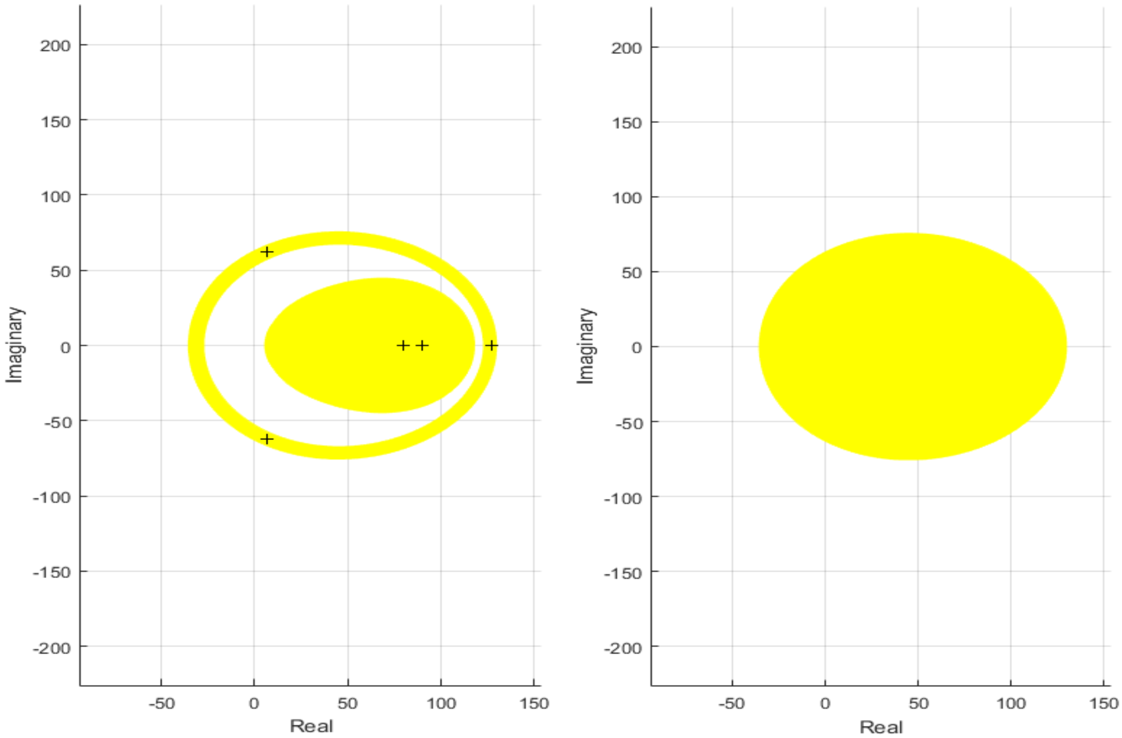

Figure 4: Eigenvalue localization sets $\Theta(\mathcal{A})$ (left) and $\mathcal{B}(\mathcal{A})$ (right).

Then we also can not use Corollary 1 of [13] to determine the nonsingularity of $\mathcal{A}$. However, we can derive the following results by Theorem 3.1.

$$
\begin{aligned}
&|a_{111}|^2|a_{222}| = 192000 > 119164 = (r_1(\mathcal{A}))^2 r_2(\mathcal{A}),\\
&|a_{111}|^2|a_{333}| = 576000 > 2883 = (r_1(\mathcal{A}))^2 r_3(\mathcal{A}),\\
&(|a_{222}| + \bar{r}_2^1(\mathcal{A}))^2|a_{111}| = 92480 < 417600 = |a_{221}|^2(2|a_{122}| - r_1(\mathcal{A})),\\
&|a_{222}|^2|a_{333}| = 81000 > 46128 = (r_2(\mathcal{A}))^2 r_3(\mathcal{A}),\\
&|a_{333}|^2|a_{111}| = 648000 > 279 = (r_3(\mathcal{A}))^2 r_1(\mathcal{A}),\\
&|a_{333}|^2|a_{222}| = 243000 > 1116 = (r_3(\mathcal{A}))^2 r_2(\mathcal{A}),
\end{aligned}
$$

which means that the tensor $\mathcal{A}$ satisfies the conditions of Theorem 3.1, and hence $\mathcal{A}$ is nonsingular. We depict the eigenvalue localization sets $\Theta(\mathcal{A})$ in Theorem 2.1, $\mathcal{B}(\mathcal{A})$ in Lemma 1.2 and the eigenvalues of $\mathcal{A}$ in Figure 4, where the eigenvalues of $\mathcal{A}$ are represented by the black plus. It can be observed that the new set $\Theta(\mathcal{A})$ can work, and $(0,0) \notin \Theta(\mathcal{A})$ while $(0,0) \in \mathcal{B}(\mathcal{A})$, which is in accordance with the results in Theorems 2.1, 2.2 and 3.1.

Theorem 3.3. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$. If the following conditions hold:*
(i) $a_{i \dots i} \neq 0$ for any $i \in N$;
(ii) for all $a_{i_1 i_2 \dots i_m} \neq 0$ and $\delta_{i_1 i_2 \dots i_m} = 0$, $\prod_{j=1}^{m} |a_{i_j \dots i_j}| > \prod_{j=1}^{m} r_{i_j}(\mathcal{A})$ or $\prod_{j=1}^{m} |a_{i_j \dots i_j}| < \prod_{j=1}^{m} (2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}))$ and $2|\bar{a}_{i_j}| - r_{i_j}(\mathcal{A}) \geq 0$,
where $|\bar{a}_{i_j}| = \max_{\pi \in \Pi_{m-1}} \{|a_{i_j \pi(i_1 \dots i_{j-1} i_{j+1} \dots i_m)}|\}$ with Π_{m-1} being the permutation group of $m-1$ indices,
then $\mathcal{A}$ is nonsingular.

Proof. Assume that λ is the eigenvalue of $\mathcal{A}$. From Theorem 2.5, it has $\lambda \in \Psi(\mathcal{A}) = (\Psi_1(\mathcal{A})) \bigcup (\Psi_2(\mathcal{A}))$, which leads to $\lambda = a_{p \dots p}$ for some $p \in N$, or there exist $k_1, k_2, \dots, k_n$ satisfying $a_{k_1 k_2 \dots k_m} \neq 0$ and $\delta_{k_1 k_2 \dots k_m} = 0$ such that

$$\prod_{j=1}^{m}|\lambda - a_{k_j\ldots k_j}| \le \prod_{j=1}^{m} r_{k_j}(\mathcal{A}), \text{ and } \prod_{j=1}^{m}|\lambda - a_{k_j\ldots k_j}| \ge \prod_{j=1}^{m}(2|\bar{a}_{k_j}| - r_{k_j}(\mathcal{A})),\ 2|\bar{a}_{k_j}| - r_{k_j}(\mathcal{A}) \ge 0.$$

If $\lambda = 0$, then we deduce that $a_{p\ldots p} = 0$ for some $p \in N$ or

$$\prod_{j=1}^{m}|a_{k_j\ldots k_j}| \le \prod_{j=1}^{m} r_{k_j}(\mathcal{A}) \text{ and } \prod_{j=1}^{m}|a_{k_j\ldots k_j}| \ge \prod_{j=1}^{m}(2|\bar{a}_{k_j}| - r_{k_j}(\mathcal{A})),\ 2|\bar{a}_{k_j}| - r_{k_j}(\mathcal{A}) \ge 0,$$

which contradicts with the conditions of this theorem. Hence, $\lambda \neq 0$ and $\mathcal{A}$ is nonsingular. □

Example 3.4. *Consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,4]}$ with elements defined as follows:*

$$a_{111} = 6,\ a_{222} = 2,\ a_{333} = 33,\ a_{444} = 20,\ a_{122} = 10,\ a_{133} = 3,$$
$$a_{221} = 30,\ a_{233} = 1,\ a_{311} = 1,\ a_{344} = 5,\ a_{443} = 40$$

and other elements of $\mathcal{A}$ are zeros.

Note that $a_{122} = 10 \neq 0$ and

$$|a_{111}||a_{222}|^2 = 24 < 12493 = r_1(\mathcal{A})(r_2(\mathcal{A}))^2.$$

It follows from Corollaries 3.2 and 3.4 in [1] that they can not be applied to identify the nonsingularity of the tensor $\mathcal{A}$. Besides, it can be seen that

$$(|a_{111}| - r_1^2(\mathcal{A}))|a_{222}| = 6 < 310 = |a_{122}|r_2(\mathcal{A}),$$
$$(|a_{111}| - r_1^3(\mathcal{A}))|a_{333}| = -132 < 18 = |a_{133}|r_3(\mathcal{A}),$$
$$(|a_{111}| - r_1^4(\mathcal{A}))|a_{444}| = -140 < 0 = |a_{144}|r_4(\mathcal{A}),$$
$$(|a_{111}| + r_1^2(\mathcal{A}))|a_{222}| = 18 > -310 = |a_{122}|(2|a_{211}| - r_2(\mathcal{A})),$$
$$(|a_{111}| + r_1^3(\mathcal{A}))|a_{333}| = 528 > -12 = |a_{133}|(2|a_{311}| - r_3(\mathcal{A})),$$
$$(|a_{111}| + r_1^4(\mathcal{A}))|a_{444}| = 380 > 0 = |a_{144}|(2|a_{411}| - r_4(\mathcal{A})),$$

hence Corollary 3 of [13] and Corollary 2.4 of [23] are invalid. While by Theorem 2.9, we obtain

$$a_{111} = 6 \neq 0,\ a_{222} = 2 \neq 0,\ a_{333} = 33 \neq 0,\ a_{444} = 20 \neq 0,$$
$$|a_{111}||a_{222}|^2 = 24 < 5887 = (2|\bar{a}_1| - r_1(\mathcal{A}))(2|\bar{a}_2| - r_2(\mathcal{A}))^2,$$
$$|a_{111}||a_{333}|^2 = 6534 > 468 = r_1(\mathcal{A})(r_3(\mathcal{A}))^2,$$
$$|a_{222}|^2|a_{111}| = 24 < 5887 = (2|\bar{a}_2| - r_2(\mathcal{A}))^2(2|\bar{a}_1| - r_1(\mathcal{A})),$$
$$|a_{222}||a_{333}|^2 = 2178 > 1116 = r_2(\mathcal{A})(r_3(\mathcal{A}))^2,$$
$$|a_{333}||a_{111}|^2 = 1188 > 1014 = r_3(\mathcal{A})(r_1(\mathcal{A}))^2,$$
$$|a_{333}||a_{444}|^2 = 13200 > 9600 = r_3(\mathcal{A})(r_4(\mathcal{A}))^2,$$
$$|a_{444}||a_{333}|^2 = 21780 > 1440 = r_4(\mathcal{A})(r_3(\mathcal{A}))^2.$$

Therefore we conclude that the tensor $\mathcal{A}$ is nonsingular.

To illustrate the correctness of Theorem 3.3, the eigenvalue localization sets $\Psi(\mathcal{A})$ are drawn in Figure 5, where $\Psi(\mathcal{A})$, the exact eigenvalues of $\mathcal{A}$ and the point $(0,0)$ are represented by the blue zones, the black plus and the red asterisk, respectively. From Figure 5, it is easy to see that $(0,0) \notin \Psi(\mathcal{A})$.

Theorem 3.5. *Let $\mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{[m,n]}$. If the following conditions hold:*
(i) $|a_{i\ldots i}| > r_i^{\Delta_i}(\mathcal{A})$ for any $i \in N$;
(ii) $(|a_{i\ldots i}| - r_i^{\Delta_i}(\mathcal{A}))(|a_{j\ldots j}| - r_j^{\overline{\Delta}_i}(\mathcal{A})) > r_i^{\overline{\Delta}_i}(\mathcal{A}) r_j^{\Delta_i}(\mathcal{A})$ or $(|a_{i\ldots i}| + r_i^{j}(\mathcal{A}))(|a_{j\ldots j}| + r_j^{\overline{\Delta}_i}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\Delta_i}(\mathcal{A}))$ for all $i, j \in N$ and $i \neq j$,
then $\mathcal{A}$ is nonsingular.

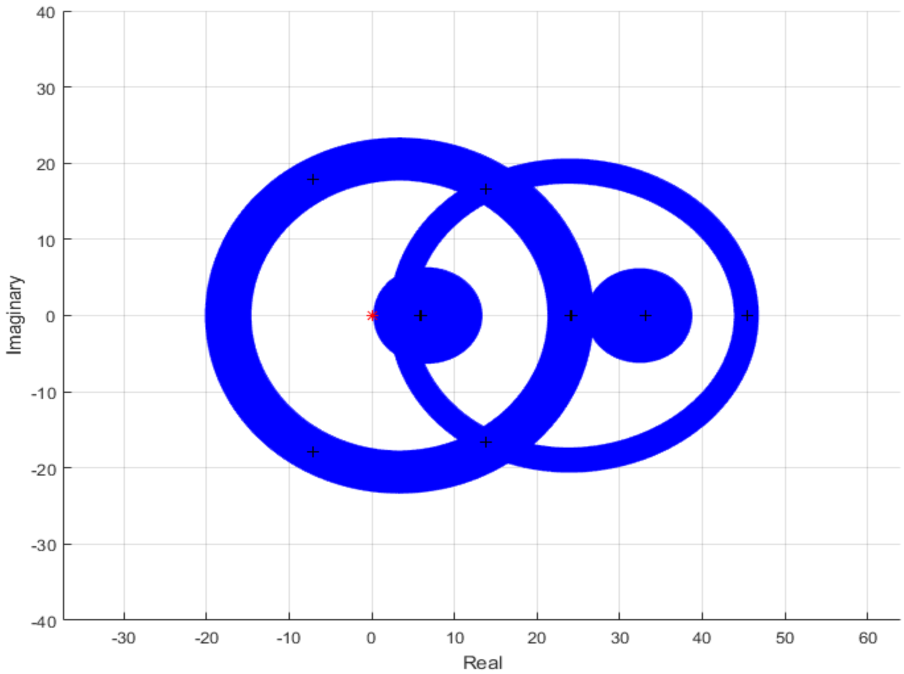

Figure 5: Eigenvalue localization set $\Psi(\mathcal{A})$.

Proof. By Theorem 2.9 and using the method applied in Theorems 3.1–3.3, we can prove the conclusion of this theorem. □

We will verify the advantages of Theorem 3.5 by Example 3.6.

Example 3.6. *Consider the tensor* $\mathcal{A} = (a_{ijk}) \in \mathbb{C}^{[3,2]}$ *with elements defined as follows:*

$$a_{111} = 12,\ a_{222} = 3.5,\ a_{112} = 1,\ a_{121} = 1,$$
$$a_{122} = 9,\ a_{211} = 10,\ a_{212} = 2,\ a_{221} = 0.1$$

and other elements of $\mathcal{A}$ *are zeros.*

Since $a_{221} = 0.1 \neq 0$, by direct computations, it follows that

$$|a_{222}|^2|a_{111}| = 147 < 1610.5 = (r_2(\mathcal{A}))^2 r_1(\mathcal{A}),$$
$$(|a_{222}| + \bar{r}_2^1(\mathcal{A}))^2|a_{111}| = 2883 > 0.7 = |a_{221}|^2(2|a_{122}| - r_1(\mathcal{A})),$$

which illustrates that the conditions of Corollaries 3.2 and 3.4 in [1] and Theorem 3.1 of this paper are not satisfied. According to Theorem 3.5 in this paper, we get

$$|a_{111}| = 12 > 11 = r_1^{\Delta_1}(\mathcal{A}),\ |a_{222}| = 3.5 > 2.1 = r_2^{\Delta_2}(\mathcal{A}),$$
$$(|a_{111}| + r_1^2(\mathcal{A}))(|a_{222}| + r_2^{\overline{\Delta}_1}(\mathcal{A})) = 49 < 71.1 = |a_{122}|(2|a_{211}| - r_2^{\Delta_1}(\mathcal{A}))),$$
$$(|a_{222}| + r_2^1(\mathcal{A}))(|a_{111}| + r_1^{\overline{\Delta}_2}(\mathcal{A})) = 67.2 < 70 = |a_{211}|(2|a_{122}| - r_1^{\Delta_2}(\mathcal{A}))),$$

which confirms that the tensor $\mathcal{A}$ is nonsingular. To further verify this fact, the eigenvalue localization set $\bar{\Omega}(\mathcal{A})$, the exact eigenvalues of $\mathcal{A}$ and the point $(0, 0)$ are drawn in Figure 6, where $\bar{\Omega}(\mathcal{A})$, the exact eigenvalues of $\mathcal{A}$ and the point $(0, 0)$ are represented by the green zones, the black plus and the red asterisk, respectively. As observed in Figure 6, $(0, 0) \notin \bar{\Omega}(\mathcal{A})$ and the tensor $\mathcal{A}$ is nonsingular.

Figure 6: Eigenvalue localization set $\bar{\Omega}(\mathcal{A})$.

4. Conclusions

In this paper, some improved Brauer-type eigenvalue localization sets for tensors are established, which are sharper than those in [1, 17]. Based on these sets, some new sufficient criterias are given, which have wider scope of applications compared with those of [1, 13, 18, 23] for the nonsingularity of tensors. In addition, we should investigate more tighter eigenvalue localization sets for tensors. Finally, based on the exclusion set for the pseudospectrum of tensors put forward recently [5], we should try to extend the proposed exclusion sets in this paper for the pseudospectrum of tensors, and investigate the more accurate exclusion sets for the pseudospectrum of tensors in our future work.

Competing interests

The authors declare that they have no competing interests.

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References

[1] C. J. Bu, X. Q. Jin, H. F. Li and C. L. Deng, Brauer-type eigenvalue inclusion sets and the spectral radius of tensors, Linear Algebra Appl. 512 (2017) 234–248.
[2] C. J. Bu, Y. P. Wei, L. Z. Sun and J. Zhou, Brualdi-type eigenvalue inclusion sets of tensors, Linear Algebra Appl. 480 (2015) 168–175.
[3] M. L. Che, G. Y. Li, L. Q. Qi and Y. M. Wei, Pseudo-spectra theory of tensors and tensor polynomial eigenvalue problems, Linear Algebra Appl. 533 (2017) 536–572.
[4] J. Cooper and A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268–3292.

[5] J. He, C. Q. Li and Y. M. Wei, Pseudospectra localization sets of tensors with applications, J. Comput. Appl. Math. 369 (2020) 112580.

[6] Z.-G. Huang, L.-G. Wang, Z. Xu and J.-J. Cui, A new *S*-type eigenvalue inclusion set for tensors and its applications, J. Inequal. Appl. 2016 (2016) 254.

[7] Z. G. Huang, L. G. Wang, Z. Xu and J. J. Cui, A new *S*-type eigenvalue localization set for tensors and its applications, preprint, arXiv:1602.07568v1, 2016.

[8] Z. G. Huang, L. G. Wang, Z. Xu and J. J. Cui, A modified *S*-type eigenvalue localization set of tensors with its applications, Filomat 32 (2018) 6395–6416.

[9] Z. G. Huang, L. G. Wang, Z. Xu and J. J. Cui, Some new inclusion sets for eigenvalues of tensors with application, Filomat 32 (2018) 3899–3916.

[10] Z. G. Huang, L. G. Wang, Z. Xu and J. J. Cui, Some new Z-eigenvalue localization sets for tensors and their applications, Revista de la Union Matematica Argentina 60 (2019) 99–119.

[11] C. Q. Li, Z. Chen and Y. T. Li, A new eigenvalue inclusion set for tensors and its applications, Linear Algebra Appl. 481 (2015) 36–53.

[12] C. Q. Li, A. Q. Jiao and Y. T. Li, An *S*-type eigenvalue localization set for tensors, Linear Algebra Appl. 493 (2016) 469–483.

[13] C. Q. Li, S. H. Li, Q. B. Liu and Y. T. Li, Exclusion sets in eigenvalue inclusion sets for tensors, preprint, arXiv:1706.00944v1, 2017.

[14] C. Q. Li and Y. T. Li, An eigenvalue localization set for tensors with applications to determine the positive (semi-)definiteness of tensors, Linear Multilin. Algebra 64 (2016) 587–601.

[15] C. Q. Li, Y. T. Li and X. Kong, New eigenvalue inclusion sets for tensors, Numer. Linear Algebra Appl. 21 (2014) 39–50.

[16] C. Q. Li, F. Wang, J. X. Zhao, Y. Zhu and Y. T. Li, Criterions for the positive definiteness of real supersymmetric tensors, J. Comput. Appl. Math. 255 (2014) 1–14.

[17] C. Q. Li, J. J Zhou and Y. T. Li, A new Brauer-type eigenvalue localization set for tensors, Linear Multilin. Algebra 64 (2016) 727–736.

[18] Y. T. Li and S. H. Li, Exclusion sets in the Δ-type eigenvalue inclusion set for tensors, Journal of Industrial and Management Optimization 15 (2019) 507–516.

[19] L. H. Lim, Singular values and eigenvalues of tensors: a variational approach, in: Proc. IEEE Int. Workshop on Comput. Advances of Multitensor Adaptive Processing. pages 129–132, 2005.

[20] M. Ng, L. Q. Qi and G. L. Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM J. Matrix Anal. Appl. 31 (2009) 1090–1099.

[21] L. Q. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.

[22] L. Q. Qi, G. H. Yu and E. X. Wu, Higher order positive semidefinite diffusion tensor imaging, SIAM J. Imaging Sci. 3 (2010) 416–433.

[23] C. L. Sang and C. Q. Li, Exclusion sets in eigenvalue localization sets for tensors, Linear and Multilinear Algebra 67 (2019) 2399–2409.

[24] Y. N. Yang and Q. Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31 (2010) 2517–2530.