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New Fixed Point Results on α_L^{ψ}-Rational Contraction Mappings in Metric-Like Spaces

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Abstract. We present some new results for $\alpha_L^{\psi}-$rational contractive and cyclic $\alpha_L^{\psi}-$rational contractive mappings defined in d_l-complete metric-like spaces (also known as dislocated metric spaces). We have showed that established results for both types of contractive mappings are in the fact equivalent. By using this result obtained so far we discuss some examples at the end of this paper. All these examples show the advantage of our results.

1. Introduction and preliminaries

Let X be a nonempty set and $f : X \to X$ a self-mapping of it. A solution of an equation $fx = x$ is called a fixed point of f. Results dealing with the existence and construction of a solution to an operator equation $fx = x$ form the part of so-called Fixed Point Theory. It is well known that the Banach contraction principle [8] is one of the most important and attractive results in nonlinear analysis and in mathematical analysis in general. Also, whole fixed point theory is a significant subject in different fields like geometry, differential equations, informatics, physics, economics, engineering, etc. After the existence of the solutions is guaranteed the numerical methodology will be established in order to obtain an approximated solution to the fixed point problem.

Fixed point of functions depend heavily on the considered spaces that are defined using intuitive axioms. These are mostly metric spaces introduced in 1906 by the French mathematician Maurice René Fréchet [12]. In this paper we will consider some recent results from the context of so-called metric-like spaces (or dislocated metric spaces) which represent one generalization of standard metric spaces.

Now, we recall some basic concepts, notations and known results from this concept, that is, we give the definitions of partial metric and metric-like spaces.

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Definition 1.1. *[30] Let X be a nonempty set. A mapping $p : X \times X \to [0, +\infty)$ is said to be a partial metric on X if for all $u, v, w \in X$ the following four conditions hold:*

($p1$) $u = v$ *if and only if* $p(u, u) = p(u, v) = p(v, v)$;

($p2$) $p(u, u) \leq p(u, v)$;

($p3$) $p(u, v) = p(v, u)$;

($p4$) $p(u, w) \leq p(u, v) + p(v, w) - p(v, v)$.

In this case, the pair (X, p) is called a partial metric space. Obviously, each metric space is a partial metric space. The inverse is not true. Indeed, let $X = [0, +\infty)$ and $p(u, v) = \max\{u, v\}$. Then (X, p) is a partial metric space but it is not a metric space because $p(1, 1) = 1 > 0$.

Definition 1.2. *[17] Let X be a nonempty set. Then a mapping $d_l : X \times X \to [0, +\infty)$ is said to be a metric-like mapping on X if for all $u, v, w \in X$ the following three conditions hold:*

(d_l1) $d_l(u, v) = 0$ *implies* $u = v$;

(d_l2) $d_l(u, v) = d_l(v, u)$;

(d_l3) $d_l(u, w) \leq d_l(u, v) + d_l(v, w)$.

Then the pair (X, d_l) is called a metric-like space or dislocated metric space.

A metric-like mapping on X satisfies all the conditions of a metric except that $d_l(u, u)$ may be positive for some $u \in X$. Such metric-like mappings are for instance:

1) $(\mathbb{R}, d_l)$, where $d_l(u, v) = \max\{|u|, |v|\}$ for all $u, v \in \mathbb{R}$. We see that $(\mathbb{R}, d_l)$ is a metric-like space which is not a metric space because for instance $d_l(|-2|, |-2|) = 2 > 0$. Otherwise, $(\mathbb{R}, d_l)$ is a partial metric space.

2) $([0, +\infty), d_l)$, where $d_l(u, v) = u + v$ for all $u, v \in [0, +\infty)$. It is clear that $([0, +\infty), d_l)$ is a metric-like space where $d_l(u, u) > 0$ for each $u > 0$. Since, $d_l(2, 2) = 2 + 2 = 4 > 3 = 2 + 1 = d_l(2, 1)$, it follows that ($p2$) does not hold. Hence, $([0, +\infty), d_l)$ is not a partial metric space.

3) (X, d_l), where $X = \{0, 1, 2\}$ and $d_l(0, 0) = d_l(1, 1) = 0$, $d_l(2, 2) = \frac{5}{2}$, $d_l(0, 2) = d_l(2, 0) = 2$, $d_l(1, 2) = d_l(2, 1) = 3$, $d_l(0, 1) = d_l(1, 0) = \frac{3}{2}$. We have that (X, d_l) is a metric-like (that is a dislocated metric) space with $d_l(2, 2) > 0$. This means that (X, d_l) is not a standard metric space. However, (X, d_l) is also not a partial metric space because $d_l(2, 2) \nleq d_l(2, 0)$.

4) (X, d_l), where $X = C([0, 1], \mathbb{R})$ is the set of real continuous functions on $[0, 1]$ and $d_l(f, g) = \sup_{t \in [0,1]} \left(\left|f(t)\right| + \left|g(t)\right| \right)$ for all $f, g \in C([0, 1], \mathbb{R})$. This is one example of metric-like space which is not a partial metric space. Indeed, for $f(t) = 2t$, we obtain $d_l(f, f) = \sup_{t \in [0,1]} 2 \cdot 2t = 4 > 0$. Putting $g(t) \equiv 0$ for all $t \in [0, 1]$, we obtain that $d_l(f, f) = 4 \nleq d_l(f, g) = d_l(f, 0) = 2$.

Now we shall give the definitions of convergence and Cauchyness of the sequences in metric-like space.

Definition 1.3. *[17] Let $\{u_n\}$ be a sequence in a metric-like space (X, d_l).*

(i) *The sequence $\{u_n\}$ is said to be convergent to $u \in X$ if $\lim_{n\to\infty} d_l(u_n, u) = d_l(u, u)$;*

(ii) *The sequence $\{u_n\}$ is said to be d_l–Cauchy in (X, d_l) if $\lim_{n,m\to\infty} d_l(u_n, u_m)$ exists and is finite;*

(iii) *One say that a metric-like space (X, d_l) is d_l– complete if for every d_l–Cauchy sequence $\{u_n\}$ in X there exists an $u \in X$ such that $\lim_{n,m\to\infty} d_l(u_n, u_m) = d_l(u, u) = \lim_{n\to\infty} d_l(u_n, u)$.*

For more details on partial metric and metric-like spaces the reader can see [14, 17, 27, 31, 35, 36, 42, 44]. Otherwise, for other classes of generalized metric spaces as well as for contractive mappings, the reader has the following literature: [1–10, 13, 15, 18, 20–26, 28, 29, 33, 37–41].

Remark 1.4. *In metric like space (as in the partial metric space) the limit of a sequence need not be unique and a convergent sequence need not be a d_l-Cauchy sequence (see Examples in Remark 1.4 (1) and (2) in [36]). However, if the sequence $\{u_n\}$ is d_l-Cauchy such that $\lim_{n,m\to\infty} d_l(u_n, u_m) = 0$ in the d_l-complete metric-like space (X, d_l), then the limit of such sequence is unique. Indeed, in such a case if $u_n \to u$ as $n \to \infty$ we get that $d_l(u, u) = 0$ (by (iii) of Definition 1.3). Now, if $u_n \to u$, $u_n \to v$ and $u \neq v$, we obtain*

$$d_l(u, v) \le d_l(u, u_n) + d_l(u_n, v) \to d_l(u, u) + d_l(v, v) = 0 + 0 = 0. \tag{1}$$

By (d_l1) it follows that $u = v$, which is a contradiction.

Definition 1.5. *[32, 36, 44] Let (X, d_l) be a metric-like space. A sequence $\{u_n\}$ is called $0 - d_l-$Cauchy sequence if $\lim_{n,m\to\infty} d_l(u_n, u_m) = 0$. The space (X, d_l) is said to be $0 - d_l-$complete if every $0 - d_l-$Cauchy sequence in X converges to a point $u \in X$ such that $d_l(u, u) = 0$.*

It is obvious that every $0-d_l-$Cauchy sequence is a d_l-Cauchy sequence in (X, d_l) and every d_l-complete metric-like space is a $0 - d_l-$complete metric-like space. Also, every $0-$complete partial metric space (X, p) is a $0 - d_l-$complete metric-like space.

In the sequel we give some results from metric-like spaces for which the proofs are immediate.

Proposition 1.6. *Let (X, d_l) be a metric-like space. Then we have the following:*

(i) *If the sequence $\{u_n\}$ converges to $u \in X$ as $n \to \infty$ and if $d_l(u, u) = 0$, then for all $v \in X$ it follows that $d_l(u_n, v) \to d_l(u, v)$;*

(ii) *If $d_l(u, v) = 0$ then $d_l(u, u) = d_l(v, v) = 0$;*

(iii) *If $\{u_n\}$ is a sequence such that $\lim_{n\to\infty} d_l(u_n, u_{n+1}) = 0$ then $\lim_{n\to\infty} d_l(u_n, u_n) = \lim_{n\to\infty} d_l(u_{n+1}, u_{n+1}) = 0$;*

(iv) *If $u \neq v$ then $d_l(u, v) > 0$;*

(v) *$d_l(u, u) \le \frac{2}{n} \sum_{i=1}^{n} d_l(u, u_i)$ holds for all $u, u_i \in X$, where $1 \le i \le n$;*

(vi) *Let $\{u_n\}$ be a sequence such that $\lim_{n\to\infty} d_l(u_n, u_{n+1}) = 0$. If $\lim_{n,m\to\infty} d_l(u_n, u_m) \neq 0$, then there exists $\varepsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$, and the following sequences tend to ε when $k \to \infty$:*

$$d_l\left(u_{n(k)}, u_{m(k)}\right),\ d_l\left(u_{n(k)+1}, u_{m(k)}\right),\ d_l\left(u_{n(k)}, u_{m(k)-1}\right),\ d_l\left(u_{n(k)+1}, u_{m(k)-1}\right). \tag{2}$$

Notice that, if the condition of (vi) is satisfied then the sequences $d_l\left(u_{n(k)+q}, u_{m(k)}\right)$ and $d_l\left(u_{n(k)+q}, u_{m(k)+1}\right)$ also converge to ε when $k \to \infty$, where $q \in \mathbb{N}$. For more details on (i)-(vi) the reader can see [14, 27, 42].

In [43] authors introduced $\alpha-$admissible mapping:

Definition 1.7. *Let (X, d_l) be a metric-like space and let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. f is said to be an $\alpha-$admissible mapping if*

$$\alpha(u, v) \ge 1 \text{ implies } \alpha(fu, fv) \ge 1 \text{ for all } u, v \in X.$$

In [19], the concept of $\alpha-$continuous mapping was introduced:

Definition 1.8. *Let (X, d_l) be a metric-like space, $\alpha : X \times X \to [0, +\infty)$ and $f : X \to X$ an $\alpha-$admissible mapping. It is said that f is $\alpha-$continuous on X if*

$$\lim_{n\to\infty} u_n = u \text{ implies } \lim_{n\to\infty} fu_n = fu, \text{ for any sequence } \{u_n\} \text{ from } Y \text{ for which } \alpha(u_n, u_{n+1}) \ge 1; n \in \mathbb{N}.$$

Also, in [14] authors introduced and proved the following:

Let Ψ denote the class of all function $\psi : [0, +\infty) \to [0, +\infty)$, satisfying the following conditions:

(i) ψ is non-decreasing and continuous such that $\psi(t) < t$ for all $t > 0$;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

Definition 1.9. *Let (X, d_l) be a metric-like space, $q \in \mathbb{N}$, $B_1, B_2, \ldots, B_q$ be d_l-closed subsets of X, $Y = B_1 \cup \ldots \cup B_q$ and $\alpha : Y \times Y \to [0, +\infty)$ be a mapping. We say that $f : Y \to Y$ is a cyclic $\alpha_L^{\psi}-$rational contractive mapping if:*

(a) *$f(B_i) \subseteq B_{i+1}$, $i = 1, 2, \ldots, q$, where $B_{q+1} = B_1$;*

(b) *for any $u \in B_i$ and $v \in B_{i+1}$, $i = 1, 2, \ldots, q$, where $B_{q+1} = B_1$ and $\alpha(u, fu)\alpha(v, fv) \geq 1$, holds*

$$\psi(d_l(fu, fv)) \leq \psi(M_{d_l}(u, v)) - L \cdot M_{d_l}(u, v), \tag{3}$$

where $\psi \in \Psi$, $0 < L < 1$ and

$$M_{d_l}(u, v) = \max\left\{d_l(u, v), \frac{d_l(u, fu)\, d_l(v, fv)}{d_l(u, v)}, \frac{d_l(v, fv)\,[d_l(u, fu) + 1]}{1 + d_l(u, v)}, \frac{d_l(u, fv) + d_l(v, fu)}{4}\right\}. \tag{4}$$

If we take $X = B_i$, $i = 1, 2, \ldots, q$, then we say that f is an $\alpha_L^{\psi}-$rational contractive mapping.

We denote the set of all fixed points of f by $Fix(f)$, that is $Fix(f) = \{u \in X : fu = u\}$.

Remark 1.10. *If $f : X \to X$ is a cyclic $\alpha_L^{\psi}-$rational contractive mapping, $u \in Fix(f)$ and $\alpha(u, u) \geq 1$, then $d_l(u, u) = 0$. Indeed, suppose $d_l(u, u) > 0$. First, we get*

$$M_{d_l}(u, u) = \max\left\{d_l(u, u), \frac{d_l(u, u)\, d_l(u, u)}{d_l(u, u)}, \frac{d_l(u, u)\,[d_l(u, u) + 1]}{1 + d_l(u, u)}, \frac{d_l(u, u) + d_l(u, u)}{4}\right\} = d_l(u, u). \tag{5}$$

Now, from (3) we can write $\psi(d_l(u, u)) = \psi(d_l(fu, fu)) \leq \psi(d_l(u, u)) - L \cdot d_l(u, u) < \psi(d_l(u, u))$, which is a contradiction.

There are some doubts about the structure of the function M_{d_l}, that follows from Remark 1.10 as well as from [14], Examples 2.3 and 2.4. Namely, in both mentioned examples we can see that $u = 0$ and $u = 1$ are fixed points. For those points we have that $M_{d_l}(0, 0) = M_{d_l}(1, 1) = \max\left\{0, \frac{0}{0}\right\}$. This means that either Examples 2.3 and 2.4 do not support Theorem 2.1 from [14] or the structure of the function M_{d_l} is not right (for more informations the reader can see [11, 14–16]). Hence, in our present paper we modify the function M_{d_l} as follows:

$$M_{d_l}(u, v) = \max\left\{d_l(u, v), \frac{1}{2}d_l(v, fu), \frac{d_l(u, v)\, d_l(v, fv)}{1 + d_l(u, fu)}, \frac{d_l(v, fv)\,[1 + d_l(u, fu)]}{1 + d_l(u, v)}, \frac{d_l(u, fv) + d_l(v, fu)}{4}\right\}. \tag{6}$$

This new definition of the function M_{d_l} significantly improve several results of [14]. With this new approach, the correct formulation of Theorem 2.1 from [14] is the following:

Theorem 1.11. *Let (X, d_l) be a complete metric-like space, q be a positive integer, $B_1, B_2, \ldots, B_q$ be nonempty d_l-closed subsets of X, $Y = B_1 \cup \ldots \cup B_q$ and $\alpha : Y \times Y \to [0, +\infty)$ be a mapping. Assume that $f : Y \to Y$ is a cyclic $\alpha_L^{\psi}-$rational contractive mapping satisfying the following conditions:*

(i) *f is an $\alpha-$admissible mapping;*

(ii) *there exists $u_0 \in Y$ such that $\alpha(u_0, fu_0) \geq 1$;*

(iii) *either f is $\alpha-$continuous, or for any sequence $\{u_n\}$ in Y with $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \geq 0$ and $u_n \to u$ as $n \to \infty$, one has $\alpha(u, fu) \geq 1$.*

Then f has a fixed point $u \in B_1 \cap \ldots \cap B_q$. Moreover, if

(iv) *for all $u \in Fix(f)$ we have $\alpha(u, u) \geq 1$,*

then f has a unique fixed point $u \in B_1 \cap \ldots \cap B_q$.

Remark 1.12. *It is clear that Theorem 1.11 is true if we consider a metric space (X, d) or a partial metric space (X, p) instead of a metric-like space (X, d_l).*

2. Main result

In the sequel of this paper we generalize, complement, extend, unify, enrich and improve several recent results announced in [4, 11, 14, 17, 27, 28, 32–38, 42–44]. Our first new result begins with the fixed point of $\alpha_L^{\psi}-$rational contractive mapping. In all our results the set $M_{d_l}(u,v)$ is defined by (6).

Theorem 2.1. *Let (X,d_l) be a d_l-complete metric-like space and $\alpha : X\times X\to[0,\infty)$ be a mapping. Assume that $f : X\to X$ is an $\alpha_L^{\psi}-$rational contractive mapping satisfying the following conditions:*

(i) *f is an $\alpha-$admissible mapping;*

(ii) *there exists $u_0\in X$ such that $\alpha(u_0,fu_0)\geq 1$;*

(iii) *either f is $\alpha-$continuous, or for any sequence $\{u_n\}$ in X with $\alpha(u_n,u_{n+1})\geq 1$ for all $n\geq 0$ and $u_n\to u$ as $n\to\infty$, then $\alpha(u,fu)\geq 1$.*

Then f has a fixed point $u\in X$. Moreover, if

(iv) *for all $u\in Fix(f)$ we have $\alpha(u,u)\geq 1$,*

then f has a unique fixed point $u\in X$.

Proof. First, we shall consider uniqueness of a possible fixed point. To prove that fixed point is unique, if it exists, suppose that f has two distinct fixed points $u^*,v^*\in X$. Then we get

$$\psi(d_l(u^*,v^*))=\psi(d_l(fu^*,fv^*))\leq\psi(M_{d_l}(u^*,v^*))-L\cdot M_{d_l}(u^*,v^*),\tag{7}$$

where

$$\begin{aligned}M_{d_l}(u^*,v^*)&=\max\left\{d_l(u^*,v^*),\frac{1}{2}d_l(v^*,fu^*),\frac{d_l(u^*,v^*)\,d_l(v^*,fv^*)}{1+d_l(u^*,fu^*)},\right.\\&\qquad\left.\frac{d_l(v^*,fv^*)\left[1+d_l(u^*,fu^*)\right]}{1+d_l(u^*,v^*)},\frac{d_l(u^*,fv^*)+d_l(v^*,fu^*)}{4}\right\}\\&=\max\left\{d_l(u^*,v^*),\frac{d_l(u^*,v^*)}{2},0,0,\frac{d_l(u^*,v^*)}{2}\right\}\\&=d_l(u^*,v^*).\end{aligned}\tag{8}$$

Now from (7) follows

$$\psi(d_l(u^*,v^*))\leq\psi(d_l(u^*,v^*))-L\cdot d_l(u^*,v^*),\tag{9}$$

that is $d_l(u^*,v^*)=0$. By (d_l1) we get a contradiction.

Let us define Picard's sequence $u_n=f^nu_0$, where u_0 is the given point for which $\alpha(u_0,fu_0)\geq 1$. Since, f is an $\alpha-$admissible mapping, we get that $\alpha(u_1,fu_1)=\alpha(fu_0,f(fu_0))\geq 1$. Again, from the same reason, it follows that $\alpha(u_2,fu_2)=\alpha(fu_1,f(fu_1))\geq 1$. Continuing this process we have that $\alpha(u_n,fu_n)\geq 1$ for all $n\in\mathbb{N}_0$, and so $\alpha(u_n,fu_n)\alpha(u_{n-1},fu_{n-1})\geq 1$ for all $n\in\mathbb{N}$. In the case when $u_{n-1}=u_n$ for some $n\in\mathbb{N}$, it is clear that u_n is a unique fixed point of f. Therefore, assume that $u_{n-1}\neq u_n$ for all $n\in\mathbb{N}$. Hence, by Proposition 1.6 (iv), we have $d_l(u_{n-1},u_n)>0$ for all $n\in\mathbb{N}$.

In order to prove that the sequence $\{u_n\}$ is a d_l-Cauchy we shall first check that it is a non-increasing one. This means that for all $n\in\mathbb{N}$ we have $d_l(u_n,u_{n+1})\leq d_l(u_{n-1},u_n)$. According to (3) we get

$$\psi(d_l(u_n,u_{n+1}))=\psi(d_l(fu_{n-1},fu_n))\leq\psi(M_{d_l}(u_{n-1},u_n))-L\cdot M_{d_l}(u_{n-1},u_n),\tag{10}$$

where

$$\begin{aligned} M_{d_l}(u_{n-1},u_n) &= \max\left\{d_l(u_{n-1},u_n), \frac{1}{2}d_l(u_n,u_n), \frac{d_l(u_{n-1},u_n)\,d_l(u_n,u_{n+1})}{1+d_l(u_{n-1},u_n)},\right. \\ &\qquad \left.\frac{d_l(u_n,u_{n+1})\left[1+d_l(u_{n-1},u_n)\right]}{1+d_l(u_{n-1},u_n)}, \frac{d_l(u_{n-1},u_{n+1})+d_l(u_n,u_n)}{4}\right\} \\ &\leq \max\left\{d_l(u_{n-1},u_n), d_l(u_n,u_{n+1}), \frac{3}{4}d_l(u_{n-1},u_n)+\frac{1}{4}d_l(u_n,u_{n+1})\right\} \\ &\leq \max\left\{d_l(u_{n-1},u_n), d_l(u_n,u_{n+1})\right\}. \end{aligned} \tag{11}$$

Hence, at the end we obtain

$$\psi(d_l(u_n,u_{n+1})) \leq \psi(\max\{d_l(u_{n-1},u_n), d_l(u_n,u_{n+1})\}) - L\cdot\max\{d_l(u_{n-1},u_n), d_l(u_n,u_{n+1})\}. \tag{12}$$

If $d_l(u_n,u_{n+1}) > d_l(u_{n-1},u_n)$ for some $n\in\mathbb{N}$, the condition (12) becomes

$$\psi(d_l(u_n,u_{n+1})) \leq \psi(d_l(u_n,u_{n+1})) - L\cdot d_l(u_n,u_{n+1}), \tag{13}$$

which is a contradiction. Hence, $d_l(u_n,u_{n+1}) \leq d_l(u_{n-1},u_n)$ for all $n\in\mathbb{N}$ and

$$\psi(d_l(u_n,u_{n+1})) \leq \psi(d_l(u_{n-1},u_n)) - L\cdot d_l(u_{n-1},u_n). \tag{14}$$

Also, it follows that there exists $\lim_{n\to\infty} d_l(u_n,u_{n+1}) = d_l^*$. Suppose that $d_l^* > 0$. Letting the limit in the relation (14) we get $d_l^* \leq 0$, which is a contradiction again. Thus, $\lim_{n\to\infty} d_l(u_n,u_{n+1}) = 0$.

Now, if $\lim_{n,m\to\infty} d_l(u_n,u_m) \neq 0$, based on Proposition 1.6 (vi), we have sequences $\{m_k\}$ and $\{n_k\}$ such that $\lim_{k\to\infty} d_l(u_{n_k},u_{m_k}) = \varepsilon > 0$. By putting $u = u_{n_k},\ v = u_{m_k}$ in (3) we obtain

$$\psi(d_l(u_{n_k+1},u_{m_k+1})) \leq \psi(M_{d_l}(u_{n_k},u_{m_k})) - L\cdot M_{d_l}(u_{n_k},u_{m_k}), \tag{15}$$

where

$$\begin{aligned} M_{d_l}(u_{n_k},u_{m_k}) &= \max\left\{d_l(u_{n_k},u_{m_k}), \frac{1}{2}d_l(u_{m_k},u_{n_k+1}), \frac{d_l(u_{n_k},u_{m_k})\,d_l(u_{m_k},u_{m_k+1})}{1+d_l(x_{n_k},x_{n_k+1})},\right. \\ &\qquad \left.\frac{d_l(u_{m_k},u_{m_k+1})\left[1+d_l(u_{n_k},u_{n_k+1})\right]}{1+d_l(u_{n_k},u_{m_k})}, \frac{d_l(u_{n_k},u_{m_k+1})+d_l(u_{m_k},u_{n_k+1})}{4}\right\} \\ &\to \max\left\{\varepsilon, \frac{\varepsilon}{2}, 0, 0, \frac{\varepsilon}{2}\right\} = \varepsilon \ \text{ as } \ k\to\infty. \end{aligned} \tag{16}$$

Letting the limit in (15) we have that $\psi(\varepsilon) \leq \psi(\varepsilon) - L\cdot\varepsilon$, which is a contradiction. Hence, the sequence $\{u_n\}$ is a d_l-Cauchy and $\lim_{n,m\to\infty} d_l(u_n,u_m) = 0$. This means that there exists a unique point $\overline{u}\in X$ such that

$$d_l(\overline{u},\overline{u}) = \lim_{n\to\infty} d_l(u_n,\overline{u}) = \lim_{n,m\to\infty} d_l(u_n,u_m) = 0. \tag{17}$$

Now, we will show that $\overline{u}$ is a fixed point of f, i.e. $f\overline{u} = \overline{u}$. This is clear in the case that the mapping f is $\alpha-$continuous. Further, suppose that for any sequence $\{u_n\}$ in X and for all $n\geq 0$, if $\alpha(u_n,u_{n+1})\geq 1$ and $\lim_{n\to\infty} u_n = \overline{u}$, then $\alpha(\overline{u},f\overline{u})\geq 1$. Let $d_l(\overline{u},f\overline{u}) > 0$. Since $\alpha(u_n,fu_n)\,\alpha(\overline{u},f\overline{u})\geq 1$, according to the given contractive condition, we have

$$\psi(d_l(fu_n,f\overline{u})) \leq \psi(M_{d_l}(u_n,\overline{u})) - L\cdot M_{d_l}(u_n,\overline{u}), \tag{18}$$

where

$$\begin{aligned} M_{d_l}(u_n,\overline{u}) &= \max\left\{d_l(u_n,\overline{u}),\frac{1}{2}d_l(\overline{u},u_{n+1}),\frac{d_l(u_n,\overline{u})\,d_l(\overline{u},f\overline{u})}{1+d_l(u_n,u_{n+1})},\right.\\ &\quad\left.\frac{d_l(\overline{u},f\overline{u})[1+d_l(u_n,u_{n+1})]}{1+d_l(u_n,u_{n+1})},\frac{d_l(u_n,f\overline{u})+d_l(\overline{u},u_{n+1})}{4}\right\}\\ &\le \max\left\{d_l(u_n,\overline{u}),\frac{1}{2}d_l(\overline{u},u_{n+1}),\frac{d_l(u_n,\overline{u})\,d_l(\overline{u},f\overline{u})}{1+d_l(u_n,u_{n+1})},\right.\\ &\quad\left.d_l(\overline{u},f\overline{u}),\frac{d_l(u_n,\overline{u})+d_l(\overline{u},f\overline{u})+d_l(\overline{u},u_{n+1})}{4}\right\}\\ &\to \max\left\{0,0,0,d_l(\overline{u},f\overline{u}),\frac{d_l(\overline{u},f\overline{u})}{4}\right\}=d_l(\overline{u},f\overline{u})\ \text{ as } n\to\infty. \end{aligned} \tag{19}$$

Now, letting the limit in (18) for $n\to\infty$, we get

$$\psi(d_l(\overline{u},f\overline{u}))\le\psi(d_l(\overline{u},f\overline{u}))-L\cdot d_l(\overline{u},f\overline{u}), \tag{20}$$

which is a contradiction again. This means that $d_l(\overline{u},f\overline{u})=0$. By (d_l1) it follows $f\overline{u}=\overline{u}$, i.e. $\overline{u}$ is a unique fixed point of the mapping f. □

Remark 2.2. *For the proof that* $\lim_{n\to\infty}\psi(d_l(fu_n,f\overline{u}))=\psi(d_l(\overline{u},f\overline{u}))$ *we used the property of the function* ψ *as well as the next claim: If* $\lim_{n\to\infty}d_l(u_n,u)=d_l(u,u)=0$ *then* $\lim_{n\to\infty}d_l(u_n,v)=d_l(u,v)$ *for each* $v\in X$*, where* (X,d_l) *is a metric-like space.*

Our second new result is in the fact an improvement of Theorem 2.1 from [14], where we use the modification of the function M_{d_l} given by (6). Namely, we give the proof of Theorem 1.11 with the new M_{d_l}. We will use the following well known lemma [32, 34, 42]:

Lemma 2.3. *Let* (X,d_l) *be a metric-like space,* $f:X\to X$ *be a mapping and let* $X=A_1\cup A_2\cup\ldots\cup A_p$ *be a cyclic representation of* X *with respect to* f*. Assume that*

$$\lim_{n\to\infty}d_l(u_n,u_{n+1})=0, \tag{21}$$

where $u_{n+1}=fu_n$ *and* $u_1\in A_1$*. If* $\{u_n\}$ *is not a* d_l-*Cauchy sequence then there exist an* $\varepsilon>0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that the following sequences tend to* ε *when* $k\to\infty$ *:*

$$d_l\left(u_{m_k-j_k},u_{n_k}\right),\ d_l\left(u_{m_k-j_k+1},u_{n_k}\right),\ d_l\left(u_{m_k-j_k},u_{n_k+1}\right),\ d_l\left(u_{m_k-j_k+1},u_{n_k+1}\right), \tag{22}$$

where $j_k\in\{1,2,...,p\}$ *is chosen so* $n_k-m_k+j_k\equiv 1 \pmod p$*, for each* $k\in\mathbb{N}$*.*

Proof. of Theorem 1.11. We can suppose that $u_0\in B_1$. Then the proof follows the lines of one for Theorem 2.1, except that obtained Picard's sequence is a d_l-Cauchy. Now, by the previous Lemma, putting $u=u_{m_k-j_k}$ and $v=u_{n_k}$ in (3) we get

$$\psi\left(d_l\left(u_{m_k-j_k+1},u_{n_k+1}\right)\right)\le\psi\left(M_{d_l}\left(u_{m_k-j_k},u_{n_k}\right)\right)-L\cdot M_{d_l}\left(u_{m_k-j_k},u_{n_k}\right), \tag{23}$$

where

$$\begin{aligned} M_{d_l}\left(u_{m_k-j_k},u_{n_k}\right) &= \max\left\{d_l\left(u_{m_k-j_k},u_{n_k}\right),\frac{1}{2}d_l\left(u_{n_k},u_{m_k-j_k+1}\right),\frac{d_l\left(u_{m_k-j_k},u_{n_k}\right)d_l\left(u_{n_k},u_{n_k+1}\right)}{1+d_l\left(u_{m_k-j_k},u_{m_k-j_k+1}\right)},\right.\\ &\quad\left.\frac{d_l\left(u_{n_k},u_{n_k+1}\right)\left[1+d_l\left(u_{m_k-j_k},u_{m_k-j_k+1}\right)\right]}{1+d_l\left(u_{m_k-j_k},u_{n_k}\right)},\frac{d_l\left(u_{m_k-j_k},u_{n_k+1}\right)+d_l\left(u_{n_k},u_{m_k-j_k+1}\right)}{4}\right\}. \end{aligned} \tag{24}$$

Taking the limit in (24) as $k \to \infty$ we have

$$\lim_{k\to\infty} M_{d_l}\left(u_{m_k-j_k}, u_{n_k}\right) = \max\left\{\varepsilon, \frac{\varepsilon}{2}, 0, 0, \frac{\varepsilon}{2}\right\} = \varepsilon. \tag{25}$$

Taking the limit as $k \to \infty$, but now in (23), we obtain

$$\psi(\varepsilon) \le \psi(\varepsilon) - L \cdot \varepsilon, \tag{26}$$

which is a contradiction. This completes the proof that the sequence $\{u_n\}$ is a d_l-Cauchy. Since Y is d_l-closed in (X, d_l), this means that there exists a unique $\overline{u} \in Y$ such that

$$d_l(\overline{u}, \overline{u}) = \lim_{n\to\infty} d_l(u_n, \overline{u}) = \lim_{n,m\to\infty} d_l(u_n, u_m) = 0. \tag{27}$$

Further, because $f(B_i) \subseteq B_{i+1},\ B_{p+1} = B_1$ it follows that the sequence $\{u_n\}$ has infinitely many terms in each B_i for $i \in \{1, 2, \ldots, p\}$. Hence, we have the subsequences $\{u_{n_i}\}$ of $\{u_n\}$ where $\{u_{n_i}\} \subseteq B_i,\ i = 1, 2, \ldots, p$. It is clear that each u_{n_i} converges to $\overline{u}$. From this it follows that $B = B_1 \cap B_2 \cap \ldots \cap B_p \neq \emptyset$ because it contains at least the element $\overline{u}$. Obviously, (B, d_l) is a d_l-complete metric-like space and $f : B \to B$. It is not hard to check that the restriction $f|_B$ of f on B satisfies all conditions of our Theorem 2.1. Hence, f has a unique fixed point $\overline{u}$ in B. This completes the proof of Theorem 1.11. □

According to the two previous Theorems we can formulate the following immediate corollary (see the corresponding results for b-metric like spaces in [11]):

Corollary 2.4. *Theorem 1.11 and Theorem 2.1 are equivalent.*

Now, by Corollary 2.4 we shall discuss some examples.

Example 2.5. *([14], Example 2.3) Let $X = \mathbb{R}$ be equipped with the metric-like mapping $d_l(u, v) = \max\{|u|, |v|\}$ for all $u, v \in X$. Suppose $B_1 = (-\infty, 0]$, $B_2 = [0, +\infty)$ and $Y = \mathbb{R}$. Define $f : Y \to Y$ and $\alpha : Y \times Y \to [0, +\infty)$ by*

$$f(u) = \begin{cases} -3u, & \text{if } u < -1 \\ -\frac{u}{5}, & \text{if } -1 \le u \le 0 \\ \frac{-u^3}{4}, & \text{if } 0 \le u \le 1 \\ -6u, & \text{if } u > 1 \end{cases} \quad \text{and} \quad \alpha(u, v) = \begin{cases} u^2 + v^2 + 2, & \text{if } (u, v) \in [-1, 1]^2 \\ 0, & \text{otherwise} \end{cases}. \tag{28}$$

Also, define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$ and $L = \frac{1}{8} \in (0, 1)$.

From ([14], Example 2.3) we know that this example satisfies all the conditions of Theorem 1.11 above (Theorem 2.1 from [14]). The verification is rather long. According to Corollary 2.4, that is our proof of Theorem 1.11 above, it is sufficient to check that Theorem 2.1 holds for all $u, v \in B_1 \cap B_2 = \{0\}$. Hence, our verification is much shorter and nicer than the one presented in Example 2.3 of [14].

Example 2.6. *([14], Example 2.5) Let $X = \mathbb{R}^+$ be equipped with the metric-like mapping $d_l(u, v) = \max\{u, v\}$ for all $u, v \in X$. Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be defined by*

$$f(u) = \begin{cases} \frac{u^3}{4}, & \text{if } 0 \le u < \frac{1}{4} \\ \frac{u^2}{8}, & \text{if } \frac{1}{4} \le u \le 1 \\ \frac{u}{24}, & \text{if } 1 < u \le 3 \\ 3u^3 + 2, & \text{if } u > 3 \end{cases} \quad \text{and} \quad \alpha(u, v) = \begin{cases} 6, & \text{if } (u, v) \in [0, 3]^2 \\ 0, & \text{otherwise} \end{cases}. \tag{29}$$

Also, define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$ and $L = \frac{3}{8} \in (0, 1)$.

From [14] follows that this example support our Theorem 2.1.

Remark 2.7. *In [42] Salimi et al. introduced the notions of $\alpha - \psi\ \phi-$contractive and cyclic $\alpha - \psi\ \phi-$contractive mappings, and established the existence and uniqueness of fixed points for such mappings in d_l-complete metric-like spaces. Also, in [27] Karapinar and Salimi introduced cyclic generalized $\phi - \psi-$contractive and generalized $\phi - \psi-$contractive mappings, and proved the corresponding fixed point result (Theorem 1.8). It is worth noticing that the cyclic and usual fixed point results are equivalent in both cited paper. The proofs for this claim are the same as in [32–34], as well as in Corollary 2.4.*

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