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Some Inequalities Involving Hilbert-Schmidt Numerical Radius on 2×2 Operator Matrices

Monire Hajmohamadi[a], **Rahmatollah Lashkaripour**[a]

[a]*Department of Mathematics, University of Sistan and Baluchestan, Zahedan, I.R.Iran*

Abstract. We present some inequalities related to the Hilbert-Schmidt numerical radius of 2×2 operator matrices. More precisely, we present a formula for the Hilbert-Schmidt numerical radius of an operator as follows:

$$w_2(T) = \sup_{\alpha^2+\beta^2=1} \|\alpha A + \beta B\|_2,$$

where $T = A + iB$ is the Cartesian decomposition of $T \in HS(\mathcal{H})$.

1. Introduction

Let $(\mathcal{H}, \langle .\,, .\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. The numerical radius of $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$w(T) := \sup\{| \langle Tx, x\rangle |: x \in \mathcal{H}, \| x \|= 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\,.\,\|$. In fact, for any $T \in \mathbb{B}(\mathcal{H})$, $\frac{1}{2}\|T\| \le w(T) \le \|T\|$; see [8]. For more facts about the numerical radius, we refer the reader to [4–6, 8]. A norm $N(\cdot)$ on $\mathbb{B}(\mathcal{H})$ is an algebra norm if $N(AB) \le N(A)N(B)$ for every $A, B \in \mathbb{B}(\mathcal{H})$. For $T \in \mathbb{B}(\mathcal{H})$, $\|T\|_2$ is the Hilbert-Schmidt norm of T and say that T belongs to the Hilbert-Schmidt class, $HS(\mathcal{H})$, if $\|T\|_2 = (\text{tr } (T^*T))^{1/2} < \infty$. Note that $\|\cdot\|_2$ is unitarily invariant, that is for every $T \in HS(\mathcal{H})$ and unitaries $U, V \in \mathbb{B}(\mathcal{H})$, we have $\|UTV\|_2 = \|T\|_2$.
Recently Abu-Omar et.al [1] defined the Hilbert-Schmidt numerical radius as follows:

$$w_2(T) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\|_2,$$

in which $w_2(\cdot)$ is a norm on $\mathbb{B}(\mathcal{H})$. This norm is equivalent to the Hilbert-Schmidt norm $\|\,.\,\|_2$. In fact, for any $T \in HS(\mathcal{H})$,

$$\frac{1}{\sqrt{2}}\|T\|_2 \le w_2(T) \le \|T\|_2. \tag{1}$$

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Email addresses: monire.hajmohamadi@yahoo.com (Monire Hajmohamadi), lashkari@hamoon.usb.ac.ir (Rahmatollah Lashkaripour)

If T is normal and the sequence of its nonzero eigenvalues have the same argument, then $w_2(T) = \|T\|_2$ and if tr $(T)^2 = 0$, then $w_2(T) = \frac{1}{\sqrt{2}}\|T\|_2$; see[1]. Hence, the inequalities in (1) are sharp. There is more properties about the Hilbert-Schmidt numerical radius. For example $w_2(\cdot)$ is self-adjoint, that is for any $T \in \mathbb{B}(\mathcal{H})$, we have $w_2(T) = w_2(T^*)$. Also, $w_2(\cdot)$ is weakly unitarily invariant, that is for any unitary $U \in \mathbb{B}(\mathcal{H})$, $w_2(UTU^*) = w_2(T)$.

Let $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ be Hilbert spaces, and consider $\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j$. With respect to this decomposition, every operator $T \in \mathbb{B}(\mathcal{H})$ has an $n \times n$ operator matrix representation $T = [T_{ij}]$ with entries $T_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, the space of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$. Operator matrices provide a usual tool for studying Hilbert space operators, which have been extensively studied in the literatures.
The authors in [2] obtained several Hilbert-Schmidt numerical radius inequalities, including lower and upper bounds for 2×2 operator matrices. For example, on off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, we have the following inequalities:

$$\frac{\max(w_2(A+B), w_2(A-B))}{\sqrt{2}} \le w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \frac{w_2(A+B) + w_2(A-B)}{\sqrt{2}}, \tag{2}$$

where $A, B \in HS(\mathcal{H})$.

In this paper we establish some Hilbert-Schmidt numerical radius inequalities, which are based on off-diagonal parts of 2×2 operator matrices. We also, find some upper bounds for 2×2 operator matrices.

2. Main results

In this section, we state some the Hilbert-Schmidt numerical radius inequalities for 2×2 operator matrices defined on $\mathcal{H}_1 \oplus \mathcal{H}_2$. To prove our results, we need the following lemma, which known in [1].

Lemma 2.1. *Let A, B, C, D belongs to the Hilbert-Schmidt class $HS(\mathcal{H})$. Then the following statements hold:*

(a) $w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \le \sqrt{w_2^2(A) + w_2^2(D)}$. *In particular, if A, D are self-adjoint, then* $w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \sqrt{w_2^2(A) + w_2^2(D)}$;

(b) $w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right)$;

(c) $w_2\left(\begin{bmatrix} 0 & B \\ e^{i\theta}C & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$, $\forall \theta \in \mathbb{R}$;

(d) $w_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \le \sqrt{w_2^2(A+B) + w_2^2(A-B)}$. *In the cases A, B are self-adjoint the inequality becomes equality.*

In particular,

$$w_2\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = \sqrt{2}w_2(B).$$

Lemma 2.2. *Let $A_i, X_i \in HS(\mathcal{H}) (1 \le i \le n)$. Then*

$$w_2\left(\sum_{i=1}^{n} A_i X_i A_i^*\right) \le \left(\sum_{i=1}^{n} \|A_i\|_2 \|A_i^*\|_2) w_2(X_i\right).$$

In particular for any $A, X \in HS(\mathcal{H})$,

$$w_2(AXA^*) \le \|A\|_2^2 w_2(X). \tag{3}$$

Proof. We have,

$$\left\| Re(e^{i\theta} \sum_{i=1}^{n} A_i X_i A_i^*) \right\|_2 = \left\| (\sum_{i=1}^{n} A_i Re(e^{i\theta} X_i) A_i^*) \right\|_2 \leq \sum_{i=1}^{n} \|A_i\|_2 \|A_i^*\|_2 \|Re(e^{i\theta}) X_i\|_2.$$

So by taking the supremum over θ, we obtain

$$w_2\left(\sum_{i=1}^{n} A_i X_i A_i^*\right) \leq \left(\sum_{i=1}^{n} \|A_i\|_2 \|A_i^*\|_2\right) w_2(X_i)$$

as required. □

Now, we present our first result.

Theorem 2.3. *Let $A, B, X \in HS(\mathcal{H})$. Then*

$$w_2(BX^*A^* + AXB^*) \leq (2\|A\|_2\|B\|_2)\frac{[w_2(X + X^*) + w_2(X - X^*)]}{\sqrt{2}}. \tag{4}$$

Proof. Assume that $C = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$, we have

$$\begin{aligned} w_2(BX^*A^* + AXB^*) &= w_2\left(\begin{bmatrix} BX^*A^* + AXB^* & 0 \\ 0 & 0 \end{bmatrix}\right) && \text{(by Lemma 2.1}(a)) \\ &= w_2(CZC^*) \\ &\leq \|C\|_2^2 w_2(Z) && \text{(by (3))} \\ &= (\|A\|_2^2 + \|B\|_2^2) w_2(Z) \\ &\leq (\|A\|_2^2 + \|B\|_2^2)\frac{w_2(X + X^*) + w_2(X - X^*)}{\sqrt{2}} && \text{(by (2))} \end{aligned}$$

Note that, if we replace A by tA and B by $\frac{1}{t}B$ for any $t > 0$, then $\min_{t>0} t^2\|A\|_2^2 + \frac{1}{t^2}\|B\|_2^2 = \min_{t>0} \frac{t^4\|A\|_2^2+\|B\|_2^2}{t^2} = 2\|A\|_2\|B\|_2$. So

$$w_2(BX^*A^* + AXB^*) \leq (2\|A\|_2\|B\|_2)\frac{w_2(X + X^*) + w_2(X - X^*)}{\sqrt{2}}.$$

□

Remark 2.4. *By putting $X^* = -X$ in* (4) *and for $A \in HS(\mathcal{H})$, we have the following inequality:*

$$w_2(AX - XA^*) \leq 2\sqrt{2}\|A\|_2 w_2(X). \tag{5}$$

Remark 2.5. *For any self-adjoint operator X, we have the following inequality:*

$$w_2(BXA^* + AXB^*) \leq 2\sqrt{2}(\|A\|_2\|B\|_2) w_2(X).$$

In the following we obtain an upper bound for an 2×2 off-diagonal operator matrix.

Theorem 2.6. *Let $A, B \in HS(\mathcal{H})$. Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq w_2(A) + w_2(B).$$

Proof. Note that $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}$, so by applying the properties of w_2, we have

$$\begin{aligned} w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq w_2\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right\|_2 + \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}\right\|_2 \\ &= \frac{1}{\sqrt{2}}\|A\|_2 + \frac{1}{\sqrt{2}}\|B\|_2 \\ &\leq w_2(A) + w_2(B). \end{aligned}$$

□

Aldalabih and Kittaneh in [2] obtained some upper bounds for the Hilbert-Schmidt numerical radius of operator matrix $\begin{bmatrix} A & B \\ A & B \end{bmatrix}$. Now, we find an upper bound for the Hilbert-Schmidt numerical radius of the operator matrix $\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}$.

Theorem 2.7. *Let $A, B \in HS(\mathcal{H})$. Then*

$$\begin{aligned} \frac{1}{\sqrt{2}}\max(w_2(A-B), w_2(A+B)) &\leq w_2\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right) \\ &\leq \frac{1}{\sqrt{2}}\sqrt{w_2^2(A) + w_2^2(B)}\max(w_2(A-B), w_2(A+B)). \end{aligned} \tag{6}$$

Proof. Notice

$$\begin{aligned} w_2\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right) &\geq w_2\left(\begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix}\right) \\ &\geq \frac{\max(w_2(A-B), w_2(A+B)}{\sqrt{2}}. \qquad \text{(by (2))} \end{aligned}$$

For the second inequality in (6), we have

$$\begin{aligned} w_2\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right) &\leq w_2\left(\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & B \\ -A & 0 \end{bmatrix}\right) \\ &\leq \sqrt{w_2^2(A) + w_2^2(B)}\frac{\max(w_2(A-B), w_2(A+B))}{\sqrt{2}} \\ &\qquad \text{(by Lemma 2.1}(a)\text{ and (2)).} \end{aligned}$$

□

Remark 2.8. *If $A, B \in HS(\mathcal{H})$ are self-adjoint, then*

$$\sqrt{w_2^2(A)+w_2^2(B)} \le w_2\left(\begin{bmatrix} A & B \\ -A & -B \end{bmatrix}\right) \le \frac{1}{\sqrt{2}}\sqrt{w_2^2(A)+w_2^2(B)}\max(w_2(A-B), w_2(A+B)).$$

Remark 2.9. *Note that in the proof of [2, Theorem 4] was seen if* $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, *then* $\frac{1}{2}\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix} = U^*\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U.$
So $\frac{1}{2}w_2\left(\begin{bmatrix} A+B & A-B \\ -(A-B) & -(A+B) \end{bmatrix}\right) = w_2\left(U^*\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}U\right) = w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$. *Thus Theorem 2.7 and [2, Theorem 4] are equivalent.*

Remark 2.10. *For* $A = B$, *we have* $\sqrt{2}w_2(A) \le w_2\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right) \le 2w_2(A)$. *Since* $\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}^2 = 0$ *so* $w_2\left(\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right\|_2$. *Thus* $w_2(A) \le \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right\|_2 \le 2w_2(A)$. *Also, its known that* $\left\|\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}\right\|_2 = 2\|A\|_2$, *so*
$\|A\|_2 \le \sqrt{2}w_2(A)$. *We reach to first inequality in* (1).

In the next theorem we obtain some new upper and lower bounds for $w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

Theorem 2.11. *Let $A, B, C, D \in HS(\mathcal{H})$.*
(i) *If A, D are self-adjoint, then*

$$w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \ge \max\left(\sqrt{w_2^2(A)+w_2^2(D)}, \frac{w_2(B+C)}{\sqrt{2}}, \frac{w_2(B-C)}{\sqrt{2}}\right).$$

(ii)

$$w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \sqrt{w_2^2(A)+w_2^2(D)} + \frac{w_2(B+C)+w_2(B-C)}{\sqrt{2}}.$$

Proof. (i) Let $A, D \in HS(\mathcal{H})$ be self-adjoint. Since $w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \ge w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)$ and $w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \ge w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$. So

$$\begin{aligned} w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\ge \max\left(w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right), w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right) \\ &\ge \max\left(\sqrt{w_2^2(A)+w_2^2(D)}, \frac{\max(w_2(B+C), w_2(B-C))}{\sqrt{2}}\right) \\ &\qquad \text{(by Lemma 2.1}(a)\text{ and (2))} \\ &= \max\left(\sqrt{w_2^2(A)+w_2^2(D)}, \frac{w_2(B+C)}{\sqrt{2}}, \frac{w_2(B-C)}{\sqrt{2}}\right). \end{aligned}$$

(ii) We have

$$\begin{aligned} w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &\leq w_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \\ &\leq \sqrt{w_2^2(A) + w_2^2(D)} + \frac{w_2(B+C) + w_2(B-C)}{\sqrt{2}} \\ &\qquad \text{(by Lemma 2.1}(a)\text{ and (2)).} \end{aligned}$$

□

Remark 2.12. *By letting $A, B \in HS(\mathcal{H})$ with A be self-adjoint, we have*

$$\sqrt{2}\max(w_2(A), w_2(B)) \leq w_2\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right) \leq \sqrt{2}(w_2(A) + w_2(B)). \tag{7}$$

Lemma 2.13. *Let $A, B \in HS(\mathcal{H})$. Then*

$$w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}} \sup_{\theta\in\mathbb{R}} \|e^{i\theta}A + e^{-i\theta}B^*\|_2.$$

Proof. For any $T \in \mathbb{B}(\mathcal{H})$, we have $w_2(T) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\|_2 = \frac{1}{2}\sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + e^{-i\theta}T^*\|_2$. By letting $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, we have

$$\begin{aligned} w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}} \left\| Re\left(\begin{bmatrix} 0 & e^{i\theta}A \\ e^{i\theta}B & 0 \end{bmatrix}\right)\right\|_2 \\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & e^{i\theta}(A + e^{-2i\theta}B^*) \\ e^{-i\theta}(A + e^{-2i\theta}B)^* & 0 \end{bmatrix}\right\|_2 \\ &= \frac{\sqrt{2}}{2}\sup_{\theta\in\mathbb{R}} \|e^{i\theta}A + e^{-i\theta}B^*\|_2 \qquad \left(\text{since } \left\|\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right\|_2 = \sqrt{2}\|A\|_2\right). \end{aligned}$$

□

3. Applications

In this section, we present some applications of some given results. At first we start by an application of [1, Theorem 5].

Lemma 3.1. *[1] Let $A, B, X \in \mathbb{B}(\mathcal{H})$. If $N(\cdot)$ is an algebra norm, then*

$$w_N(AXB + B^*XA^*) \leq (N(A)N(B) + N(B^*)N(A^*))w_N(X). \tag{8}$$

There is an special case of (8), when $X = I$(identity operator matrix) and $N(\cdot)$ the Hilbert Schmidt norm $\|\cdot\|_2$ as following:

$$w_2(AX + XA^*) \leq 2\|A\|_2 w_2(X). \tag{9}$$

Now, as an application of (9) we obtain a lower bound for $w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

Theorem 3.2. *Let $A, B, C, D \in HS(\mathcal{H})$ such that B, C be self-adjoint. Then*

$$w_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geq \frac{1}{2}\max(w_2(A+D), w_2(B+C)).$$

Proof. Put $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $A = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ in (9). So

$$\begin{aligned} w_2(X) &\geq \frac{1}{2\sqrt{2}} w_2\left(\begin{bmatrix} B+C & A+D \\ A+D & B+C \end{bmatrix}\right) \\ &\geq \frac{1}{2\sqrt{2}}\max\left(w_2\left(\begin{bmatrix} B+C & 0 \\ 0 & B+C \end{bmatrix}\right), w_2\left(\begin{bmatrix} 0 & A+D \\ A+D & 0 \end{bmatrix}\right)\right) \\ &\geq \frac{1}{2\sqrt{2}}\max(\sqrt{2}w_2(B+C), \frac{2}{\sqrt{2}}w_2(A+D)) \\ &= \frac{1}{2}\max(w_2(B+C), w_2(A+D)). \end{aligned}$$

□

Applying (7) in the next theorem, we state an application of (5).

Theorem 3.3. *Let $A, B \in HS(\mathcal{H})$ such that B be self-adjoint. Then*

$$w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \geq \frac{1}{4}\max(w_2(A), w_2(B)).$$

Proof. Let $X = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. We have

$$\begin{aligned} w_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) = w_2(X) &\geq \frac{1}{4}w_2(YX - XY^*) \\ &= \frac{1}{4}w_2\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix}\right) \\ &\geq \frac{\sqrt{2}}{4}\max(w_2(A), w_2(B)). \end{aligned}$$

□

The following result gives a form of the Hilbert-Schmidt numerical radius by using Cartesian decomposition. A related result has been given in [7].

Theorem 3.4. *Let $T = A + iB$ be the Cartesian decomposition of $T \in HS(\mathcal{H})$. Then for any $\alpha, \beta \in \mathbb{R}$*

$$w_2(T) = \sup_{\alpha^2+\beta^2=1} \|\alpha A + \beta B\|_2.$$

In particular,

$$w_2(T) \geq \frac{1}{2}\|T + T^*\|_2 \text{ and } w_2(T) \geq \frac{1}{2}\|T - T^*\|_2. \tag{10}$$

Proof. It is known $w_2(T) = \sup_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\|_2$. We have

$$\begin{aligned} Re(e^{i\theta}T) &= \frac{e^{i\theta}T + e^{-i\theta}T^*}{2} \\ &= \frac{(\cos\theta + i\sin\theta)T + (\cos\theta - i\sin\theta)T^*}{2} \\ &= \cos\theta(\frac{T+T^*}{2}) + \sin\theta(\frac{T-T^*}{2i}) = A\cos\theta + B\sin\theta. \end{aligned}$$

By putting $\alpha = \cos\theta$ and $\beta = \sin\theta$, we get the desired result. In particular for $\alpha = 1, \beta = 0$ and for $\alpha = 0, \beta = 1$ we get the result. □

Lemma 3.5. *[10] Let $X \geq mI > 0$ for some positive real number m and Y be in the associated ideal corresponding to a unitarily invariant norm $|||\cdot|||$. Then*

$$m|||Y||| \leq \frac{1}{2}|||XY + YX|||. \tag{11}$$

Proposition 3.6. *Let $A, B, X \in \mathbb{M}_2$ be Hermitian and $0 < mI_2 \leq X$ for some positive real number m. Then*

$$\frac{m}{\sqrt{2}}\|A - B\|_2 \leq w_2(AX - XB) \leq \|AX - XB\|_2. \tag{12}$$

Proof. The proof is similar to the technique used in reference [7]. □

Theorem 3.7. *Let $A, B, X \in HS(\mathcal{H})$ and $0 < mI_2 \leq X$ for some positive real number m. Then*

$$m\|A - B\|_2 \leq w_2\left(\begin{bmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{bmatrix}\right) \leq \frac{\|AX - XB\|_2 + \|A^*X - XB^*\|_2}{\sqrt{2}}. \tag{13}$$

Proof. By applying inequality (12) for self-adjoint operator matrices $A_1 = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$, and positive operator matrix $X_1 = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$, we get

$$\frac{m}{\sqrt{2}}\|A_1 - B_1\|_2 \leq w_2(A_1X_1 - X_1B_1).$$

So

$$\begin{aligned} m\|A - B\|_2 &\leq w_2\left(\begin{bmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{bmatrix}\right) \\ &\leq w_2\left(\begin{bmatrix} 0 & AX - XB \\ 0 & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & 0 \\ A^*X - XB^* & 0 \end{bmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\|AX - XB\|_2 + \frac{1}{\sqrt{2}}\|A^*X - XB^*\|_2 \\ &\quad (\text{since } \begin{bmatrix} 0 & AX - XB \\ 0 & 0 \end{bmatrix}^2 = 0 \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ A^*X - XB^* & 0 \end{bmatrix}^2 = 0). \end{aligned}$$

□

Remark 3.8. *Note that inequalities* (12) *are special cases of inequalities* (13).

We have another version of Theorem 3.7 as follows.

Theorem 3.9. *Let $A, B \in HS(\mathcal{H})$ and $0 < mI \leq X$ for some positive real number m. Then*

$$\frac{m}{\sqrt{2}}\|Re(A) - Re(B)\|_2 \leq w_2(Re(A)X - XRe(B)) \leq \frac{1}{2}\|AX - XB\|_2 + \|XA - BX\|_2.$$

For its proof we use from $w_2(A+B) = \frac{1}{\sqrt{2}}w_2\left(\begin{bmatrix} 0 & A+B \\ A+B & 0 \end{bmatrix}\right) \leq \frac{1}{\sqrt{2}}\left[w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) + w_2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)\right] = \sqrt{2}w_2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$.

Proof. From Proposition 3.6, we have

$$\begin{aligned} \frac{m}{\sqrt{2}}\|Re(A) - Re(B)\|_2 &\leq w_2(Re(A)X - XRe(B)) \\ &= \frac{w_2((AX - XB) + (A^*X - XB^*))}{2} \\ &\leq \frac{\sqrt{2}}{2}w_2\left(\begin{bmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{bmatrix}\right) \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\sup_{\theta\in\mathbb{R}}\|e^{i\theta}AX - XB + e^{-i\theta}(A^*X - XB^*)^*\|_2 \\ &\quad \text{(by Lemma 2.13)} \\ &\leq \frac{1}{2}\|AX - XB\|_2 + \|XA - BX\|_2. \end{aligned}$$

□

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