



On the Complex Hermite Polynomials

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Abstract. In this paper we use a set of partial differential equations to prove an expansion theorem for multiple complex Hermite polynomials. This expansion theorem allows us to develop a systematic and completely new approach to the complex Hermite polynomials. Using this expansion, we derive the Poisson Kernel, the Nielsen type formula, the addition formula for the complex Hermite polynomials with ease. A multilinear generating function for the complex Hermite polynomials is proved.

1. Introduction and preliminary

With the aid of a system of partial differential equations, we proved an expansion theorem for the bivariate Hermite polynomials in [13, Theorem 1.8]. This expansion theorem allows us to develop a systematic method to prove the identities involving the Hermite polynomials. I find the idea of [13] has universal significance, which stimulates us to develop a new method to treat the complex Hermite polynomials.

Definition 1.1. For complex numbers x, y and non-negative integers m, n , the complex Hermite polynomials are defined by

$$H_{m,n}(x, y) = \sum_{k=0}^{m \wedge n} (-1)^k k! \binom{m}{k} \binom{n}{k} x^{m-k} y^{n-k},$$

where $m \wedge n = \min\{m, n\}$.

The polynomials $H_{m,n}(z, \bar{z})$ were first considered by Itô [12] in his study of complex multiple Wiener integrals and their applications to normal stochastic processes. These polynomials are also applied in [1] to coherent states, and in [19], [20] to quantum optics and quasi-probabilities respectively. Several papers about this topic have been published in recent year, see for example [6], [9], [10], [11].

For our purpose, we need extend slightly the complex Hermite polynomials by adding an extra parameter to them, and for convenience, we still call the extended complex Hermite polynomials as the complex Hermite polynomials.

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Definition 1.2. For any complex numbers x, y and z , the complex Hermite polynomials $H_{m,n}(x, y, z)$ are defined as

$$H_{m,n}(x, y, z) = \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} x^{m-k} y^{n-k} z^k.$$

It is obvious that when $z = -1$, $H_{m,n}(x, y, z)$ reduce to the usual complex Hermite polynomials $H_{m,n}(x, y)$. By a simple calculation, we also find the following proposition.

Proposition 1.3. The polynomials $H_{m,n}(x, y, z)$ and the polynomials $H_{m,n}(x, y)$ satisfy

$$H_{m,n}(x, y, z) = (\sqrt{-z})^{m+n} H_{m,n}\left(\frac{x}{\sqrt{-z}}, \frac{y}{\sqrt{-z}}\right).$$

Thus we may regard $H_{m,n}(x, y, z)$ as a variant form of the usual complex Hermite polynomials $H_{m,n}(x, y)$. Although $H_{m,n}(x, y, z)$ are equivalent to the complex Hermite polynomials $H_{m,n}(x, y)$, the former have a richer mathematical structure than the latter.

Remark 1.4. The polynomials $H_{m,n}(x, y, -z)$ have been considered by Datolli et al. [5, pp.23–24], and several basic properties about $H_{m,n}(x, y, -z)$ were obtained by them.

To state our expansion theorem, we now introduce the definition of the k -fold complex Hermite series in several variables.

Definition 1.5. The k -fold complex Hermite series are defined as

$$\sum_{m_1, n_1, \dots, m_k, n_k=0}^{\infty} \lambda_{m_1, n_1, \dots, m_k, n_k} H_{m_1, n_1}(x_1, y_1, z_1) \cdots H_{m_k, n_k}(x_k, y_k, z_k),$$

where $\lambda_{m_1, n_1, \dots, m_k, n_k}$ are complex numbers independent of $x_1, y_1, z_1, \dots, x_k, y_k, z_k$.

The principal result of this paper is the following expansion theorem for the analytic functions in several variables.

Theorem 1.6. If $f(x_1, y_1, z_1, \dots, x_k, y_k, z_k)$ is a $3k$ -variable analytic function at $(0, 0, \dots, 0) \in \mathbb{C}^{3k}$, then, f can be expanded in an absolutely and uniformly convergent k -fold complex Hermite series, if and only if, for $j \in \{1, 2, \dots, k\}$, f satisfies the partial differential equations

$$\frac{\partial f}{\partial z_j} = \frac{\partial^2 f}{\partial x_j \partial y_j}.$$

This theorem is a powerful tool for proving formulas involving the complex Hermite polynomials, which allows us to develop a systematic method to derive identities involving the complex Hermite polynomials.

2. The proof of Theorem 1.6

Using $\exp(sx + ty + stz) = \exp(sx) \exp(ty) \exp(stz)$ and the Maclaurin expansion for the exponential function, one can easily derive Proposition 2.1.

Proposition 2.1. For any complex numbers x, y, z and s, t , we have

$$\sum_{m,n=0}^{\infty} H_{m,n}(x, y, z) \frac{s^m t^n}{m!n!} = \exp(sx + ty + stz).$$

In order to prove Theorem 1.6, we need the following three propositions.

Proposition 2.2. *The complex Hermite polynomials $H_{m,n}(x, y, z)$ satisfy the partial differential equation*

$$\frac{\partial H_{m,n}}{\partial z} = \frac{\partial^2 H_{m,n}}{\partial x \partial y}.$$

Proof. Applying the partial differential operator $\partial^2 / \partial x \partial y$ to act both sides of the equation in Proposition 2.1, we find that

$$\sum_{m,n=0}^{\infty} \frac{\partial^2 H_{m,n}}{\partial x \partial y} \frac{s^m t^n}{m!n!} = st \exp(sx + ty + stz).$$

Upon differentiating both sides of the equation in Proposition 2.2 with respect to z , we arrive at

$$\sum_{m,n=0}^{\infty} \frac{\partial H_{m,n}}{\partial z} \frac{s^m t^n}{m!n!} = st \exp(sx + ty + stz).$$

A comparison of these two equations immediately gives us that

$$\sum_{m,n=0}^{\infty} \frac{\partial H_{m,n}}{\partial z} \frac{s^m t^n}{m!n!} = \sum_{m,n=0}^{\infty} \frac{\partial^2 H_{m,n}}{\partial x \partial y} \frac{s^m t^n}{m!n!}.$$

Equating the coefficients of like powers of s and t , we complete the proof of the proposition. \square

Proposition 2.3. *The following exponential operator representation for the complex Hermite polynomials holds:*

$$H_{m,n}(x, y, z) = \exp\left(z \frac{\partial^2}{\partial x \partial y}\right) \{x^m y^n\}.$$

This operational identity for the complex Hermite polynomials is equivalent to [5, Eq.(1.5.2d)].

Remark 2.4. Using the exponential operator $\exp\left(-z \frac{\partial^2}{\partial x \partial y}\right)$ to act both sides of the equation in Proposition 2.3, we have

$$\begin{aligned} x^m y^n &= \exp\left(-z \frac{\partial^2}{\partial x \partial y}\right) \{H_{m,n}(x, y)\} \\ &= \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} \exp\left(-z \frac{\partial^2}{\partial x \partial y}\right) \{x^{m-k} y^{n-k}\} z^k \\ &= \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} H_{m-k}(x, y, -z) H_{n-k}(x, y, -z) z^k. \end{aligned} \tag{2.1}$$

Proposition 2.5. *If $f(x_1, x_2, \dots, x_k)$ is analytic at the origin $(0, 0, \dots, 0) \in \mathbb{C}^k$, then, f can be expanded in an absolutely and uniformly convergent power series,*

$$f(x_1, x_2, \dots, x_k) = \sum_{n_1, n_2, \dots, n_k=0}^{\infty} \lambda_{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

This proposition can be found in the standard textbooks for complex analysis in several variables (see, for example [14, p. 5, Proposition 1]).

Now we begin to prove Theorem 1.6 with the help of the above three propositions.

Proof. The theorem can be proved by mathematical induction. We first prove the theorem for the case $k = 1$.

Since f is analytic at $(0, 0, 0)$, we know that f can be expanded in an absolutely and uniformly convergent power series in a neighborhood of $(0, 0)$. Thus there exists a sequence $\{\lambda_{m,n,p}\}$ independent of x_1, y_1 and z_1 such that

$$f(x_1, y_1, z_1) = \sum_{m,n,p=0}^{\infty} \lambda_{m,n,p} x_1^m y_1^n z_1^p. \tag{2.2}$$

The series on the right-hand side of the equation above is absolutely and uniformly convergent.

Upon substituting the equation above into the following partial differential equation:

$$\frac{\partial f}{\partial z_1} = \frac{\partial^2 f}{\partial x_1 \partial y_1},$$

and then using the identities, $D_{z_1}\{z_1^p\} = pz_1^{p-1}$, in the resulting equation, we obtain

$$\sum_{m,n,p=0}^{\infty} p\lambda_{m,n,p} x_1^m y_1^n z_1^{p-1} = \frac{\partial^2}{\partial x_1 \partial y_1} \left\{ \sum_{m,n,p=0}^{\infty} \lambda_{m,n,p} x_1^m y_1^n z_1^p \right\}.$$

Upon equating the coefficients of z_1^{p-1} on both sides of the equation, we deduce that

$$p \sum_{m,n=0}^{\infty} \lambda_{m,n,p} x_1^m y_1^n = \frac{\partial^2}{\partial x_1 \partial y_1} \left\{ \sum_{m,n=0}^{\infty} \lambda_{m,n,p-1} x_1^m y_1^n \right\}.$$

If we iterate this relation $(p - 1)$ times and interchange the order of differentiation and summation, we deduce that

$$\begin{aligned} \sum_{m,n=0}^{\infty} \lambda_{m,n,p} x_1^m y_1^n &= \frac{1}{p!} \frac{\partial^{2p}}{\partial x_1^p \partial y_1^p} \left\{ \sum_{m,n=0}^{\infty} \lambda_{m,n,0} x_1^m y_1^n \right\} \\ &= \frac{1}{p!} \sum_{m,n=0}^{\infty} \lambda_{m,n,0} \frac{\partial^{2p}}{\partial x_1^p \partial y_1^p} \{x_1^m y_1^n\}. \end{aligned}$$

Substituting this equation into (2.2) and using a simple calculation, we conclude that

$$\begin{aligned} f(x_1, y_1, z_1) &= \sum_{p=0}^{\infty} z_1^p \sum_{m,n=0}^{\infty} \lambda_{m,n,p} x_1^m y_1^n \\ &= \sum_{p=0}^{\infty} \frac{z_1^p}{p!} \sum_{m,n=0}^{\infty} \lambda_{m,n,0} \frac{\partial^{2p}}{\partial x_1^p \partial y_1^p} \{x_1^m y_1^n\}. \end{aligned}$$

Interchanging the order of summation and using Proposition 2.3, we deduce that

$$\begin{aligned} f(x_1, y_1, z_1) &= \sum_{m,n=0}^{\infty} \lambda_{m,n,0} \exp\left(z_1 \frac{\partial^2}{\partial x_1 \partial y_1}\right) \{x_1^m y_1^n\} \\ &= \sum_{m,n=0}^{\infty} \lambda_{m,n,0} H_{m,n}(x_1, y_1, z_1). \end{aligned}$$

This indicates that $f(x_1, y_1, z_1)$ can be expanded in terms of $H_{m,n}(x_1, y_1, z_1)$.

Conversely, if $f(x_1, y_1, z_1)$ can be expanded in terms of $H_{m,n}(x_1, y_1, z_1)$, then, using Proposition 2.2, we find that $f(x_1, y_1, z_1)$ satisfies the partial differential equation

$$\frac{\partial f}{\partial z_1} = \frac{\partial^2 f}{\partial x_1 \partial y_1}.$$

This shows that Theorem 1.6 holds for the case with $k = 1$.

Now, we assume that the theorem is true for the case $k - 1$ and consider the case k . If we regard $f(x_1, y_1, z_1, \dots, x_k, y_k, z_k)$ as a function of x_1, y_1 and z_1 , then, f is analytic at $(0, 0, 0)$ and satisfies the partial differential equation

$$\frac{\partial f}{\partial z_1} = \frac{\partial^2 f}{\partial x_1 \partial y_1}.$$

Hence there exists a sequence $\{c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)\}$ independent of x_1, y_1 and z_1 such that

$$\begin{aligned} f(x_1, y_1, z_1, \dots, x_k, y_k, z_k) & \tag{2.3} \\ &= \sum_{m_1, n_1=0}^{\infty} c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k) H_{m_1, n_1}(x_1, y_1, z_1). \end{aligned}$$

Setting $z_1 = 0$ in the equation and using the obvious equation $H_{m_1, n_1}(x_1, y_1, 0) = x_1^{m_1} y_1^{n_1}$, we obtain

$$\begin{aligned} f(x_1, y_1, 0, \dots, x_k, y_k, z_k) & \\ &= \sum_{m_1, n_1=0}^{\infty} c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k) x_1^{m_1} y_1^{n_1}. \end{aligned}$$

Using the Maclaurin expansion for analytic functions of two variables, we immediately deduce that

$$\begin{aligned} c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k) & \\ &= \frac{\partial^{m_1+n_1} f(x_1, y_1, 0, \dots, x_k, y_k, z_k)}{m_1! n_1! \partial x_1^{m_1} \partial y_1^{n_1}} \Big|_{x_1=y_1=0}. \end{aligned}$$

Since $f(x_1, y_1, z_1, \dots, x_k, y_k, z_k)$ is analytic at $(0, \dots, 0) \in \mathbb{C}^{2k}$, from the above equation, we know that $c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)$ is analytic at

$$(x_2, y_2, z_2, \dots, x_k, y_k, z_k) = (0, \dots, 0) \in \mathbb{C}^{3k-3}.$$

Substituting (2.3) into the partial differential equations in Theorem 1.6, we find that for $j = 2, \dots, k$,

$$\begin{aligned} \sum_{n_1=0}^{\infty} \frac{\partial c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)}{\partial z_j} H_{m_1, n_1}(x_1, y_1, z_1) & \\ &= \sum_{m_1, n_1=0}^{\infty} \frac{\partial^2 c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)}{\partial x_j \partial y_j} H_{m_1, n_1}(x_1, y_1, z_1). \end{aligned}$$

By equating the coefficients of $H_{m_1, n_1}(x_1, y_1, z_1)$ in the above equation, we find that for $j = 2, \dots, k$,

$$\frac{\partial c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)}{\partial z_j} = \frac{\partial^2 c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k)}{\partial x_j \partial y_j}.$$

Thus by the inductive hypothesis, there exists a sequence $\lambda_{m_1, n_1, \dots, m_k, n_k}$ independent of $x_2, y_2, z_2, \dots, x_k, y_k, z_k$ (of course independent of x_1, y_1 and z_1) such that

$$c_{m_1, n_1}(x_2, y_2, z_2, \dots, x_k, y_k, z_k) = \sum_{m_1, n_1, \dots, m_k, n_k=0}^{\infty} \lambda_{m_1, n_1, \dots, m_k, n_k} H_{m_2, n_2}(x_2, y_2, z_2) \dots H_{m_k, n_k}(x_k, y_k, z_k).$$

Substituting this equation into (2.3), we find that f can be expanded into a k -fold complex Hermite series. Conversely, if f is a k -fold complex Hermite series, then it satisfies the partial differential equations in Theorem 1.6 by using Proposition 2.2. Hence we complete the proof of the theorem. \square

To determine if a given function is an analytic functions in several complex variables, we can use the following theorem due to Hartogs (see, for example, [17, p. 28]).

Theorem 2.6. *If a complex valued function $f(z_1, z_2, \dots, z_n)$ is holomorphic (analytic) in each variable separately in a domain $U \in \mathbb{C}^n$, then, it is holomorphic (analytic) in U .*

3. The Poisson Kernel for the complex Hermite polynomials

In this section we will use Theorem 1.6 to give a completely new proof of the following Poisson kernel for the complex Hermite polynomials. This formula was first derived by Carlitz [4, p.13] in 1978, and rediscovered by Wünsche [20] without proof in 1999. Ismail [9, Theorem 3.3] recovered it as a specific case of his Kibble–Slepian formula. For other different proofs, please see [7, Theorem 4.1], [10]. Our proof is brand new.

Theorem 3.1. *For $|stz_1z_2| < 1$, the Mehler formula for the complex Hermite polynomials states that*

$$\sum_{m, n=0}^{\infty} \frac{H_{m, n}(x_1, y_1, z_1)H_{m, n}(x_2, y_2, z_2)}{m!n!} s^m t^n = \frac{1}{1 - stz_1z_2} \exp\left(\frac{sx_1x_2 + ty_1y_2 + (z_1x_2y_2 + z_2x_1y_1)st}{1 - stz_1z_2}\right).$$

Proof. If we use $f(x_1, y_1, z_1)$ to denote the right-hand side of the equation in Theorem 3.1, then, it is easily seen that $f(x_1, y_1, z_1)$ is an analytic function of x_1, y_1, z_1 for any x_1, y_1 and $|stz_1z_2| < 1$. Hence $f(x_1, y_1, z_1)$ is analytic at $(x_1, y_1, z_1) = (0, 0, 0)$. By a direct computation, we find that

$$\frac{\partial f}{\partial z_1} = \frac{\partial^2 f}{\partial x_1 \partial y_1} = \left(\frac{z_2st}{(1 - stz_1z_2)^2} + \frac{st(x_2 + y_1z_2)(y_2 + x_1z_2)}{(1 - stz_1z_2)^2} \right) f.$$

Thus, by Theorem 1.6, there exists a sequence $\{\lambda_{m, n}\}$ independent of x_1, y_1 and z_1 such that

$$\begin{aligned} & \frac{1}{1 - stz_1z_2} \exp\left(\frac{sx_1x_2 + ty_1y_2 + (z_1x_2y_2 + z_2x_1y_1)st}{1 - stz_1z_2}\right) \\ &= \sum_{m, n=0}^{\infty} \lambda_{m, n} H_{m, n}(x_1, y_1, z_1). \end{aligned} \tag{3.1}$$

Setting $z_1 = 0$ in this equation and using $H_{m, n}(x_1, y_1, 0) = x_1^m y_1^n$ in the resulting equation, we immediately find that

$$\exp(sx_1x_2 + ty_1y_2 + x_1y_1z_2st) = \sum_{m, n=0}^{\infty} \lambda_{m, n} x_1^m y_1^n.$$

Using the generating function for the complex Hermite polynomials in Proposition 2.1, we have

$$\exp(sx_1x_2 + ty_1y_2 + x_1y_1z_2st) = \sum_{m,n=0}^{\infty} \frac{H_{m,n}(x_2, y_2, z_2)}{m!n!} (sx_1)^m (ty_1)^n.$$

Comparing the right-hand sides of these two equations, we conclude that

$$\lambda_{m,n} = \frac{H_{m,n}(x_2, y_2, z_2)}{m!n!} s^m t^n.$$

Substituting this into (3.1), we complete the proof of Theorem 3.1. \square

Using Proposition 2.3, we easily find that the Poisson kernel for the complex Hermite polynomials is equivalent to the following exponential operational identity, which is equivalent to [21, Equation (5.1)].

Theorem 3.2. For $|stz_1z_2| < 1$, we have the exponential operator identity

$$\begin{aligned} & \exp\left(z_2 \frac{\partial^2}{\partial x_2 \partial y_2}\right) \left\{ \exp(sx_1x_2 + ty_1y_2 + ty_1y_2 + stz_1x_2y_2) \right\} \\ &= \frac{1}{1 - stz_1z_2} \exp\left(\frac{sx_1x_2 + ty_1y_2 + (z_1x_2y_2 + z_2x_1y_1)st}{1 - stz_1z_2}\right). \end{aligned}$$

4. The Nielsen type formulas for the complex Hermite polynomials

We begin this section with the following formula for the complex Hermite polynomials.

Theorem 4.1. For any complex numbers x, y, z, s_1, s_2, t_1 and t_2 , we have

$$\begin{aligned} & \exp((s_1 + s_2)x + (t_1 + t_2)y + (s_1 + s_2)(t_1 + t_2)z) \\ &= \sum_{m_1, n_1, m_2, n_2=0}^{\infty} H_{m_1+m_2, n_1+n_2}(x, y, z) \frac{s_1^{m_1} s_2^{m_2} t_1^{n_1} t_2^{n_2}}{m_1! m_2! n_1! n_2!}. \end{aligned}$$

Proof. Denote the left-hand side of the equation in Theorem 4.1 by $f(x, y, z)$. It is easily seen that $f(x, y, z)$ is analytic at $(0, 0, 0)$. A simple computation shows that

$$\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x \partial y} = (s_1 + s_2)(t_1 + t_2)f(x, y, z).$$

Thus, by Theorem 1.6, there exists a sequence $\{\lambda_{k,l}\}$ independent of x, y and z such that

$$\begin{aligned} & \exp((s_1 + s_2)x + (t_1 + t_2)y + (s_1 + s_2)(t_1 + t_2)z) \\ &= \sum_{k_1, l=0}^{\infty} \lambda_{k,l} H_{k,l}(x, y, z). \end{aligned} \tag{4.1}$$

Upon setting $z = 0$ in the equation and using $H_{k,l}(x, y, 0) = x^k y^l$, we deduce that

$$\exp((s_1 + s_2)x + (t_1 + t_2)y) = \sum_{k_1, l=0}^{\infty} \lambda_{k,l} x^k y^l.$$

Equating the coefficients of $x^k y^l$ on both sides of this equation, we find that $k!l!\lambda_{k,l} = (s_1 + s_2)^k (t_1 + t_2)^l$. Substituting this into the right-hand side of (4.1), expanding $(s_1 + s_2)^k (t_1 + t_2)^l$ using the binomial theorem and interchanging the order of summation, we complete the proof of Theorem 4.1. \square

Using Theorem 4.1 and method of equating the coefficients of like power, we can derive the following Nielsen type formula for the complex Hermite polynomials, which is equivalent to [6, Equation (3.11)] and [9, Equation (4.7)].

Theorem 4.2. For any non-negative integers m_j, n_j, p_j $j \in \{1, 2\}$, we have

$$\frac{H_{m_1+m_2, n_1+n_2}(x, y, z)}{m_1!m_2!n_1!n_2!} = \sum_{p_1=0}^{m_1 \wedge n_1} \sum_{p_2=0}^{n_1 \wedge m_2} \frac{H_{m_1-p_1, n_1-p_2}(x, y, z)H_{m_2-p_2, n_2-p_1}(x, y, z)z^{p_1+p_2}}{p_1!p_2!(m_1-p_1)!(m_2-p_2)!(n_1-p_2)!(n_2-p_1)!}.$$

Upon multiplying both sides of the equation in Theorem 4.1 by $\exp(-s_1t_2 - s_2t_1)z$ and then equating the coefficients of like power, we can also derive the following formula due to Ismail [9, Theorem 4.1].

Theorem 4.3. For any non-negative integers m_j, n_j, p_j $j \in \{1, 2\}$, we have

$$\frac{H_{m_1, n_1}(x, y, z)H_{m_2, n_2}(x, y, z)}{m_1!m_2!n_1!n_2!} = \sum_{p_1=0}^{m_1 \wedge n_1} \sum_{p_2=0}^{n_1 \wedge m_2} \frac{H_{m_1+m_2-p_1-p_2, n_1+n_2-p_1-p_2}(x, y, z)(-z)^{p_1+p_2}}{p_1!p_2!(m_1-p_1)!(m_2-p_2)!(n_1-p_1)!(n_2-p_2)!}.$$

5. Addition formula for the complex Hermite polynomials

Theorem 5.1. If M, N are two non-negative integers, then, we have the following addition formula for the complex Hermite polynomials:

$$\begin{aligned} &H_{M, N}(a_1x_1 + \dots + a_kx_k, b_1y_1 + \dots + b_ky_k, a_1b_1z_1 + \dots + a_kb_kz_k) \\ &= \sum_{m_1, n_1, \dots, m_k, n_k} \frac{M!N!}{m_1!n_1! \dots m_k!n_k!} a_1^{m_1} \dots a_k^{m_k} b_1^{n_1} \dots b_k^{n_k} \\ &\quad \times H_{m_1, n_1}(x_1, y_1, z_1) \dots H_{m_k, n_k}(x_k, y_k, z_k). \end{aligned}$$

The sum is taken over all combinations of non-negative integers indices m_1 through m_k and n_1 through n_k such that

$$m_1 + \dots + m_k = M, \text{ and } n_1 + \dots + n_k = N.$$

Proof. Upon denoting the left-hand side of the equation in Theorem 5.1 by

$$f(x_1, y_1, z_1, \dots, x_k, y_k, z_k),$$

it is obvious that this function is analytic at $(0, \dots, 0) \in \mathbb{C}^{3k}$. For simplicity, we temporarily denote

$$\begin{aligned} x &= a_1x_1 + \dots + a_kx_k, \\ y &= b_1y_1 + \dots + b_ky_k, \\ z &= a_1b_1z_1 + \dots + a_kb_kz_k. \end{aligned}$$

By a simple calculation, we find that for $j = 1, \dots, k$,

$$\frac{\partial f}{\partial z_j} = \frac{\partial^2 f}{\partial x_j \partial y_j} = a_j b_j \frac{\partial H_{M, N}}{\partial z}.$$

Thus, by Theorem 1.6, there exists a sequence $\{\lambda_{m_1, n_1, \dots, m_k, n_k}\}$ independent of

$$x_1, y_1, z_1, \dots, x_k, y_k, z_k$$

such that

$$\begin{aligned}
 & H_{M,N}(a_1x_1 + \dots + a_kx_k, b_1y_1 + \dots + b_ky_k, a_1b_1z_1 + \dots + a_kb_kz_k) \\
 &= \sum_{m_1, n_1, \dots, m_k, n_k=0}^{\infty} \lambda_{m_1, n_1, \dots, m_k, n_k} H_{m_1, n_1}(x_1, y_1, z_1) \cdots H_{m_k, n_k}(x_k, y_k, z_k).
 \end{aligned}$$

Setting $z_1 = \dots = z_k = 0$ and in the resulting equation using the fact that

$$H_{m_j, n_j}(x_j, y_j, 0) = x_j^{m_j} y_j^{n_j},$$

we deduce that

$$\begin{aligned}
 & (a_1x_1 + \dots + a_kx_k)^M (b_1y_1 + \dots + b_ky_k)^N \\
 &= \sum_{m_1, n_1, \dots, m_k, n_k=0}^{\infty} \lambda_{m_1, n_1, \dots, m_k, n_k} x_1^{m_1} y_1^{n_1} \cdots x_k^{m_k} y_k^{n_k}.
 \end{aligned}$$

Expanding the left-hand side by the multinomial theorem and then equating the coefficients of multiple power series, we complete the proof of Theorem 5.1. \square

6. A multilinear generating function for the complex Hermite polynomials

Theorem 6.1. *If $|s_1t_1z_1 + \dots + s_rt_rz_r| < 1$ and a, b, c are defined by*

$$\begin{aligned}
 a &= s_1x_1 + \dots + s_rx_r, \\
 b &= t_1y_1 + \dots + t_ry_r, \\
 c &= s_1t_1z_1 + \dots + s_rt_rz_r,
 \end{aligned}$$

then, we have the following multilinear generating function for the complex Hermite polynomials:

$$\frac{1}{(1 - cz)} \exp\left(\frac{ax + by + cxy + abz}{1 - cz}\right) \tag{6.1}$$

$$\begin{aligned}
 &= \sum_{m_1, n_1, \dots, m_r, n_r=0}^{\infty} H_{m_1, n_1}(x_1, y_1, z_1) \cdots H_{m_r, n_r}(x_r, y_r, z_r) \\
 &\quad \times H_{m_1+\dots+m_r, n_1+\dots+n_r}(x, y, z) \frac{s_1^{m_1} t_1^{n_1} \cdots s_r^{m_r} t_r^{n_r}}{m_1! n_1! \cdots m_r! n_r!}.
 \end{aligned}$$

Proof. If we use $f(x, y, z)$ to denote the left-hand side of (6.1), then, it is easily seen that f is an analytic function of x, y, z such that $|s_1t_1z_1 + \dots + s_rt_rz_r| < 1$. Hence $f(x, y, z)$ is analytic at $(x, y, z) = (0, 0, 0)$. By a straightforward computation, we conclude that

$$\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x \partial y} = \left(\frac{c}{1 - cz} + \frac{(a + cy)(b + cz)}{(1 - cz)^2} \right) f.$$

Thus, by Theorem 1.6, there exists a sequence $\lambda_{k,l}$ independent of x, y, z such that

$$f(x, y, z) = \sum_{k,l=0}^{\infty} \lambda_{k,l} H_{k,l}(x, y, z). \tag{6.2}$$

Setting $z = 0$ in the above equation and using the fact that $H_{k,l}(x, y, 0) = x^k y^l$, we find that

$$f(x, y, 0) = \sum_{k,l=0}^{\infty} \lambda_{k,l} x^k y^l. \tag{6.3}$$

On other hand, from the definition of $f(x, y, z)$, it is easily seen that

$$f(x, y, 0) = \prod_{j=1}^r \exp(s_j x_j x + t_j y_j y + s_j t_j z_j xy).$$

Using the generating function of the exponential type for the complex Hermite polynomials in Proposition 2.1, we find that

$$f(x, y, 0) = \sum_{m_1, n_1, \dots, m_r, n_r=0}^{\infty} H_{m_1, n_1}(x_1, y_1, z_1) \cdots H_{m_r, n_r}(x_r, y_r, z_r) \times \frac{(s_1 x)^{m_1} (t_1 y)^{n_1} \cdots (s_r x)^{m_r} (t_r y)^{n_r}}{m_1! n_1! \cdots m_r! n_r!}.$$

Comparing this equation with (6.3) and equating the coefficients of $x^k y^l$, we conclude that

$$\lambda_{k,l} = \sum_{\substack{m_1 + \dots + m_r = k \\ n_1 + \dots + n_r = l}}^{\infty} H_{m_1, n_1}(x_1, y_1, z_1) \cdots H_{m_r, n_r}(x_r, y_r, z_r) \times \frac{s_1^{m_1} t_1^{n_1} \cdots s_r^{m_r} t_r^{n_r}}{m_1! n_1! \cdots m_r! n_r!}.$$

Substituting this into (6.2), we complete the proof of Theorem 6.1. \square

7. A generating function for the products of the Hermite polynomials and the complex Hermite polynomials

As usual, for any real number x , we use $[x]$ to denote the greatest integer function. For any complex number x , the Hermite polynomials are defined by

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(n-2k)!} (2x)^{n-2k}. \tag{7.1}$$

The exponential generating function for the Hermite polynomials $H_n(x)$ is given by

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \quad |t| < \infty. \tag{7.2}$$

The following formula is equivalent to Wünsche [21, Equation (7.4)]. In his paper Professor Wünsche just said that his formula can be proved by using auxiliary formulae prepared in Appendix A, but lacks sufficient details. Now we will use Theorem 1.6 to give a very simple proof of Theorem 7.1.

Theorem 7.1. For $|2stz| < 1$, we have the following generating function for the Hermite polynomials and the complex Hermite polynomials.

$$\begin{aligned} & \sum_{m,n=0}^{\infty} (-1)^{m+n} H_{m,n}(x, y, z) H_m(u) H_n(v) \frac{s^m t^n}{m! n!} \\ &= \frac{\exp(u^2 + v^2)}{\sqrt{1 - 4s^2 t^2 z^2}} \exp\left(\frac{4stz(sx + u)(ty + v) - (sx + u)^2 - (ty + v)^2}{1 - 4s^2 t^2 z^2}\right). \end{aligned}$$

Proof. If we use $f(x, y, z)$ to denote the right-hand side of the equation in Theorem 7.1, then, it is easily seen that f is analytic at $(0, 0, 0)$. A elementary calculation shows that

$$\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x \partial y} = \left\{ \frac{4s^2 t^2 z}{1 - 4s^2 t^2 z^2} + \frac{4st(2stz(sx + u) - (yv + t))(2stz(ty + v) - (sx + u))}{(1 - 4s^2 t^2 z^2)^2} \right\} f$$

Hence, by Theorem 1.6, there exists a sequence $\lambda_{m,n}$ independent of x, y, z such that

$$f(x, y, z) = \sum_{m,n=0}^{\infty} \lambda_{m,n} H_{m,n}(x, y, z). \tag{7.3}$$

Setting $z = 0$ in the above equation and using the fact that $H_{m,n}(x, y, 0) = x^m y^n$, we deduce that

$$\exp(-(sx)^2 - 2sxu - (ty)^2 - 2tyv) = \sum_{m,n=0}^{\infty} \lambda_{m,n} x^m y^n. \tag{7.4}$$

Using the exponential generating function for the Hermite polynomials, we find that

$$\begin{aligned} \exp(-(sx)^2 - 2sxu) &= \sum_{m=0}^{\infty} H_m(u) \frac{(-sx)^m}{m!}, \\ \exp(-(ty)^2 - 2tyv) &= \sum_{n=0}^{\infty} H_n(v) \frac{(-ty)^n}{n!}. \end{aligned}$$

Upon substituting these two equations into the left-hand side of (7.4) and equating the coefficients of like power, we obtain

$$\lambda_{m,n} = (-1)^{m+n} H_m(u) H_n(v) \frac{s^m t^n}{m! n!}.$$

Combining this equation with (7.3), we complete the proof of Theorem 7.1. \square

Theorem 7.1 contains the Mehler formula for the Hermite polynomials as a special case, which was discovered by Mehler [15, p.174, Equation(18)] in 1866. One can also find this important formula in most books on special functions, for example, [2, p.280, Equation (6.1.13)], [3, p.111, Equation(4.417)], [8, p.108, Equation (4.7.6)], [16, p. 198, Equation (2)]. One very simple proof of this formula can be found in [13].

Theorem 7.2. For $|2t| < 1$, we have the Mehler formula for the Hermite polynomials:

$$\sum_{n=0}^{\infty} \frac{H_n(u)H_n(v)}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4tuv - 4(u^2 + v^2)t^2}{1 - 4t^2}\right).$$

Proof. Upon taking $x = y = 0$ in the equation in Theorem 7.1 and using the fact that

$$H_{m,n}(0, 0, z) = \delta_{m,n} n! z^n,$$

in the resulting equation, we immediately conclude that

$$\sum_{n=0}^{\infty} \frac{H_n(u)H_n(v)}{n!} (stz)^n = \frac{\exp(u^2 + v^2)}{\sqrt{1 - 4s^2 t^2 z^2}} \exp\left(\frac{4stuv - (u^2 + v^2)}{1 - 4s^2 t^2 z^2}\right).$$

Putting $s = z = 1$ in this equation and simplifying we complete the proof of Theorem 7.2. \square

In the same way we can prove the following more general generating function formula, which appeared to be new.

Theorem 7.3. *If k is a non-negative integer and $|2stz| < 1$, we have the following generating function for the Hermite polynomials and the complex Hermite polynomials:*

$$\sum_{m,n=0}^{\infty} (-1)^{m+n} H_{m,n}(x, y, z) H_{m+k}(u) H_n(v) \frac{s^m t^n}{m!n!} \\ = \frac{\exp(u^2 + v^2)}{(1 - 4s^2 t^2 z^2)^{(k+1)/2}} H_k \left(\frac{u + sx - 2stz(v + ty)}{\sqrt{1 - 4s^2 t^2 z^2}} \right) \exp \left(\frac{4stz(sx + u)(ty + v) - (sx + u)^2 - (ty + v)^2}{1 - 4s^2 t^2 z^2} \right).$$

Upon putting $x = y = 0$ in Theorem 7.3 and using the fact that

$$H_{m,n}(0, 0, z) = \delta_{m,n} n! z^n,$$

in the resulting equation and finally setting $s = z = 1$, we derive the following formula due to Weisner [18, Equation (4.9)].

Theorem 7.4. *For $|2t| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(u) H_n(v)}{n!} t^n = \frac{1}{(1 - 4t^2)^{(k+1)/2}} H_k \left(\frac{u - 2tv}{\sqrt{1 - 4t^2}} \right) \exp \left(\frac{4tuv - 4(u^2 + v^2)t^2}{1 - 4t^2} \right).$$

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References

- [1] S.T. Ali, F. Bagarello, G. Honnouvo, Modular structures on trace class operators and applications to Landau levels, *J. Phys. A* 43 (2010) 105–202.
- [2] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [3] R. Beals and R. Wong, *Special Functions*, Cambridge University Press, Cambridge, 2010.
- [4] L. Carlitz, A set of polynomials in three variables, *Houston J. Math.* 4 (1) (1978) 11–33.
- [5] G. Dattoli, P.L. Ottaviani, A. Torre, L. Vazquez, Evolution operator equations, integration with algebraic and finite difference methods: applications to physical problems in classical and quantum mechanics, *Riv. Nuovo Cimento* 20 (1997) 1–133.
- [6] A. Ghanmi, Operational formulae for the complex Hermite polynomials $H_{p,q}(z, \bar{z})$, *Integral Transforms Spec. Funct.* 24 (2013) 884–895.
- [7] A. Ghanmi, Mehler’s formulas for the univariate complex Hermite polynomials and applications. *Math. Methods Appl. Sci.* 40 (2017) 7540–7545.
- [8] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications, Vol. 98, Cambridge University Press, Cambridge, 2009.
- [9] M. E. H. Ismail, Analytic properties of complex Hermite polynomials. *Trans. Amer. Math. Soc.* 368 (2016) 1189–1210.
- [10] M.E.H. Ismail, R. Zhang, Kibble-Slepian formula and generating functions for 2D polynomials. *Adv. in Appl. Math.* 80 (2016) 70–92.
- [11] M. E. H. Ismail, J. Zeng, A two variable extension of the Laguerre and disc polynomials, *J. Math. Anal. Appl.* 424 (2015) 289–303.
- [12] K. Itô, Complex multiple Wiener integral, *Jpn. J. Math.* 22 (1952) 63–86.
- [13] Z.-G. Liu, On a system of partial differential equations and the bivariate Hermite polynomials, *J. Math. Anal. Appl.* 454 (2017) 1–17.
- [14] B. Malgrange, *Lectures on functions of several complex variables*, Springer-Verlag, Berlin, 1984.
- [15] F. G. Mehler, Ueber die Entwicklung einer Function von beliebig vielen Variablen nach Laplaceschen Functionen höherer Ordnung, *J. Reine Angew. Math.* 66 (1866) 161–176.
- [16] E. D. Rainville, *Special Functions*, The Macmillan Company, New York 1960.
- [17] J. Taylor, *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups*, Graduate Studies in Mathematics, vol. 46. Am. Math. Soc., Providence, 2002.
- [18] L. Weisner, Generating functions for Hermite functions, *Canad. J. Math.* 11 (1959) 141–147.
- [19] A. Wünsche, Laguerre 2D-functions and their application in quantum optics, *J. Phys. A* 31 (1998) 8267–8287.
- [20] A. Wünsche, Transformations of Laguerre 2D-polynomials and their applications to quasiprobabilities, *J. Phys. A* 21 (1999) 3179–3199.
- [21] A. Wünsche, Generating functions for products of special Laguerre 2D and Hermite 2D polynomials, *Applied Mathematics* 6 (2015) 2142–2168.