Filomat 34:2 (2020), 433–441
https://doi.org/10.2298/FIL2002433A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Construction of Fuzzy Topology by Using Fuzzy Metric

Abdülkadir Aygünoğlu[a], Ebru Aydoğdu[a], Halis Aygün[a]

[a] *Kocaeli University, Department of Mathematics, 41380, Kocaeli, Turkey*

Abstract. In this work, we construct a stratified fuzzy topological space induced by a fuzzy metric in the sense of Kramosil and Michalek. Our special interests are to investigate bases for such spaces and to study continuity and compactness.

1. Introduction and Preliminaries

1.1. Fuzzy Metric Spaces

In this subsection, we recall the definition of fuzzy metric space and its basic properties. A fuzzy metric which introduced by Kramosil and Michalek [10] is a certain kind of mapping that associates two points with a value in $[0,1]$, which intuitively means "the degree of nearness between these points according to a parameter t".

Definition 1.1. *[10] A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty)set, $*$ is a continuous t-norm and $M : X \times X \times [0,\infty) \to [0,1]$ is a map satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:*

(F1) $M(x, y, 0) = 0$

(F2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$

(F3) $M(x, y, t) = M(y, x, t)$

(F4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ *for all* $t, s > 0$

(F5) $M(x, y, \cdot) : [0,\infty) \to [0,1]$ *is left continuous.*

Usual product (denoted by $\cdot$) and minimum,(denoted by $\wedge$) are examples of continuous triangular-norms. One can easily show that $x * y \leq x \wedge y$ for each x, y and for all (continuous)triangular-norm $*$. Consequently, if $(X, F, \wedge)$ is a fuzzy metric space then $(X, F, *)$ is a fuzzy metric space [8].

From now on, we use the infimum t-norm since some proofs require the idempotency property.

The family $\{B(x, \alpha, t) : x \in X,\ t \in [0,\infty), \alpha \in (0,1)\}$ generates a crisp topology denoted by T^M, where the open ball $B(x, \alpha, t) = \{y \in X : M(x, y, t) > 1 - \alpha\}$ [4, 10] .

2010 *Mathematics Subject Classification*. 54A40
Keywords. fuzzy metric, fuzzy topology, continuity, compactness.
Received: 25 October 2019; Accepted: 04 June 2020
Communicated by Miodrag Spalević
Email addresses: abdulkadir.aygunoglu@kocaeli.edu.tr (Abdülkadir Aygünoğlu), ebru.diyarbakirli@kocaeli.edu.tr (Ebru Aydoğdu), halis@kocaeli.edu.tr (Halis Aygün)

Definition 1.2. *[8] Let $(X, M_1, \wedge)$ and $(Y, M_2, \wedge)$ be two fuzzy metric spaces. A mapping f from X to Y is called continuous at $x \in X$ if for each $\varepsilon \in (0,1)$ and $t > 0$, there exist $\delta \in (0,1)$ and $s > 0$ such that $M_2(f(x), f(y), t) > 1-\varepsilon$ whenever $M_1(x, y, s) > 1-\delta$.*

Proposition 1.3. *[4] Let $(X, M_1, \wedge)$ and $(Y, M_2, \wedge)$ be two fuzzy metric spaces. Then the followings are equivalent.*

a) $f : (X, M_1, \wedge) \to (Y, M_2, \wedge)$ is continuous.

b) $f : (X, T^{M_1}) \to (Y, T^{M_2})$ is continuous.

Definition 1.4. *[5] Let $(X, M, \wedge)$ be a fuzzy metric space. A sequence (x_n) in X is called convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, t) = 1$, for all $t > 0$.*

Definition 1.5. *[7] A fuzzy metric space $(X, M, \wedge)$ is called co-principal if the family $\{B(x, r, t) : t > 0\}$ is a local base at $x \in X$, for each $x \in X$ and each $r \in (0,1)$.*

Definition 1.6. *[1] Let $(X, M, \wedge)$ be a fuzzy metric space, and $\alpha \in (0,1)$. A sequence (x_n) in X is called α-convergent to $x_0 \in X$ if for each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) \geq 1-\alpha$, for all $n \geq n_0$.*

Definition 1.7. *[1] Let $(X, M, \wedge)$ be a fuzzy metric space and $\alpha \in (0,1)$. A sequence (x_n) in X is called α-Cauchy sequence if for each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) \geq 1-\alpha$, for all $m, n \geq n_0$.*

Definition 1.8. *[1] A fuzzy metric space $(X, M, \wedge)$ is called α-complete if every α-Cauchy sequence in X is α-convergent to some point of X.*

Definition 1.9. *[6] A fuzzy metric space $(X, M, \wedge)$ is called totally bounded if for each $\alpha \in (0,1)$, and $t > 0$, there is a finite subset A of X, such that $X = \bigcup_{x \in A} B(x, \alpha, t)$.*

Definition 1.10. *[6] A fuzzy metric space $(X, M, \wedge)$ is called compact if $\left(X, T^M\right)$ is a compact topological space.*

According to the previous definitions we give the definitions of α-totally boundedness and α-compactness in the following way.

Definition 1.11. *Let $(X, M, \wedge)$ be a fuzzy metric space and $\alpha \in (0,1)$. X is called α-totally bounded if for each $t > 0$ and $r \in (\alpha, 1)$ there exists a finite subset A of X such that $X = \bigcup_{x \in A} B(x, r, t)$.*

Definition 1.12. *Let $\alpha \in (0,1)$. A fuzzy metric space $(X, M, \wedge)$ is called α-compact if every sequence in X has a α-convergent subsequence.*

Proposition 1.13. *A compact fuzzy metric spaces is α-compact, for all $\alpha \in (0,1)$.*

1.2. Fuzzy Topological Spaces Determined by Level-Topologies

In the following, we summarize the construction of fuzzy topology with the help of level topologies by giving related definitions and properties. We begin with the definition of fuzzy topology which was given by Chang [2].

Let $\overline{c} : X \to [0,1]$ be a mapping defined by $\overline{c}(x) = c, \forall x \in X$, i.e, $\overline{c} \in I^X$ is a constant fuzzy set.

Definition 1.14. *[2] A fuzzy topological space is an ordered pair $(X, \mathcal{T})$ such that X be a set and $\mathcal{T} \subset I^X$ satisfies the following conditions*

CT1 $\overline{0}, \overline{1} \in \mathcal{T}$

CT2 *If $\lambda_1, \lambda_2 \in \mathcal{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathcal{T}$*

CT3 *If* $\lambda_i \in \mathcal{T}$ *for all* $i \in I$ *then* $\bigvee_{i\in I} \lambda_i \in \mathcal{T}$.

In [11], Lowen proposed a more natural definition of fuzzy topology, called stratified fuzzy topology, since constant functions may not be continuous in general with Chang's definition.

Definition 1.15. *[11] A stratified fuzzy topological space is an ordered pair* $(X, \mathcal{T})$ *such that* X *be a set and* $\mathcal{T} \subset I^X$ *satisfies the following conditions*

LT1 $\bar{c} \in \mathcal{T}$, *for all* $c \in [0, 1]$

LT2 *If* $\lambda_1, \lambda_2 \in \mathcal{T}$, *then* $\lambda_1 \wedge \lambda_2 \in \mathcal{T}$

LT3 *If* $\lambda_i \in \mathcal{T}$ *for all* $i \in I$ *then* $\bigvee_{i\in I} \lambda_i \in \mathcal{T}$.

Definition 1.16. *Let* $(X, \mathcal{T})$ *be a fizzy topological space. A subfamily* $\mathcal{B}$ *of* $\mathcal{T}$ *is a base for* $\mathcal{T}$ *if and only if for each* $\lambda \in \mathcal{T}$ *there exist* $(\mu_i)_{j\in J} \subset \mathcal{B}$ *such that* $\lambda = \sup_{j\in J} \mu_j$.

Definition 1.17. *[11] Let* $(X, \mathcal{T}_1)$ *and* $(Y, \mathcal{T}_2)$ *be fuzzy topological spaces. A mapping from* X *to* Y *is continuous if* $f^{-1}(\mu) \in \mathcal{T}_1$ *whenever* $\mu \in \mathcal{T}_2$. *(*$f^{-1}(\mu)(x) = \mu(f(x))$ *for all* $x \in X$*).*

Lowen[11] gave a relation between the category of classical topologies $\mathbb{TOP}$ and the category of stratified fuzzy topologies $\mathbb{SFUZTOP}$ by introducing two functors given in the next definition.

Definition 1.18. *[11] Let* X *be a nonempty set,* T *be a topology on* X *and* $\mathcal{T}$ *be an stratified fuzzy topology on* X. *Define* $w(T) = \{\lambda : \lambda$ *is lower semicontinuous*$\}$ *and* $\imath(\mathcal{T}) = \{\lambda^{-1}(\varepsilon, 1) : \varepsilon \in [0, 1), \lambda \in \mathcal{T}\}$.

Proposition 1.19. *[11] Let* X *be a nonempty set,* T *be a topology on* X *and* $\mathcal{T}$ *be an stratified fuzzy topology on* X.

a) $w(T) = \{\lambda : \lambda$ *is lower semicontinuous*$\}$ *is an stratified fuzzy topology on* X

b) $\imath(\mathcal{T}) = \{\lambda^{-1}(\varepsilon, 1) : \varepsilon \in [0, 1), \lambda \in \mathcal{T}\}$ *is a topology on* X.

Proposition 1.20. *[12] Let* $(X, \mathcal{T})$ *be a fuzzy topological space and* $0 \le \alpha < 1$. *The family* $\imath_\alpha(\mathcal{T}) = \{[\lambda]^\alpha : \lambda \in \mathcal{T}\}$, *is a topology on* X, *which is called the* α*-level topology of* $\mathcal{T}$, *where* $[\lambda]^\alpha = \{x \in X : \lambda(x) > \alpha\}$. *On the other hand,* $\bigcup \{\imath_\alpha(\mathcal{T}) : \alpha \in [0, 1)\}$ *is a subbase for the topology* $\imath(\mathcal{T})$.

Let $\{T_\alpha : \alpha \in [0, 1)\}$ be a family of topologies on X. In order to guarantee the existence of at least one fuzzy topology $\mathcal{T}$ on X such that $\imath_\alpha(\mathcal{T}) = T_\alpha$ for all $\alpha \in [0, 1)$, a necessary and sufficient condition was given in [18, 19] .

Proposition 1.21. *[19] Let* $\{T_\alpha : \alpha \in [0, 1)\}$ *be a family of topologies on a set* X. *Then the followings are equivalent:*

a) There exists at least one fuzzy topology $\mathcal{T}$ *on* X *such that* $\forall \alpha \in [0, 1) : \imath_\alpha(\mathcal{T}) = T_\alpha$.

b) LT-property: $\forall \alpha \in [0, 1), \forall G \in T_\alpha, \exists \left(G_\beta\right)_{\beta\in(\alpha,1)} \in \prod_{\beta\in(\alpha,1)} T_\beta$ *descending and* $G = \bigcup_{\beta\in(\alpha,1)} G_\beta$.

Theorem 1.22. *[18]Let* $\mathcal{F} = \{T_\alpha : \alpha \in [0, 1)\}$ *be a family of topologies on a set* X. *Then there exist fuzzy topologies* $\mathcal{T}$ *on* X *having* $\mathcal{F}$ *as their level topologies, i.e. such that*

$$\imath_\alpha(\mathcal{T}) = T_\alpha, \ \forall \alpha \in [0, 1) \qquad (*)$$

if and only if $\mathcal{F}$ *has the LT-property. Moreover, the fuzzy topology* $\mathcal{T}(\mathcal{F}) = \{\lambda : [\lambda]^\alpha \in T_\alpha$, *for all* $\alpha \in [0, 1)\}$ *is the finest of all stratified fuzzy topologies on* X *if* (*) *holds.*

Proposition 1.23. *[19] Let* $(X, \mathcal{T})$ *be a fuzzy topological space. Then the following are equivalent:*

a) All level topologies are equal and $\mathcal{T}$ is maximal;

b) $(X,\mathcal{T})$ is topologically generated, i.e. $\mathcal{T} = w(\imath(\mathcal{T}))$

In the following, we recall some concepts related to compactness of fuzzy topological spaces.

Definition 1.24. *[12] A fuzzy topological space $(X,\mathcal{T})$ is called fuzzy compact if for all family $\delta \subset \mathcal{T}$, $c \in (0,1]$ satisfying $\bigvee_{\mu\in\delta} \mu \geq c$ and for all $\varepsilon \in (0,c]$ there exists a finite subfamily $\delta_0 \subset \delta$ such that $\bigvee_{\mu\in\delta_0} \mu \geq c-\varepsilon$.*

Theorem 1.25. *[12] A topological space (X,T) is compact if and only if $(X,w(T))$ is fuzzy compact.*

Definition 1.26. *[12] A fuzzy topological space $(X,\mathcal{T})$ is called ultra fuzzy compact if $(X,\imath(\mathcal{T}))$ is compact.*

Definition 1.27. *[3] Let $\alpha \in [0,1)$. A family $\{\lambda_i : i \in \Delta\}$ of fuzzy subsets of a fuzzy topological space $(X,\mathcal{T})$ is called an α-shading of X if for each $x \in X$, there exists a $\lambda_{i_0} \in \{\lambda_i : i \in \Delta\}$ such that $\lambda_{i_0}(x) > \alpha$.*

Definition 1.28. *[3] A fuzzy topological space $(X,\mathcal{T})$ is called α-compact if every α-shading has an open α-subshading.*

Definition 1.29. *[12] A fuzzy topological space $(X,\mathcal{T})$ is called strong fuzzy compact if it is α-compact for all $\alpha \in [0,1)$.*

Theorem 1.30. *[12] Let $\alpha \in [0,1)$. A fuzzy topological space $(X,\mathcal{T})$ is α-compact if and only if $(X,\imath_\alpha(\mathcal{T}))$is compact topological space.*

2. Fuzzy Topology Induced by Fuzzy Metric

2.1. Construction of Fuzzy Topology

Many researchers are interested in topological structure of fuzzy metric [9, 13, 16]. In most study, the topology induced by a fuzzy metric was crisp topology on an underlying set. Recently few researchers [9, 13, 14, 20] have addressed the problem of construction of fuzzy-type topological structures induced by a fuzzy metric. In general, most study consist of to determine a fuzzifying topology.

In this section, we intend to construct a stratified fuzzy topology with the help of level topologies. Therefore, we construct a base of a topology on X for each parameter $\alpha \in [0,1)$, so that we obtain a parameterized family of topologies.

Lemma 2.1. *Let $(X,M,\wedge)$ be a fuzzy metric space and $\alpha \in [0,1)$. Then the family*

$$B_\alpha = \{B(x,r,t) : x \in X,\ r \in (\alpha,1),\ t > 0\}$$

is a base for a topology on X.

Proof. Let $z \in B(x,r_1,t) \cap B(y,r_2,s)$ where $r_1,r_2 \in (\alpha,1)$. Thus $M(x,z,t) > 1-r_1$ and $M(y,z,s) > 1-r_2$. Then there exists $t_0 < t$ and $s_0 < s$ such that $M(x,z,t_0) > 1-r_1$ and $M(y,z,s_0) > 1-r_2$.
Let $r = \min\{r_1,r_2\}$ and $p = \min\{t-t_0, s-s_0\}$. We claim that $B(z,1-r,p) \subset B(x,r_1,t) \cap B(y,r_2,s)$. Let $u \in B(z,r,p) \subset B(z,r,t-t_0)$. Then $M(u,z,t-t_0) > 1-r$.
Therefore $M(x,u,t) \geq M(x,z,t_0) \wedge M(z,u,t-t_0) > (1-r_1) \wedge (1-r) = 1-r_1$. Then $u \in B(x,r_1,t)$ and we have $B(z,r,p) \subset B(x,r_1,t)$.
On the other hand let $u \in B(z,r,p) \subset B(z,r,s-s_0)$. Then $M(u,z,s-s_0) > 1-r$. Therefore $M(y,u,s) \geq M(y,z,s_0) \wedge M(z,u,s-s_0) > (1-r_2) \wedge (1-r) = 1-r_2$. Then $u \in B(y,r_2,s)$. We have $B(z,r,p) \subset B(y,r_2,s)$.
Hence $B(z,r,p) \subset B(x,r_1,t) \cap B(y,r_2,s)$. □

The topology, call T_α, generated by B_α is characterized in the following theorem.

Theorem 2.2. *Let $(X, M, \wedge)$ be a fuzzy metric space, $\alpha \in [0, 1)$ and $G \subset X$. Then $G \in T_\alpha$ if and only if for each $x \in G$ there exists $t > 0$ and $r \in (\alpha, 1)$ such that $B(x, r, t) \subset G$.*
Equivalently;

$$G \in T_\alpha \text{ if and only if } G = \bigcup_{\substack{r>\alpha \\ B(x,r,t)\subset G}} B(x, r, t).$$

By the definition of T_α, we have $T_0 = T^M$.

The family $\{T_\alpha : \alpha \in [0, 1)\}$ is a decreasing family of topologies, since $B_\alpha \subset B_\beta$ whenever $\beta < \alpha$. Besides this family satisfies LT-property by the Theorem 2.2. This allows us to construct a fuzzy topology by the following theorem.

Theorem 2.3. *Let $(X, M, \wedge)$ be a fuzzy metric space and $\{T_\alpha : \alpha \in [0, 1)\}$ be the family of topologies induced by this metric. Then*

$$\mathcal{T}^M := \{\lambda : [\lambda]^\alpha \in T_\alpha \text{ for all } \alpha \in [0, 1)\}$$

is the finest stratified fuzzy topology satisfying $\iota_\alpha\left(\mathcal{T}^M\right) = T_\alpha$.

As a consequence, we have $\iota\left(\mathcal{T}^M\right) = T^M$.

Definition 2.4. *[16] Let $(X, M, \wedge)$ be a fuzzy metric space, $x \in X$, $r \in (0, 1)$, $t > 0$ and $\beta \in (0, 1)$. Then the fuzzy set $\beta B(x, r, t)$ is called β open ball with the center x and radius r, where*

$$\beta B(x, r, t)(y) = \begin{cases} \beta, & y \in B(x, r, t) \\ 0, & \text{other} \end{cases}.$$

In the following proposition, we show that the collection of β open balls is a base for $w\left(T^M\right)$.

Proposition 2.5. *Let $(X, M, \wedge)$ be a fuzzy metric space. Then the family*

$$\mathcal{B}_1 = \{\beta B(x, r, t) : x \in X,\ r \in (0, 1), t > 0,\ \beta \in (0, 1)\}$$

is a base for $w\left(T^M\right)$.

Proof. Obviously, $\mathcal{B}_1 \subset w\left(T^M\right)$. Let $\lambda \in w\left(T^M\right)$ and $\lambda(x) > 0$. Because of lower semicontinuity of λ, for all $\varepsilon \in (0, 1)$ satisfying $\lambda(x) - \varepsilon > 0$ there exists $r \in (0, 1)$ and $t > 0$ such that $\lambda(y) \geq \lambda(x) - \varepsilon$ for all $y \in B(x, r, t)$. Choose $\beta = \lambda(x) - \varepsilon$, we get $\beta B(x, r, t) \leq \lambda$. □

On the other hand, if M is co-principle then it can be easily shown that $T_\alpha = T_\beta$ for all $\alpha \neq \beta$. However, we have the following corollary by the Proposition 1.23 and Theorem 2.3.

Corollary 2.6. *The fuzzy topological space $(X, \mathcal{T}^M)$ is topologically generated, i.e $\mathcal{T}^M = w\left(\iota\left(\mathcal{T}^M\right)\right)$, if M is co-principle.*

In the next example, we show that $\mathcal{T}^M \neq w\left(\iota\left(\mathcal{T}^M\right)\right)$ in general.

Example 2.7. *Let $f : [0, 1) \to \left(\frac{1}{2}, 1\right]$ be a nondecreasing left continuous surjective function. Consider the fuzzy metric space $(X, M, \wedge)$ in [15]. M is defined by*

$$M(x, y, t) = \begin{cases} 0, & t = 0 \\ f\left(\frac{t}{|x-y|}\right) & x \neq y, t \geq 0 \\ 1 & x = y,\ t \geq 0 \end{cases}.$$

Let $\alpha = \frac{1}{4}$*and* $r = \frac{1}{3} > \alpha$. *Then* $B\left(x, \frac{1}{3}, t\right) = \{x\}$ *and* $\frac{2}{3}B\left(x, \frac{1}{3}, t\right) \in w\left(\imath\left(\mathcal{T}^M\right)\right)$.
On the other hand for $\alpha = \frac{1}{2} < \beta = \frac{2}{3}$ *we have* $\left[\frac{2}{3}B\left(x, \frac{1}{3}, t\right)\right]^{\frac{1}{2}} = \{x\} \notin T_{\frac{1}{2}}$. *Then* $T_{\frac{1}{2}}$ *is trivial topology, since* $B(x, r, t) = X$ *for* $r > \frac{1}{2}$. *It follows that* $\frac{2}{3}B\left(x, \frac{1}{3}, t\right) \notin \mathcal{T}^M$.

Moreover, by considering a relation between β and r, we can construct a base for the fuzzy topology $\mathcal{T}^M$ in the following proposition:

Proposition 2.8. *Let* $(X, M, \wedge)$ *be a fuzzy metric space. Then the family*

$$\mathcal{B}_2 = \{\beta B(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0,\ \beta \in (0, r)\}$$

is a base for $\mathcal{T}^M$.

Proof. We first show that $\mathcal{B}_2 \subset \mathcal{T}^M$. If $\alpha < \beta$ then $[\beta B(x, r, t)]^\alpha = B(x, r, t)$.
Since $r > \beta$, we have $r > \alpha$. It follows that $B(x, r, t) \in \mathcal{B}_2 \subset T_\alpha$. Hence $\beta B(x, r, t) \in \mathcal{T}^M$.
Let $\lambda \in \mathcal{T}^M$ and $\lambda(x) > 0$. Then $[\lambda]^\alpha \in T_\alpha$ and $x \in [\lambda]^\alpha$ for all $\alpha \in [0, 1)$ satisfy $\lambda(x) > \alpha > 0$. By the definition of T_α, there exists $r > \alpha$ satisfying $B(x, r, t) \subset [\lambda]^\alpha$. That is, $\lambda(y) > \alpha$ for each $y \in B(x, r, t)$. It follows that $\alpha B(x, r, t) \leq \lambda$. □

2.2. Continuity

Contrary to expectations, the continuity between fuzzy metric spaces and between induced fuzzy topological spaces are incompatible. In this manner, we propose an adjusted form of continuity as follows:

Definition 2.9. *Let* $(X, M_1, \wedge)$ *and* $(Y, M_2, \wedge)$ *be two fuzzy metric spaces and* $\alpha \in [0, 1)$. *A mapping* f *from* X *to* Y *is called* $\overline{\alpha}$*-continuous if for all* $\varepsilon > 0$ *and* $r \in (\alpha, 1)$ *there exist* $\delta \in (0, 1)$ *and* $s \in (\alpha, 1)$ *such that* $M_1(x, y, \delta) > 1 - s$ *implies* $M_2(f(x), f(y), \varepsilon) > 1 - r$.

If f *is* $\overline{\alpha}$*-continuous for all* $\alpha \in [0, 1)$, *then it is called* $\star$*-continuous.*

Remark 2.10. *Notice that* $\overline{\alpha}$*-continuity is a stronger version of continuity and the definition of* $\overline{0}$*-continuity coincides with the continuity.*

Theorem 2.11. *A mapping* $f : \left(X, \mathcal{T}^{M_1}\right) \to \left(Y, \mathcal{T}^{M_2}\right)$ *is continuous if and only if the mapping* $f : (X, M_1, \wedge) \to (Y, M_2, \wedge)$ *is* $\star$*-continuous.*

Proof. Let $f : \left(X, \mathcal{T}^{M_1}\right) \to \left(Y, \mathcal{T}^{M_2}\right)$ be a continuous mapping, $x \in X$, $\varepsilon > 0$ and $\alpha \in [0, 1)$. Then $B(f(x), r, \varepsilon) \in T_\alpha^{M_2}$ where $r > \alpha$. Choose $\beta \in (0, 1)$ satisfying $r > \beta > \alpha$. Thus $\beta B(f(x), r, \varepsilon) \in \mathcal{T}^{M_2}$. Then $f^{-1}(\beta B(f(x), r, \varepsilon)) \in \mathcal{T}^{M_1}$, since f is continuous. That is $\left[f^{-1}(\beta B(f(x), r, \varepsilon))\right]^\alpha = f^{-1}(B(f(x), r, \varepsilon)) \in T_\alpha^{M_1}$.
Then there exists $\delta > 0$ and $s > \alpha$ such that $B(x, s, \delta) \subset f^{-1}(B(f(x), r, \varepsilon))$. For $y \in B(x, s, \delta)$ we have $f(y) \in B(f(x), r, \varepsilon)$. Then $M_1(x, y, \delta) > 1 - s$ implies $M_2(f(x), f(y), \varepsilon) > 1 - r$ with $r, s > \alpha$. Hence $f : (X, M_1, \wedge) \to (Y, M_2, \wedge)$ is $\star$-continuous.

Let $f : (X, M_1, \wedge) \to (Y, M_2, \wedge)$ be $\overline{\alpha}$-continuous for all $\alpha \in [0, 1)$. Suppose that $f : \left(X, \mathcal{T}^{M_1}\right) \to \left(Y, \mathcal{T}^{M_2}\right)$ is not continuous. There exists $\mu \in \mathcal{T}^{M_2}$ such that $f^{-1}(\mu) \notin \mathcal{T}^{M_1}$. That is, $\left[f^{-1}(\mu)\right]^\alpha \notin T_\alpha^{M_1}$ for some $\alpha \in [0, 1)$.
Then there exists $x \in \left[f^{-1}(\mu)\right]^\alpha$ such that $B(x, u, t) \subseteq \left[f^{-1}(\mu)\right]^\alpha$ for all $t > 0$ and $u \in (\alpha, 1)$.
On the other hand, $f^{-1}(\mu)(x) = \mu(f(x)) > \alpha$ and we have $f(x) \in [\mu]^\alpha \in \mathcal{T}_\alpha^{M_2}$. It follows that, there exists $\varepsilon > 0$ and $r \in (\alpha, 1)$ such that $B(f(x), r, \varepsilon) \subset [\mu]^\alpha$. Since f is $\overline{\alpha}$-continuous, then there exists $\delta > 0$ and $s > \alpha$ such that $M_1(x, y, \delta) > 1 - s$ implies $M_2(f(x), f(y), \varepsilon) > 1 - r$. That is, $y \in B(x, \delta, s)$ implies $f(y) \in B(f(x), r, \varepsilon)$, which is a contradiction. □

We get the category $\mathbb{FUZFMS}$ of fuzzy metric spaces $(X, M, \wedge)$ as objects and their $\star$-continuous mappings as morphism. By the Theorem 2.11 one can easily define a faithful functor

$$\varphi : \mathbb{FUZFMS} \to \mathbb{SFUZTOP}$$

2.3. Compactness

In the next lemma, we give a condition for the α-convergence of a sequence.

Lemma 2.12. *Let $(X, M, \wedge)$ be a fuzzy metrics space and (x_n) be a sequence in X. If $M(x_n, x, t) > 1 - \left(\alpha + \frac{1}{kn}\right)$ for all $n \in \mathbb{N}$, where $k \in \mathbb{N}$ satisfies $\alpha + \frac{1}{k} < 1$, then (x_n) is α-convergent to x.*

Proof. Let $M(x_n, x, t) > 1 - \left(\alpha + \frac{1}{kn}\right)$ for all $n \in \mathbb{N}$, where $k \in \mathbb{N}$ satisfies $\alpha + \frac{1}{k} < 1$. Suppose that x_n is not α-convergent to x_0. Then for some $\varepsilon \in (0, 1-\alpha)$ the following holds:

For all $n \in \mathbb{N}$, there exist $n_1 > n$ such that $M(x_{n_1}, x_0, t) \leq 1 - \alpha - \varepsilon$.

We have $M(x_{n_1}, x_0, t) \leq 1 - \alpha - \varepsilon < 1 - \alpha - \frac{1}{n_2}$ for some $n_2 \in \mathbb{N}$ satisfying $\frac{1}{n_2} < \varepsilon$. Then there exists $n_0 \geq n_2$ such that $M(x_{n_2}, x_0, t) \leq 1 - \alpha - \varepsilon$. We chose $n_0 = \max\{n_1, n_2\}$. It follows that $M(x_{n_0}, x_0, t) \leq 1 - \alpha - \varepsilon \leq 1 - \alpha - \frac{1}{n_0} \leq 1 - \alpha - \frac{1}{kn_0}$ where $\alpha + \frac{1}{k} < 1$. This is a contradiction. □

Definition 2.13. *Let $(X, M, \wedge)$ be a fuzzy metrics space, $A \subset X$ and $x \in A$. x is called an α-isolated point of A if there exists $t > 0$ and $r \in (\alpha, 1)$ such that $B(x, r, t) \cap A = \{x\}$.*

Theorem 2.14. *Let $(X, M, \wedge)$ be a fuzzy metric space, and (X, T_α) be an α-level topology of fuzzy topology induced by this fuzzy metric, where $\alpha \in [0, 1)$. The followings are equivalent:*

a) (X, T_α) *is compact.*

b) $(X, M, \wedge)$ *is α-compact.*

c) $(X, M, \wedge)$ *is α-totally bounded and α-complete.*

Proof. "$(a) \Rightarrow (b)$" Let (X, T_α) be compact. Assume that, we have a sequence (x_n) in X with no α-convergent subsequence. No term in the sequence can occure infinitely many times. We can assume without loss of generality that $x_i \neq x_j$ whenever $i \neq j$.
Notice that, each terms of the sequence (x_n) is an α-isolated point of $\{x_n\}_{n\in\mathbb{N}}$. Indeed, suppose that x_{i_0} is not α-isolated point of $\{x_n\}_{n\in\mathbb{N}}$. Then for each $r \in (\alpha, 1)$ and $t > 0$ there exist $x_j \in \{x_n : n \in \mathbb{N}\}$ such that $M\left(x_{i_0}, x_j, t\right) > 1 - r$.
If we choose $r = \alpha + \frac{1}{kn}$ for each $n \in \mathbb{N}$, where $k \in \mathbb{N}$ satisfies $\alpha + \frac{1}{k} < 1$, then there exists $x_{j_n} \in \{x_n\}_{n\in\mathbb{N}}$ such that $M\left(x_{i_0}, x_{j_n}, t\right) > 1 - \left(\alpha + \frac{1}{kn}\right)$. By the Lemma 2.12, x_{j_n} is an α-convergent subsequence of (x_n). Which is a contradiction. Hence, for each i, there exists $r_i \in (\alpha, 1)$ and $t > 0$ such that $x_j \notin B(x_i, r_i, t_i)$ for $i \neq j$.
Let $U_0 = X/\{x_n\}_{n\in\mathbb{N}}$. Then $U_0 \in T_\alpha$. Indeed, assume that $U_0 = X/\{x_n\}_{n\in\mathbb{N}}$ is not open. Then there exists $y \in U_0$ such that $B(y, r, t) \cap \{x_n\}_{n\in\mathbb{N}} \neq \emptyset$ for all $r \in (\alpha, 1)$ and $t > 0$. By taking $r = \alpha + \frac{1}{kn}$ with $k \in \mathbb{N}$ satisfies $\alpha + \frac{1}{k} < 1$, we have $B\left(y, \alpha + \frac{1}{kn}, t\right) \cap \{x_n\}_{n\in\mathbb{N}} \neq \emptyset$. It follows that, $x_{i_n} \in B\left(y, \alpha + \frac{1}{kn}, t\right)$ for each $n \in \mathbb{N}$. Hence, $M\left(x_{i_n}, y, t + \frac{1}{n}\right) > 1 - \left(\alpha + \frac{1}{kn}\right)$ for each $n \in \mathbb{N}$. That is x_{i_n} is α- convergent. Which is a contradiction. As a consequence $\{U_0\} \cup \{B(x_n, r_n, t_n) : n \in \mathbb{N}\}$ is an open cover of X. However it has no finite subcover, since any finite subcover of this would fail to include infinitely many terms of the sequence (x_n).

"$(b) \Rightarrow (c)$" Let $(X, M, \wedge)$ be α-compact and (x_n) be a α-Cauchy sequence in X. Since $(X, M, \wedge)$ is α-compact, we have an α-convergent subsequence. Hence $(X, M, \wedge)$ is α-complete.

Assume that X is not α-totally bounded. Then there exists $t > 0$ and $r > \alpha$ such that X can not be covered by finitely many balls of the form $B(x, r, t)$. Take $x_1 \in X$. Since $B(x_1, r, t)$ does not cover X, there exists at least one point $x_2 \in X - B(x_1, r, t)$. Since $B(x_1, r, t) \cup B(x_2, r, t)$ does not cover X, there exists at least one point in $X - (B(x_1, r, t) \cup B(x_2, r, t))$. Continuing this process, we find a sequence (x_n) satisfying $x_{n+1} \in X - \bigcup_{i=1}^{n} B(x_i, r, t)$ for each $n \in \mathbb{N}$. However such sequence can not have an α-convergent subsequence, since $M(x_n, x_m, t) < 1 - r$ for all n, m. This is a contradiction. Therefore X is α-totally bounded.

"$(c) \Rightarrow (a)$" Let $\mathcal{U} = \{U_i\}_{i\in I} \subset T_\alpha$ be an open cover of X. Suppose that $\mathcal{U}$ does not have a finite subcover. α-totally boundedness implies that there exists a finite set of closed balls $\overline{B}\left(x_1^1, r, \frac{1}{2}\right), \ldots, \overline{B}\left(x_n^1, r, \frac{1}{2}\right)$ that cover

X, for all $r > \alpha$. One of these sets can not be covered by a finite subfamily of $\mathcal{U}$. Denote this by X^1. Since X^1 is a subset of X, X^1 is α-totally bounded. Then there exists $\overline{B}\left(x_1^2, r, \frac{1}{2^2}\right), \ldots, \overline{B}\left(x_n^2, r, \frac{1}{2^2}\right)$ that cover X^1. Again, one of these sets can not be covered by a finite subfamily of $\mathcal{U}$. Denote this by X^2. Continuing this process, we obtain a sequence of closed sets X^n such that $\ldots \subset X^n \subset X^{n-1} \subset \ldots \subset X^1$, and none of which can be finitely covered by $\mathcal{U}$. Let choose the centers of these balls as a sequence such that $y_1 = x_i^1, \ldots, y_n = x_i^n, \ldots$. Then $y_n, y_m \in X^k$, for all $n, m \geq k$. It follows that $M\left(y_n, y_m, \frac{1}{2^k}\right) > 1 - r$, for all $n, m \geq k$. Then for all $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{2^k} < t$ for all $k \geq k_0$. That is, $M(y_n, y_m, t) \geq M\left(y_n, y_m, \frac{1}{2^k}\right) > 1 - r$. It follows that, y_n is an α-Cauchy sequence. Since X is α-complete, y_n is α-convergent to a point y. Since $y_n \in X^m$ for all $n > m$ and X^m is closed it follows that $y \in X^m$.

Since $\mathcal{U}$ covers X^m, the point y belongs to some U_{i_0}. This means that $B(y, r, t) \subset U_i$ for some $r \in (\alpha, 1)$ and $t > 0$. Take $m_0 \in \mathbb{N}$ satisfying $\frac{1}{2^{m_0}} < \frac{t}{2}$. If $x \in X^{m_0}$ then

$$M(x, y, t) \geq M\left(x, y_{m_0}, \frac{t}{2}\right) \wedge M\left(y_{m_0}, y, \frac{t}{2}\right)$$
$$\geq M\left(x, y_{m_0}, \frac{1}{2^{m_0}}\right) \wedge M\left(y_{m_0}, y, \frac{1}{2^{m_0}}\right)$$
$$> (1 - r) \wedge (1 - r) = 1 - r.$$

That is, $x \in B(y, r, t)$. Hence $X^{m_0} \subset U_{i_0}$, which is a contradiction. □

By the Theorem 2.14 and Proposition 1.13, we have the following corollary.

Corollary 2.15. *Let $(X, M, \wedge)$ be a fuzzy metric space and $(X, \mathcal{T}^M)$ be the induced fuzzy topological space. Then the followings are equivalent:*

a) $(X, M, \wedge)$ is compact

b) $(X, \mathcal{T}^M)$ is ultra fuzzy compact

c) $(X, \mathcal{T}^M)$ is strong fuzzy compact

d) $(X, w(T^M))$ is fuzzy compact.

Furthermore, if $(X, M, \wedge)$ is co-principle then we can extend the Corollary 2.15 by adding "(e) $(X, \mathcal{T}^M)$ is fuzzy compact".

On the other hand, the compactness of $(X, M, \wedge)$ implies the fuzzy compactness of $(X, \mathcal{T}^M)$ since $\mathcal{T}^M \subseteq w(T^M)$. Moreover, whether the inverse implication holds or not is an open problem.

3. Conclusion

In this study, we focused on to investigate the relation between fuzzy metric and Lowen-type fuzzy topology. We propose a formulation to determine how a fuzzy topology can be constructed by the help of fuzzy metric. We compare the induced fuzzy topology with the fuzzy topology induced by the Lowen functor w and show that they are different for non-coprinciple fuzzy metric spaces. Furthermore, we present a modified and stronger version of continuity for fuzzy metric spaces in order to get equivalence of continuity between fuzzy metric spaces and induced fuzzy topological spaces. Finally we study compactness for such spaces and show that compactness of fuzzy metric spaces coincides with the stronger version of fuzzy compactness of induced fuzzy topological spaces.

Acknowledgements. This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary

References

[1] T. Bag, S. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems 151 (2005) 513–547.
[2] C. L. Chang, Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 24 (1968) 182–190.
[3] T. E. Gantner, R. C. Steinlage, R. H. Warren, Compactness in fuzzy topological spaces, Journal of Mathematical Analysis and Applications 62 (1978) 547–562.
[4] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.
[5] A. George, P. Veeramani, Some theorems in fuzzy metric spaces, Journal of Fuzzy Mathematics 3 (1995) 933–940.
[6] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 485–489.
[7] V. Gregori, J. J. Minana, S. Morillas, A note on local bases and convergence in fuzzy metric spaces, Topology and its Applications 163 (2014) 142–148.
[8] V. Gregori, S. Morillas, A. Sapena, On a class of completable fuzzy metric spaces, Fuzzy Sets and Systems 161 (2010) 2193–2205.
[9] S. Grecova, A. Sostak, I. Uljane, A construction of a fuzzy topology from a strong fuzzy metric, Applied General Topology 17 (2016) –.
[10] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975) 336–344.
[11] R Lowen, Fuzzy topological spaces and fuzzy compactness, Journal of Mathematical Analysis and Applications 56 (1976) 621–633.
[12] R. Lowen, A comparison of different compactness notions in fuzzy topological spaces, Journal of Mathematical Analysis and Applications 64 (1978) 446–454.
[13] J. J. Miñana, A. Šostak, Fuzzifying topology induced by a strong fuzzy metric, Fuzzy Sets and Systems 300 (2016) 24–39.
[14] I. Mardones-Perez, M. A. Prada Vicente, Fuzzy pseudometric spaces vs fuzzifying structures, Fuzzy Sets and Systems 267 (2015) 117–132.
[15] I. Mardones-Perez, M. A. Prada Vicente, A representation theorem for fuzzy pseudometrics, Fuzzy Sets and Systems 195 (2012) 90–99.
[16] I. M. Saheli, Fuzzy Topology Generated by Fuzzy Norm, Iranian Journal of Fuzzy Systems 13 (2016) 113–123.
[17] B. Schweizer, A. Sklar, Statistical metric spaces, Pac.J.Math 10 (1960) 314–334.
[18] P. Wuyts, On the determination of fuzzy topological spaces and fuzzy neighborhood spaces by their level-topologies, Fuzzy Sets and Systems 12 (1984) 71–85.
[19] P. Wuyts, On level-topologies and maximality of fuzzy topological spaces, Fuzzy Sets and Systems 79 (1996) 337–339.
[20] Y. Yue, F. G. Shi, On fuzzy pseudo-metric spaces, Fuzzy Sets and Systems 161 (2010) 1105–1116.