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Some Identities on λ-Analogues of r-Stirling Numbers of the First Kind

Taekyun Kim[a], Dae San Kim[b]

[a]*Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[b]*Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*

Abstract. In this paper, we study λ-analogues of the r-Stirling numbers of the first kind which have close connections with the r-Stirling numbers of the first kind and λ-Stirling numbers of the first kind. Specifically, we give the recurrence relations for these numbers and show their connections with the λ-Stirling numbers of the first kind and higher-order Daehee polynomials.

1. Introduction

It is known that the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (\text{see } [1,2,6-9,14]), \tag{1}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1)$.

For $\lambda \in \mathbb{R}$, the λ-analogue of falling factorial sequence is defined by

$$\begin{aligned}&(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1),\\ &(\text{see } [2,10,14,15,17]).\end{aligned} \tag{2}$$

In view of (1), we define λ-analogues of the Stirling numbers of the first kind as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k)x^k, \quad (\text{see } [2,11-13,16,17]). \tag{3}$$

It is not difficult to show that

$$(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \binom{x}{l}_\lambda t^l = \sum_{l=0}^{\infty} \frac{(x)_{l,\lambda}}{l!} t^l, \quad (\text{see } [4,7-17]), \tag{4}$$

where $\binom{x}{l}_\lambda$ are the λ-analogues of binomial coefficients $\binom{x}{n}$ given by $\binom{x}{l}_\lambda = \frac{(x)_{l,\lambda}}{l!}$.

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Email addresses: tkkim@kw.ac.kr (Taekyun Kim), dskim@sogang.ac.kr (Dae San Kim)

The r-Stirling numbers of the first kind are defined by the generating function

$$\frac{1}{k!}\Big(\log(1+t)\Big)^k(1+t)^r=\sum_{n=k}^{\infty}S_1^{(r)}(n,k)\frac{t^n}{n!},\quad \text{(see [3,20-23])}. \tag{5}$$

where $k\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{R}$.

The unsigned r-Stirling numbers of the first kind are defined as

$$(x+r)(x+r+1)\cdots(x+r+n-1)=\sum_{k=0}^{n}{\left[n+r \atop k+r \right]}_r x^k,\quad \text{(see [1,17,22])}. \tag{6}$$

Thus, by (5), we get

$$(x+r)_n=(x+r)(x+r-1)\cdots(x+r-n+1)=\sum_{k=0}^{n}S_1^{(r)}(n,k)x^k,\quad \text{(see [1])}. \tag{7}$$

From (5) and (7), we note that

$$S_1^{(-r)}(n,k)=(-1)^{n-k}{\left[n+r \atop k+r \right]}_r. \tag{8}$$

The higher-order Daehee polynomials are defined by

$$\left(\frac{\log(1+t)}{t}\right)^k(1+t)^x=\sum_{n=0}^{\infty}D_n^{(k)}(x)\frac{t^n}{n!},\quad \text{(see [5,18,19,24])}. \tag{9}$$

When $x=0$, $D_n^{(k)}=D_n^{(k)}(0)$ are called the higher-order Daehee numbers. In particular, for $k=1$, $D_n(x)=D_n^{(1)}(x)$, $(n\geq 0)$, are called the ordinary Daehee polynomials.

In this paper, we consider λ-analogues of r-Stirling numbers of the first kind which are derived from the λ-analogues of the falling factorial sequence and investigate some properties for these numbers. Specifically, we give some identities and recurrence relations for the λ-analogues of r-Stirling numbers of the first kind and show their connections with the λ-Stirling numbers of the first kind and higher-order Daehee polynomials.

2. λ-analogues of r-Stirling numbers of the first kind

From (3) and (4), we have

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}}&=\sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}S_{1,\lambda}(k,n)x^n\right)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(n!\sum_{k=n}^{\infty}S_{1,\lambda}(k,n)\frac{t^k}{k!}\right)\frac{x^n}{n!}.\end{aligned} \tag{10}$$

On the other hand, we also have

$$(1+\lambda t)^{\frac{x}{\lambda}}=e^{\frac{x}{\lambda}\log(1+\lambda t)}=\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^n\frac{x^n}{n!}. \tag{11}$$

Therefore, by (10) and (11), we get the generating function for $S_{1,\lambda}(n,k)$, $(n,k\geq 0)$, which is given by

$$\frac{1}{n!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^n=\sum_{k=n}^{\infty}S_{1,\lambda}(k,n)\frac{t^k}{k!}. \tag{12}$$

Now, we define λ-analogues of r-Stirling numbers of the first kind as

$$\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!}, \tag{13}$$

where $k \in \mathbb{N} \cup \{0\}$, and $r \in \mathbb{R}$.

From (12) and (13), we note that $S_{1,\lambda}^{(0)}(n,k) = S_{1,\lambda}(n,k)$, $(n \geq k \geq 0)$. Also, it is easy to show that

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=0}^{\infty}(x+r)_{n,\lambda}\frac{t^n}{n!}. \tag{14}$$

By (14), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(x+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\binom{x+r}{n}_{\lambda} t^n = (1+\lambda t)^{\frac{r}{\lambda}} e^{\frac{x}{\lambda}\log(1+\lambda t)} \\
&= \sum_{k=0}^{\infty} x^k \frac{1}{k!}\left(\frac{\log(1+\lambda t}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} \\
&= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k\right)\frac{t^n}{n!}.
\end{aligned} \tag{15}$$

Therefore, by comparing the coefficients on both sides of (15), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k.$$

Now, we observe that

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^k \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} \\
&= \left(\sum_{k=0}^{\infty} x^k \sum_{m=k}^{\infty} S_{1,\lambda}(m,k)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right) \\
&= \left(\sum_{m=0}^{\infty}\sum_{k=0}^{m} S_{1,\lambda}(m,k)x^k\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}x^k\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}x^k\right)\frac{t^n}{n!}.
\end{aligned} \tag{16}$$

Thus, by (15) and (16), we get

$$\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k = \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}\right)x^k. \tag{17}$$

Therefore, by comparing the coefficients on both sides of (17), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$S_{1,\lambda}^{(r)}(n,k) = \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k)(r)_{n-m,\lambda}.$$

Now, we define λ-analogues of the unsigned r-Stirling numbers of the first kind as follows:

$$(x+r)(x+r+\lambda)(x+r+2\lambda)+\cdots(x+r+(n-1)\lambda) = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} x^k. \tag{18}$$

Note that $\lim_{\lambda\to 1} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r}$, $(n \geq k \geq 0)$.
By Theorem 2.1 and (18), we get

$$(x-r)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}^{(-r)}(n,k)x^k, \tag{19}$$

and

$$(x-r)_{n,\lambda} = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} x^k. \tag{20}$$

From (19) and (20), we can easily derive the following equation (21).

$$S_{1,\lambda}^{(-r)}(n,k) = (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda}, \ (n \geq k \geq 0). \tag{21}$$

For $n \geq 1$, by Theorem 2.1, we get

$$(x+r)_{n+1,\lambda} = \sum_{k=0}^{n+1} S_{1,\lambda}^{(r)}(n+1,k)x^k = \sum_{k=1}^{n+1} S_{1,\lambda}^{(r)}(n+1,k)x^k + (r)_{n+1,\lambda}. \tag{22}$$

On the other hand, by (2), we get

$$\begin{aligned}
(x+r)_{n+1,\lambda} &= (x+r)_{n,\lambda}(x+r-n\lambda) \\
&= x\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k - (n\lambda - r)\sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n,k)x^k \\
&= \sum_{k=1}^{n} S_{1,\lambda}^{(r)}(n,k-1)x^k - \sum_{k=1}^{n} (n\lambda - r)S_{1,\lambda}^{(r)}(n,k)x^k + (r-n\lambda)(r)_{n,\lambda} + x^{n+1} \\
&= \sum_{k=1}^{n} \left\{ S_{1,\lambda}^{(r)}(n,k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n,k) \right\} x^k + (r)_{n+1,\lambda} + x^{n+1}.
\end{aligned} \tag{23}$$

Therefore, by Theorem 2.1 and (23), we obtain the following theorem.

Theorem 2.3. *For $1 \leq k \leq n$, we have*

$$S_{1,\lambda}^{(r)}(n+1,k) = S_{1,\lambda}^{(r)}(n,k-1) - (n\lambda - r)S_{1,\lambda}^{(r)}(n,k).$$

From (13), we note that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k\sum_{l=0}^{\infty}\frac{r^l}{l!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^l\\ &= \sum_{l=0}^{\infty}\binom{k+l}{l}r^l\frac{1}{(k+l)!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k+l}\\ &= \sum_{l=0}^{\infty}\binom{k+l}{l}r^l\sum_{n=k+l}^{\infty}S_{1,\lambda}(n,k+l)\frac{t^n}{n!}\\ &= \sum_{l=0}^{\infty}r^l\binom{k+l}{l}\sum_{n=l}^{\infty}S_{1,\lambda}(n+k,k+l)\frac{t^{n+k}}{(n+k)!}\\ &= \sum_{n=0}^{\infty}\left(\frac{n!t^k}{(n+k)!}\sum_{l=0}^{n}r^l\binom{k+l}{l}S_{1,\lambda}(n+k,k+l)\right)\frac{t^n}{n!}.\end{aligned}\tag{24}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}}\\ &= \left(\sum_{l=0}^{\infty}D_l^{(k)}\frac{\lambda^l t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(r)_{m,\lambda}\frac{t^m}{m!}\right)\frac{t^k}{k!}\\ &= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda}\frac{t^n}{n!}\right)\frac{t^k}{k!}.\end{aligned}\tag{25}$$

Thus, by (24) and (25), we get

$$\sum_{l=0}^{n}r^l\frac{\binom{k+l}{l}}{\binom{n+k}{n}}S_{1,\lambda}(n+k,k+l) = \sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda}.\tag{26}$$

Therefore, by (26), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}D_l^{(k)}\lambda^l(r)_{n-l,\lambda} = \sum_{l=0}^{n}\frac{\binom{k+l}{l}}{\binom{n+k}{n}}r^l S_{1,\lambda}(n+k,k+l).$$

Now, we observe that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right)\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k\\ &= \sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(r)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{27}$$

Therefore, by (13) and (27), we obtain the following theorem.

Theorem 2.5. *For $n, k \geq 0$, with $n \geq k$, we have*

$$S_{1,\lambda}^{(r)}(n,k) = \sum_{m=k}^{n}\binom{n}{m}(r)_{n-m,\lambda}S_{1,\lambda}(m,k).$$

From (13), we note that

$$\begin{aligned}
&\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k}(1+\lambda t)^{\frac{r}{\lambda}}\\
&=\frac{(m+k)!}{m!k!}\frac{1}{(m+k)!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m+k}(1+\lambda t)^{\frac{r}{\lambda}}\\
&=\binom{m+k}{m}\sum_{n=m+k}^{\infty}S_{1,\lambda}^{(r)}(n,m+k)\frac{t^n}{n!}.
\end{aligned}\tag{28}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{m}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k}(1+\lambda t)^{\frac{r}{\lambda}}\\
&=\left(\sum_{l=m}^{\infty}S_{1,\lambda}(l,m)\frac{t^l}{l!}\right)\left(\sum_{j=k}^{\infty}S_{1,\lambda}^{(r)}(j,k)\frac{t^j}{j!}\right)\\
&=\sum_{n=m+k}^{\infty}\left(\sum_{l=k}^{n-m}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)S_{1,\lambda}(n-l,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{29}$$

Therefore, by (28) and (29), we obtain the following theorem.

Theorem 2.6. *For $m,n,k\geq 0$ with $n\geq m+k$, we have*

$$\binom{m+k}{m}S_{1,\lambda}^{(r)}(n,m+k)=\sum_{l=k}^{n-m}\binom{n}{l}S_{1,\lambda}(l,k)S_{1,\lambda}(n-l,m).$$

By (12), we get

$$\begin{aligned}
\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}&=\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^{k}(1+\lambda t)^{\frac{r}{\lambda}}(1+\lambda t)^{-\frac{r}{\lambda}}\\
&=\left(\sum_{l=k}^{\infty}S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{-\frac{r}{\lambda}}{m}\lambda^m t^m\right)\\
&=\left(\sum_{l=k}^{\infty}S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^m(r+(m-1)\lambda)_{m,\lambda}\frac{t^m}{m!}\right)\\
&=\sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)(-1)^{n-l}(r+(n-l-1)\lambda)_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}\tag{30}$$

Comparing the coefficients on both sides of (30), we have the following theorem.

Theorem 2.7. *For $n,k\geq 0$, with $n\geq k$, we have*

$$S_{1,\lambda}(n,k)=\sum_{l=k}^{n}\binom{n}{l}S_{1,\lambda}^{(r)}(l,k)(-1)^{n-l}(r+\lambda(n-l-1))_{n-l,\lambda}.$$

From (9), we have

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} \\ &= \frac{t^k}{k!}\left(\sum_{m=0}^{\infty}D_m^{(k)}\lambda^m\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(r)_{l,\lambda}\frac{t^l}{l!}\right) \\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

On the other hand, by (13), we get

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \sum_{n=k}^{\infty}S_{1,\lambda}^{(r)}(n,k)\frac{t^n}{n!} \\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}S_{1,\lambda}^{(r)}(n+k,k)\frac{n!k!}{(n+k)!}\frac{t^n}{n!}.\end{aligned} \tag{32}$$

Thus, by comparing the coefficients on both sides of (31) and (32), we get

$$\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda} = \frac{1}{\binom{n+k}{n}}S_{1,\lambda}^{(r)}(n+k,k). \tag{33}$$

Therefore, by (33), we obtain the following theorem.

Theorem 2.8. *For $n,k \geq 0$, we have*

$$S_{1,\lambda}^{(r)}(n+k,k) = \binom{n+k}{n}\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}\lambda^m(r)_{n-m,\lambda}.$$

From (9), we note that

$$\begin{aligned}\frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} &= \frac{t^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda t}\right)^k(1+\lambda t)^{\frac{r}{\lambda}} \\ &= \frac{t^k}{k!}\sum_{n=0}^{\infty}\lambda^n D_n^{(k)}\left(\tfrac{r}{\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

By (32) and (34), we get

$$S_{1,\lambda}^{(r)}(n+k,k) = \lambda^n\frac{(n+k)!}{n!k!}D_n^{(k)}\left(\tfrac{r}{\lambda}\right) = \lambda^n\binom{n+k}{n}D_n^{(k)}\left(\tfrac{r}{\lambda}\right), \ (n \geq 0). \tag{35}$$

In particular, for $r = 0$, from (30) and (35) we have

$$\begin{aligned}\lambda^n\binom{n+k}{k}D_n^{(k)} &= S_{1,\lambda}(n+k,k) \\ &= \sum_{l=k}^{n+k}\binom{n+k}{l}S_{1,\lambda}^{(r)}(l,k)(-1)^{n+k-l}(r+(n+k-l-1)\lambda)_{n+k-l,\lambda},\end{aligned} \tag{36}$$

where $n,k \geq 0$.

Therefore, by (36), we obtain the following theorem.

Theorem 2.9. *For $n, k \geq 0$, we have*

$$\lambda^n \binom{n+k}{k} D_n^{(k)} = \sum_{l=k}^{n+k} \binom{n+k}{l} S_{1,\lambda}^{(r)}(l,k)(-1)^{n+k-l}(r+(n+k-l-1)\lambda)_{n+k-l,\lambda}.$$

In addition,

$$D_n^{(k)} = \frac{1}{\binom{n+k}{k}} \sum_{l=k}^{n+k} \binom{n+k}{l}\binom{l}{k}\left(\frac{1}{\lambda}\right)^{n+k-l} \times (r+(n+k-l-1)\lambda)_{n+k-l,\lambda}(-1)^{n+k-l} D_{l-k}^{(k)}\left(\tfrac{r}{\lambda}\right).$$

Now, we observe that

$$\begin{aligned}\sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} e^{-\frac{r}{\lambda}\log(1+\lambda t)} \\ &= \left(\sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right)\sum_{m=0}^{\infty}(-1)^m r^m \frac{1}{m!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^m \\ &= \left(\sum_{l=k}^{\infty} S_{1,\lambda}^{(r)}(l,k)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^m r^m \sum_{j=m}^{\infty} S_{1,\lambda}(j,m)\frac{t^j}{j!}\right) \\ &= \sum_{n=k}^{\infty}\left(\sum_{j=0}^{n-k}\sum_{m=0}^{j}\binom{n}{j}(-1)^m r^m S_{1,\lambda}(j,m) S_{1,\lambda}(n-j,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{37}$$

Therefore, by comparing the coefficients on both sides of (37), we obtain the following theorem

Theorem 2.10. *For $n, k \geq 0$, with $n \geq k$, we have*

$$S_{1,\lambda}(n,k) = \sum_{j=0}^{n-k}\sum_{m=0}^{j}\binom{n}{j}(-1)^m r^m S_{1,\lambda}(j,m) S_{1,\lambda}(n-j,k).$$

For $m, n \geq 0$, we define λ-analogues of the Whitney's type r-Stirling numbers of the first kind as

$$\begin{aligned}(mx+r)_{n,\lambda} &= (mx+r)(mx+r-\lambda)(mx+r-2\lambda)\cdots(mx+r-(n-1)\lambda) \\ &= \sum_{k=0}^{n} T_{1,\lambda}^{(r)}(n,k|m)x^k.\end{aligned} \tag{38}$$

By (38), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(mx+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} T_{1,\lambda}^{(r)}(n,k|m)x^k\right)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} T_{1,\lambda}^{(r)}(n,k|m)\frac{t^n}{n!}\right)x^k.\end{aligned} \tag{39}$$

On the other hand, by binomial expansion, we get

$$\begin{aligned}\sum_{n=0}^{\infty}(mx+r)_{n,\lambda}\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\binom{mx+r}{n}_{\lambda} t^n \\ &= (1+\lambda t)^{\frac{mx+r}{\lambda}} = (1+\lambda t)^{\frac{r}{\lambda}} e^{mx\left(\frac{\log(1+\lambda t)}{\lambda}\right)} \\ &= \sum_{k=0}^{\infty}\frac{m^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} x^k.\end{aligned} \tag{40}$$

Comparing the coefficients on both sides of (39) and (40), the generating function for $T^{(r)}_{1,\lambda}(n,k|m)$, $(n,k \geq 0)$, is given by

$$\frac{m^k}{k!}\left(\frac{\log(1+\lambda t)}{\lambda}\right)^k (1+\lambda t)^{\frac{r}{\lambda}} = \sum_{n=k}^{\infty} T^{(r)}_{1,\lambda}(n,k|m)\frac{t^n}{n!}. \tag{41}$$

From (13) and (41), we note that

$$S^{(r)}_{1,\lambda}(n,k) = \frac{1}{m^k} T^{(r)}_{1,\lambda}(n,k|m), \ (n \geq k \geq 0). \tag{42}$$

It is known that the r-Whitney numbers are defined as

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \quad \text{(see [3])}. \tag{43}$$

By (3), we get

$$\begin{aligned}
(mx+r)_{n,\lambda} &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)(mx+r)^l \\
&= \sum_{l=0}^{n} S_{1,\lambda}(n,l) \sum_{j=0}^{l} m^j W_{m,r}(l,j)(x)_j \\
&= \sum_{j=0}^{n}\sum_{l=j}^{n} S_{1,\lambda}(n,l) m^j W_{m,.r}(l,j)(x)_j \\
&= \sum_{j=0}^{n}\sum_{l=j}^{n} S_{1,\lambda}(n,l) m^j W_{m,r}(l,j) \sum_{k=0}^{j} S_1(j,k)x^k \\
&= \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\sum_{l=j}^{n} S_{1,\lambda}(n,l)S_1(j,k)m^j W_{m,r}(l,j)\right)x^k.
\end{aligned} \tag{44}$$

Therefore, by (38) and (44), we obtain the following theorem.

Theorem 2.11. *For $n,k \geq 0$, with $n \geq k$, we have*

$$T^{(r)}_{1,\lambda}(n,k|m) = \sum_{j=k}^{n}\sum_{l=j}^{n} S_{1,\lambda}(n,l)S_1(j,k)m^j W_{m,r}(l,j).$$

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