Filomat 34:14 (2020), 4637–4647
https://doi.org/10.2298/FIL2014637M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient and Fekete-Szegö Problem Estimates for Certain Subclass of Analytic and Bi-Univalent Functions

Hesam Mahzoon[a]

[a] *Department of Mathematics, Islamic Azad University, West Tehran Branch, Tehran, Iran*

Abstract. In this paper, we obtain the Fekete-Szegö problem for the k-th ($k \geq 1$) root transform of the analytic and normalized functions f satisfying the condition

$$1+\frac{\alpha-\pi}{2\sin\alpha}<\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<1+\frac{\alpha}{2\sin\alpha}\quad(|z|<1),$$

where $\alpha \in [\pi/2, \pi)$. Afterwards, by the above two-sided inequality we introduce a certain subclass of analytic and bi-univalent functions in the disk $|z| < 1$ and obtain upper bounds for the first few coefficients and Fekete-Szegö problem for functions f belonging to this class.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Also let $\mathcal{P}$ be the class of functions p analytic in Δ which are of the form

$$p(z)=1+p_1z+p_2z^2+\cdots+p_nz^n+\cdots,$$

such that $\operatorname{Re}\{p(z)\} > 0$ for all $z \in \Delta$. The subclass of all functions f in $\mathcal{A}$ which are univalent (one-to-one) in Δ is denoted by $\mathcal{S}$. An example for the class $\mathcal{S}$ is the well-known *Koebe* function which has the following form

$$k(z):=\frac{z}{(1-z)^2}=z+2z^2+3z^3+\cdots+nz^n+\cdots\quad(z\in\Delta).$$

It is known that the Koebe function maps the open unit disk Δ onto the entire plane minus the interval $(-\infty, -1/4]$. Also, the well-known *Koebe One-Quarter Theorem* states that the image of the open unit disk Δ

2010 *Mathematics Subject Classification*. 30C45
Keywords. univalent, bi-univalent, starlike, Fekete-Szegö problem, coefficient estimates
Received: 02 February 2020; Accepted: 09 February 2020
Communicated by Hari. M. Srivastava
Email address: hesammahzoon1@gmail.com, mahzoon_hesam@yahoo.com (Hesam Mahzoon)

under every function $f \in \mathcal{S}$ contains the disk $\{w : |w| < \frac{1}{4}\}$, see [11, Theorem 2.3]. Therefore, according to the above, every function f in the class $\mathcal{S}$ has an inverse f^{-1} which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f);\ \ r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots =: g(w). \tag{2}$$

We say that a function $f \in \mathcal{A}$ is *bi-univalent* in Δ if, and only if, both f and f^{-1} are univalent in Δ. We denote by Σ the class of all bi-univalent functions in Δ. The following functions

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

with the corresponding inverse functions, respectively,

$$\frac{w}{1+w}, \quad \frac{\exp(w)-1}{\exp(w)} \quad \text{and} \quad \frac{\exp(2w)-1}{\exp(2w)+1},$$

belong to the class Σ. It is clear that the Koebe function is not a member of the class Σ, also the following functions

$$z - \frac{1}{2}z^2 \quad \text{and} \quad \frac{z}{1-z^2},$$

do not belong to the class Σ, see [35].

It should be mentioned here that the pioneering work on the subject by Srivastava et al. [35] actually revived the study of analytic and bi-univalent functions in recent years. In fact, subsequent to this important investigation by Srivastava et al. [35], many authors have introduced and studied various subclasses of analytic and bi-univalent functions (see, for example, [9, 23, 25, 28, 29, 31, 32, 36, 37, 40, 43, 44])

A function $f \in \mathcal{A}$ is called starlike (with respect to 0) if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0,1]$. We denote by $\mathcal{S}^*$ the class of all starlike functions in Δ. Also, we say that a function $f \in \mathcal{A}$ is starlike of order γ $(0 \leq \gamma < 1)$ if, and only if,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta).$$

The class of the starlike functions of order γ in Δ is denoted by $\mathcal{S}^*(\gamma)$. As usual we put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$.

We recall that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{M}(\alpha)$ if f satisfies the following two-sided inequality

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \quad (z \in \Delta),$$

where $\alpha \in [\pi/2, \pi)$. The class $\mathcal{M}(\alpha)$ was introduced by Kargar *et al.* in [13]. We define the function ϕ as follows

$$\phi(\alpha) := 1 + \frac{\alpha - \pi}{2\sin\alpha} \quad (\pi/2 \leq \alpha < \pi).$$

Since

$$2\phi'(\alpha) = [(\pi - \alpha)\cot\alpha + 1]\csc\alpha \quad (\pi/2 \leq \alpha < \pi),$$

therefore for each $\alpha \in [\pi/2, \pi)$ we see that $\phi'(\alpha) \neq 0$. On the other hand, since $\phi(\pi/2) = 1 - \pi/4 \approx 0.2146$ and

$$\lim_{\alpha \to \pi^-} \phi(\alpha) = \frac{1}{2},$$

thus the class $\mathcal{M}(\alpha)$ is a subclass of the starlike functions of order γ where $0.2146 \leq \gamma < 0.5$. By this fact that $\mathcal{S}^*(\gamma) \subset \mathcal{S}$ for each $\gamma \in [0, 1)$, thus we conclude that the members of the class $\mathcal{M}(\alpha)$ are univalent in Δ.

Now, we recall the following result for the class $\mathcal{M}(\alpha)$, see [13, Lemma 1.1].

Lemma 1.1. *Let $f(z) \in \mathcal{A}$ and $\alpha \in [\pi/2, \pi)$. Then $f \in \mathcal{M}(\alpha)$ if, and only if,*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \mathcal{B}_\alpha(z) \quad (z \in \Delta),$$

where

$$\mathcal{B}_\alpha(z) := \frac{1}{2i \sin\alpha} \log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right) \quad (z \in \Delta). \tag{3}$$

Here " $\prec$ " denotes the well known subordination relation.

The function $\mathcal{B}_\alpha(z)$ is convex univalent and has the form

$$\mathcal{B}_\alpha(z) = \sum_{n=1}^{\infty} A_n z^n \quad (z \in \Delta), \tag{4}$$

where

$$A_n := \frac{(-1)^{(n-1)} \sin n\alpha}{n \sin\alpha} \quad (n = 1, 2, \ldots).$$

Also we have $\mathcal{B}_\alpha(\Delta) = \Omega_\alpha$ (see [10]) where

$$\Omega_\alpha := \left\{\zeta \in \mathbb{C} : \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\{\zeta\} < \frac{\alpha}{2\sin\alpha}, \ \frac{\pi}{2} \leq \alpha < \pi\right\}.$$

Very recently Sun *et al.* (see [41]) and Kwon and Sim (see [17]) have studied the class $\mathcal{M}(\alpha)$. Sun *et al.* showed if the function f is of the form (1) belongs to the class $\mathcal{M}(\alpha)$, then $|a_n| \leq 1$ while the estimate is not sharp. Subsequently, Kwon and Sim obtained sharp estimates on the initial coefficients a_2, a_3, a_4 and a_5 of the functions f belonging to the class $\mathcal{M}(\alpha)$. The coefficient estimate problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n = 6, 7, \ldots)$ is still an open question. Also, the logarithmic coefficients of the function $f \in \mathcal{M}(\alpha)$ were estimated by Kargar, see [12].

It is interesting to mention this subject that Brannan and Taha [7] introduced certain subclass of the bi-univalent function class Σ, denoted by $\mathcal{S}^*_\Sigma(\gamma)$ similar to the class of the starlike functions of order γ $(0 \leq \gamma < 1)$. For each function $f \in \mathcal{S}^*_\Sigma(\gamma)$ they found non-sharp estimates for the initial Taylor-Maclaurin coefficients. Recently, motivated by the Brannan and Taha's work, many authors investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ, see for instance [8, 21, 22, 26, 27, 35, 38, 39].

In this paper, motivated by the aforementioned works, we introduce and investigate a certain subclass of Σ similar to the class $\mathcal{M}(\alpha)$ as follows.

Definition 1.2. *Let $\alpha \in [\pi/2, \pi)$. A function $f \in \Sigma$ is in the class $\mathcal{M}_\Sigma(\alpha)$, if the following inequalities hold:*

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \quad (z \in \Delta)$$

and

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{wg'(w)}{g(w)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \quad (w \in \Delta),$$

where g is defined by (2).

Remark 1.3. *Upon letting $\alpha \to \pi^-$ it is readily seen that a function $f \in \Sigma$ is in the class $\mathcal{M}_\Sigma(1/2)$ if the following inequalities are satisfied:*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \Delta)$$

and

$$\mathrm{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \frac{1}{2} \quad (w \in \Delta),$$

where g is defined by (2).

The following lemma will be useful.

Lemma 1.4. (*see [19]*) *Let the function p be of the form belongs to the class $\mathcal{P}$. Then for any complex number μ we have*

$$\left|p_2 - \mu p_1^2\right| \leq \begin{cases} -4\mu + 2, & \text{if } \mu \leq 0; \\ 2, & \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 2, & \text{if } \mu \geq 1. \end{cases}$$

The result is sharp for the cases $\mu < 0$ or $\mu > 1$ if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. For the case $\mu = 0$, the equality holds if and only if

$$p(z) = \frac{1}{2}(1+\nu)\frac{1+z}{1-z} + \frac{1}{2}(1-\nu)\frac{1-z}{1+z} \quad (0 \leq \nu \leq 1),$$

or one of its rotations. If $\mu = 1$, the equality holds if and only if

$$\frac{1}{p(z)} = \frac{1}{2}(1+\nu)\frac{1+z}{1-z} + \frac{1}{2}(1-\nu)\frac{1-z}{1+z} \quad (0 \leq \nu \leq 1),$$

or one of its rotations.

This paper is organized as follows. In Section 2 we derive the Fekete-Szegö coefficient functional associated with the k-th root transform for functions in the class $\mathcal{M}(\alpha)$. In Section 3 we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö problem for functions in the class $\mathcal{M}_\Sigma(\alpha)$ which we introduced in Definition 1.2.

2. Fekete-Szegö problem for the class $\mathcal{M}(\alpha)$

Recently, many authors have obtained the Fekete-Szegö coefficient functional associated with the k-th root transform for certain subclasses of analytic functions, see for instance [5, 14, 15]. In this section, we investigate this problem for the class $\mathcal{M}(\alpha)$. At first, we recall that for a univalent function f is of the form (1), the k-th root transform is defined by

$$F_k(z) := (f(z^k))^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta, k \geq 1). \tag{5}$$

For f given by (1), we have

$$(f(z^k))^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1} + \cdots. \tag{6}$$

Equating the coefficients of (5) and (6) yields

$$b_{k+1} = \frac{1}{k}a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{7}$$

Now we have the following.

Theorem 2.1. *Let $\alpha \in [\pi/2, \pi)$ and $f \in \mathcal{M}(\alpha)$. If F is the k-th $(k \geq 1)$ root transform of the function f defined by* (5)*, then for any complex number μ we have*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \frac{1}{2k}\left(1 - \cos\alpha - \frac{2\mu+k-1}{k}\right), & \text{if } \mu \leq \delta_1; \\ \frac{1}{2k}, & \text{if } \delta_1 \leq \mu \leq \delta_2; \\ \frac{1}{2k}\left(\cos\alpha + \frac{2\mu+k-1}{k} - 1\right), & \text{if } \mu \geq \delta_2, \end{cases} \tag{8}$$

where $\delta_1 := (1 - k(1 + \cos\alpha))/2$, $\delta_2 := (1 + k(1 - \cos\alpha))/2$ and b_{2k+1} and b_{k+1} are defined by (7)*. The result is sharp.*

Proof. Let $\alpha \in [\pi/2, \pi)$. If $f \in \mathcal{M}(\alpha)$, then by Lemma 1.1 and by definition of subordination, there exists a Schwarz function $w : \Delta \to \overline{\Delta} := \{z : |z| \leq 1\}$ with the following properties

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$\frac{zf'(z)}{f(z)} = 1 + \mathcal{B}_\alpha(w(z)) \quad (z \in \Delta), \tag{9}$$

where $\mathcal{B}_\alpha$ is defined by (3). We define

$$p(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \Delta). \tag{10}$$

It is clear that $p(0) = 1$ and $p \in \mathcal{P}$. Relationships (4) and (10) give us

$$1 + \mathcal{B}_\alpha(w(z)) = 1 + \frac{1}{2}A_1 p_1 z + \left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1\left(p_2 - \frac{1}{2}p_1^2\right)\right)z^2 + \cdots,$$

where $A_1 = 1$ and $A_2 = -\cos\alpha$. If we equate the coefficients of z and z^2 on both sides of (9), then we get

$$a_2 = \frac{1}{2}p_1 \tag{11}$$

and

$$a_3 = \frac{1}{4}\left(p_2 - \frac{1}{2}\cos\alpha p_1^2\right). \tag{12}$$

From (7), (11) and (12), we get

$$b_{k+1} = \frac{p_1}{2k},$$

and

$$b_{2k+1} = \frac{1}{4k}\left[p_2 - \frac{1}{2}\left(\cos\alpha + \frac{k-1}{k}\right)p_1^2\right],$$

where $k \geq 1$. Therefore

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{4k}\left[p_2 - \frac{1}{2}\left(\cos\alpha + \frac{2\mu + k - 1}{k}\right)p_1^2\right] \quad (\mu \in \mathbb{C}).$$

If we apply the Lemma 1.4 and letting

$$\mu' := \frac{1}{2}\left(\cos\alpha + \frac{2\mu + k - 1}{k}\right),$$

then we get the desired inequality (8).

From now, we shall show that the result is sharp. For the sharpness of the first and third cases of (8), i.e. $\mu \le \delta_1$ and $\mu \ge \delta_2$, respectively, consider the function

$$\begin{aligned} f_1(z) &:= z \exp\left\{ \int_0^z \frac{\mathcal{B}_\alpha(\xi) - 1}{\xi} \mathrm{d}\xi \right\} \quad (z \in \Delta) \\ &= z + z^2 + \frac{1}{2}(1 - \cos\alpha) z^3 + \frac{1}{18}(1 - 9\cos\alpha + 8\cos^2\alpha) z^4 + \cdots , \end{aligned}$$

or one of its rotations. It is easy to see that f_1 belongs to the class $\mathcal{M}(\alpha)$ and

$$(f_1(z^k))^{1/k} = z + \frac{1}{k} z^{k+1} + \left(\frac{1}{2k}(1 - \cos\alpha) - \frac{1}{2}\frac{k-1}{k^2} \right) z^{2k+1} + \cdots .$$

The last equation shows that these inequalities are sharp. For the sharpness of the second inequality, we consider the function

$$f_2(z) := z^2 \exp\left\{ \int_0^z \frac{\mathcal{B}_\alpha(\xi^2) - 1}{\xi} \mathrm{d}\xi \right\} = z + \frac{1}{2} z^3 + \cdots \quad (z \in \Delta).$$

A simple calculation gives that

$$(f_2(z^k))^{1/k} = z + \frac{1}{2k} z^{2k+1} + \cdots .$$

Therefore the equality in the second inequality (8) holds for the k-th root transform of the above function f_2. This completes the proof of Theorem 2.1. □

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $\mathcal{A}$ is known as the Fekete-Szegö problem. In the recent years, many scholars have investigated the Fekete-Szegö problem for some certain subclasses of analytic functions, see for example [16, 24, 30, 33, 34, 42].

Letting $k = 1$ in the Theorem 2.1 we get the Fekete-Szegö inequality for the class $\mathcal{M}(\alpha)$ which we give in the following corollary.

Corollary 2.2. *Let $\alpha \in [\pi/2, \pi)$ and $f \in \mathcal{M}(\alpha)$. Then for any complex number μ we have*

$$\left| a_3 - \mu a_2^2 \right| \le \begin{cases} \frac{1}{2}(1 - \cos\alpha) - \mu, & \text{if } \mu \le -\frac{1}{2}\cos\alpha; \\ \frac{1}{2}, & \text{if } -\frac{1}{2}\cos\alpha \le \mu \le 1 - \frac{1}{2}\cos\alpha; \\ \frac{1}{2}(\cos\alpha - 1) + \mu, & \text{if } \mu \ge 1 - \frac{1}{2}\cos\alpha. \end{cases}$$

The result is sharp.

Putting $\alpha = \pi/2$ in the Corollary 2.2 we get the following.

Corollary 2.3. *Let the function f be given by* (1) *satisfies the inequality*

$$\left| \mathrm{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} - 1 \right| < \frac{\pi}{4} \quad (z \in \Delta).$$

Then for any complex number $\mu \in \mathbb{C}$ we have the following sharp inequalities

$$\left| a_3 - \mu a_2^2 \right| \le \begin{cases} \frac{1}{2} - \mu, & \text{if } \mu \le 0; \\ \frac{1}{2}, & \text{if } 0 \le \mu \le 1; \\ \mu - \frac{1}{2}, & \text{if } \mu \ge 1. \end{cases}$$

If we let $\alpha \to \pi^-$ in the Corollary 2.2, then we have:

Corollary 2.4. *If the function f is of the form* (1) *is starlike of order* $1/2$, *then for any complex number $\mu \in \mathbb{C}$ the following sharp inequalities hold true.*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 1-\mu, & \text{if } \mu \le \frac{1}{2}; \\ \frac{1}{2}, & \frac{1}{2} \le \mu \le \frac{3}{2}; \\ \mu - 1, & \text{if } \mu \ge \frac{3}{2}. \end{cases}$$

From (11) and (12) and the first case of the Lemma 1.4 we get.

Corollary 2.5. *If a function $f \in \mathcal{A}$ is of the form* (1) *belongs to the class $\mathcal{M}(\alpha)$ $(\pi/2 \le \alpha < \pi)$, then the following sharp inequalities hold.*

$$|a_2| \le 1 \quad \text{and} \quad |a_3| \le \frac{1}{2}(1 - \cos\alpha).$$

3. Coefficient estimate and Fekete-Szegö problem for the class $\mathcal{M}_\Sigma(\alpha)$

In this section, motivated by the Zaprawa's work (see [45]) we shall obtain the Fekete-Szegö problem for the class $\mathcal{M}_\Sigma(\alpha)$. Also, we obtain upper bounds for the first coefficients $|a_2|$ and $|a_3|$ of the function f is of the form (1) belonging to the class $\mathcal{M}_\Sigma(\alpha)$. The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \ge 4)$ is an open question. Here we recall that the initial coefficients estimate of the class of bi-univalent functions Σ was studied by Lewin in 1967 and he obtained the bound 1.51 for the modulus of the second coefficient $|a_2|$, see [18]. Afterward, Brannan and Clunie conjectured that $|a_2| \le \sqrt{2}$, see [6]. Finally, in 1969, Netanyahu [20] showed that $\max_{f\in\Sigma} |a_2| = 4/3$. For the another coefficients a_n $(n \ge 3)$ the sharp estimate is presumably still an open problem.

Moreover, we apply the same technique as in [4].

Theorem 3.1. *Let the function f given by* (1) *be in the class $\mathcal{M}_\Sigma(\alpha)$ and $\alpha \in [\pi/2, \pi)$. Then*

$$|a_2| \le \sqrt{\frac{2}{2 + \cos\alpha}} \tag{13}$$

and for any real number μ we have

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{2}, & \text{if } |1-\mu| \le \frac{1}{2}\left(1 + \frac{1}{2}\cos\alpha\right); \\ \frac{|1-\mu|}{1+\frac{1}{2}\cos\alpha}, & \text{if } |1-\mu| \ge \frac{1}{2}\left(1 + \frac{1}{2}\cos\alpha\right). \end{cases}$$

Proof. Let $f \in \mathcal{M}_\Sigma(\alpha)$ be of the form (1) and $g = f^{-1}$ be given by (2). Then by Definition 1.2, Lemma 1.1 and definition of subordination there exist two Schwarz functions $u : \Delta \to \Delta$ and $v : \Delta \to \Delta$ with the properties $u(0) = 0 = v(0)$, $|u(z)| < 1$ and $|v(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \mathcal{B}_\alpha(u(z)) \quad (z \in \Delta) \tag{14}$$

and

$$\frac{wg'(w)}{g(w)} = 1 + \mathcal{B}_\alpha(v(z)) \quad (z \in \Delta), \tag{15}$$

where $\mathcal{B}_\alpha$ is defined by (3). Now we define the functions k and l, respectively as follows

$$k(z) = \frac{1+u(z)}{1-u(z)} = 1 + k_1 z + k_2 z^2 + \cdots \quad (z \in \Delta)$$

and

$$l(z) = \frac{1+v(z)}{1-v(z)} = 1 + l_1 z + l_2 z^2 + \cdots \quad (z \in \Delta)$$

or equivalently

$$u(z) = \frac{k(z)-1}{k(z)+1} = \frac{1}{2}\left(k_1 z + \left(k_2 - \frac{1}{2}k_1^2\right) z^2 + \cdots\right) \tag{16}$$

and

$$v(z) = \frac{l(z)-1}{l(z)+1} = \frac{1}{2}\left(l_1 z + \left(l_2 - \frac{1}{2}l_1^2\right) z^2 + \cdots\right). \tag{17}$$

It is clear that the functions k and l belong to class $\mathcal{P}$ and $|k_i| \le 2$ and $|l_i| \le 2$ $(i = 1, 2, \ldots)$. From (4), (14)-(17), we have

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + \mathcal{B}_\alpha\left(\frac{k(z)-1}{k(z)+1}\right) \\ &= 1 + \frac{1}{2}A_1 k_1 z + \left(\frac{1}{2}A_1\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1}{4}A_2 k_1^2\right) z^2 + \cdots, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{wg'(w)}{g(w)} &= 1 + \mathcal{B}_\alpha\left(\frac{l(z)-1}{l(z)+1}\right) \\ &= 1 + \frac{1}{2}A_1 l_1 z + \left(\frac{1}{2}A_1\left(l_2 - \frac{1}{2}l_1^2\right) + \frac{1}{4}A_2 l_1^2\right) z^2 + \cdots. \end{aligned} \tag{19}$$

where $A_1 = 1$ and $A_2 = -\cos\alpha$. Thus, upon comparing the corresponding coefficients in (18) and (19), we obtain

$$a_2 = \frac{1}{2}A_1 k_1 = \frac{1}{2}k_1, \tag{20}$$

$$2a_3 - a_2^2 = \frac{1}{2}A_1\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1}{4}A_2 k_1^2 = \frac{1}{2}\left(k_2 - \frac{1}{2}k_1^2\right) - \frac{k_1^2}{4}\cos\alpha, \tag{21}$$

$$-a_2 = \frac{1}{2}A_1 l_1 = \frac{1}{2}l_1, \tag{22}$$

and

$$3a_2^2 - 2a_3 = \frac{1}{2}A_1\left(l_2 - \frac{1}{2}l_1^2\right) + \frac{1}{4}A_2 l_1^2 = \frac{1}{2}\left(l_2 - \frac{1}{2}l_1^2\right) - \frac{l_1^2}{4}\cos\alpha. \tag{23}$$

From equations (20) and (22), we can easily see that

$$k_1 = -l_1 \tag{24}$$

and

$$8a_2^2 = (k_1^2 + l_1^2).$$

If we add (21) to (23), we get

$$2a_2^2 = \frac{1}{2}\left[\left(k_2 - \frac{1}{2}k_1^2\right) + \left(l_2 - \frac{1}{2}l_1^2\right)\right] - \frac{1}{4}\cos\alpha\left(k_1^2 + l_1^2\right). \tag{25}$$

Substituting (20), (22) and (24) into (25), we obtain

$$k_1^2 = \frac{k_2 + l_2}{2(1 + (\cos\alpha)/2)}. \tag{26}$$

Now, (20) and (26) imply that

$$a_2^2 = \frac{k_2 + l_2}{2(2 + \cos\alpha)}. \tag{27}$$

Since $|k_2| \leq 2$ and $|l_2| \leq 2$, (27) implies that

$$|a_2| \leq \sqrt{\frac{2}{2 + \cos\alpha}},$$

which proves the first assertion (13) of Theorem 3.1. Now, if we subtract (23) from (21) and use of (24), we get

$$a_3 = a_2^2 + \frac{1}{8}(k_2 - l_2). \tag{28}$$

From (27) and (28) it follows that

$$a_3 - \mu a_2^2 = \left(\frac{1}{8} + \hbar(\mu)\right)k_2 + \left(\hbar(\mu) - \frac{1}{8}\right)l_2 \quad (\mu \in \mathbb{R}),$$

where

$$\hbar(\mu) := \frac{1 - \mu}{2(2 + \cos\alpha)} \quad (\mu \in \mathbb{R}).$$

Since $|k_2| \leq 2$ and $|l_2| \leq 2$, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2}, & \text{if } 0 \leq |\hbar(\mu)| \leq \frac{1}{8}; \\ \\ 4|\hbar(\mu)|, & \text{if } |\hbar(\mu)| \geq \frac{1}{8}. \end{cases}$$

This completes the proof. □

Taking $\mu = 0$ in the above Theorem 3.1 we get.

Corollary 3.2. *Let f of the form* (1) *be in the class $\mathcal{M}_\Sigma(\alpha)$. Then*

$$|a_3| \leq \frac{1}{1 + \frac{1}{2}\cos\alpha} \quad (\pi/2 \leq \alpha < \pi).$$

If we let $\alpha \to \pi^-$ in the Theorem 3.1, we get the following.

Corollary 3.3. *If the function f is of the form* (1) *belongs to the class $\mathcal{M}_\Sigma(1/2)$, then $|a_2| \leq 1$ and*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2}, & \textit{if } |1 - \mu| \leq \frac{1}{4}; \\ \\ 2|1 - \mu|, & \textit{if } |1 - \mu| \geq \frac{1}{4}, \end{cases}$$

where μ is real.

References

[1] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.
[2] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[3] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.
[4] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions Applied Mathematics Letters 25 (2012) 344–351.
[5] R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, The Fekete-Szegö coefficient functional for transforms of analytic functions, Bulletin of the Iranian Mathematical Society 35 (2011) 119–142.
[6] D. A. Brannan, J. G. Clunie (Eds.), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1–20, 1979), Academic Press, New York and London, 1980.
[7] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), Mathematical Analysis and Its Applications, Kuwait; February 18–21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also Studia Universitatis Babeş-Bolyai / Mathematica 31 (1986) 70–77.
[8] S. Bulut, Coefficient estimates for a class of analytic bi-univalent functions related to Pseudo-starlike functions, Miskolc Mathematical Notes 19 (2018) 149–156.
[9] M. Caglar, E. Deniz, H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, Turkish Journal of Mathematics 41 (2017) 694–706.
[10] M. Dorff, Convolutions of planar harmonic convex mapping, Complex Variables and Elliptic Equations 45 (2001) 263–271.
[11] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.
[12] R. Kargar, On logarithmic coefficients of certain starlike functions related to the vertical strip, The Journal Analysis 27 (2019) 985–995.
[13] R. Kargar, A. Ebadian, J. Sokół, Radius problems for some subclasses of analytic functions, Complex Analysis and Operator Theory 11 (2017), 1639–1649.
[14] R. Kargar, A. Ebadian, J. Sokół, On Booth lemniscate and starlike functions, Analysis and Mathematical Physics 9 (2019) 143–154.
[15] R. Kargar, H. Mahzoon, N. Kanzi, Some inequalities for a certain subclass of starlike functions, arXiv:1804.06435 [math.CV]
[16] B. Kowalczyk, A. Lecko, H. M. Srivastava, A note on the Fekete-Szegö problem for close-to-convex functions with respect to convex functions, Publications de l'Institut Mathématique 101 (115) (2017) 143–149.
[17] O. S. Kwon, Y. J. Sim, On coefficient problems for starlike functions related to vertical strip domain, Communications of the Korean Mathematical Society 34 (2019) 451–464.
[18] M. Lewin, On a coefficient problem for bi-univalent function, Proceedings of the American Mathematical Society 18 (1967) 63–68.
[19] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In Proceedings of the Conference On Complex Analysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
[20] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Archive for Rational Mechanics and Analysis 32 (1969) 100–112.
[21] P. Sharma and A. Nigam, The Fekete-Szegö problem for a Ma-Minda type class of bi-univalent functions associated with the Hohlov operator, Asian-European Journal of Mathematics 10 (2017) 1750052 (15 pages).
[22] H. M. Srivastava, Ş. Altinkaya, S. Yalçin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, Iranian Journal of Science and Technology. Transaction A 43 (2019), 1873–1879.
[23] H. M. Srivastava, S. Altinkaya, S. Yalcin, Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q-derivative operator, Filomat 32 (2018) 503–516.
[24] H. M. Srivastava, M. K. Aouf, A. O. Mostafa, Some properties of analytic functions associated with fractional q-calculus operators, Miskolc Mathematical Notes 20 (2019) 1245–126.
[25] H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, Journal of the Egyptian Mathematical Society 23 (2015) 242–246.
[26] H. M. Srivastava, S. Bulut, M. Çağlar M, N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27 (2013) 831–842.
[27] H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, Afrika Matematika 28 (2017) 693–706.
[28] H. M. Srivastava, S. Gaboury, F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Mathematica Scientia. Series B. English Edition 36 (2016) 863–871.
[29] H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Universitatis Apulensis. Mathematics-Informatics 41 (2015) 153–164.
[30] H. M. Srivastava, S. Hussain, A. Raziq, M. Raza, The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination, Carpathian Journal of Mathematics 34 (2018) 103–113.
[31] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. K. Misra, Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 112 (2018) 1487–1501.
[32] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, Studia Universitatis Babeş-Bolyai Mathematica 63 (2018) 419–436.
[33] H. M. Srivastava, C. Kizilateş, A parametric kind of the Fubini-type polynomials, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113 (2019) 3253–3267.
[34] H. M. Srivastava, A. K. Mishra, M. K. Das, The Fekete-Szegö problem for a subclass of close-to-convex functions, Complex Variables, Theory and Application 44 (2001) 145–163.

[35] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Applied Mathematics Letters 23 (2010) 1188–1192.
[36] H. M. Srivastava, F. M. Sakar, H.Ö. Güney, Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination, Filomat 34 (2018) 1313–1322.
[37] H. M. Srivastava, S. Sivasubramanian, R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions,Tbilisi Mathematical Journal 7 (2014) 1–10.
[38] H. M. Srivastava, S. Sümer Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat 29:8 (2015) 1839–1845.
[39] H. M. Srivastava, S. Sümer Eker, S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bulletin of the Iranian Mathematical Society 44 (2018) 149–157.
[40] H. M. Srivastava, A. K. Wanas, Initial Maclaurin coefficient bounds for new subclasses of analytic and m-Fold symmetric bi-univalent functions defined by a linear combination, Kyungpook Mathematical Journal 59 (2019) 493–503.
[41] Y. Sun, Z. -G. Wang, A. Rasila, Janusz Sokół, On a subclass of starlike functions associated with a vertical strip domain, Journal of Inequalities and Applications (2019) 2019: 35. https://doi.org/10.1186/s13660-019-1988-8
[42] H. Tang, H. M. Srivastava, S. Sivasubramanian, P. Gurusamy, The Fekete-Szegö functional problems for some classes of m-fold symmetric bi-univalent functions, Journal of Mathematical Inequalities **10** (2016) 1063–1092.
[43] Q. -H. Xu, H. -G. Xiao, H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Applied Mathematics and Computation 218 (2012) 11461–11465.
[44] Q. -H. Xu, Y. -C. Gui, H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Applied Mathematics Letters 25 (2012) 990–994.
[45] P. Zaprawa, On the Fekete-Szegö problem for classes of bi-univalent functions, Bulletin of the Belgian Mathematical Society-Simon Stevin 21 (2014) 169–178.