Filomat 34:2 (2020), 461–467
https://doi.org/10.2298/FIL2002461C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Solution of the Word Problem for Braid Groups via the Complex Reflection Group G_{12}

Ahmet Sinan Cevik[a], **Amer Hassan Albargi**[b]

[a]*Department of Mathematics, Science Faculty, King Abdulaziz University, 21589, Jeddah-Saudi Arabia.*
(Prior) Department of Mathematics, Science Faculty, Selcuk University, 42075, Konya-Turkey.
[b]*Department of Mathematics, Science Faculty, King Abdulaziz University, 21589, Jeddah-Saudi Arabia.*

Abstract. It is known that if there exists a Gröbner-Shirshov basis for a group G, then we say that one of the decision problem, namely the word problem, is solvable for G as well. Therefore, as the main target of this paper, we will present a (non-commutative) Gröbner-Shirshov basis for the braid group associated with the congruence classes of complex reflection group G_{12} which will give us normal forms of the elements of G_{12} and so will obtain a new algorithm to solve the word problem over it.

1. Introduction and Preliminaries

The Gröbner basis theory for commutative algebras was introduced by Buchberger [10] and provides a solution to the reduction problem for commutative algebras. In [4] Bergman generalized the Gröbner basis theory to associative algebras by proving the "Diamond Lemma". On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [20]. In [6] Bokut noticed that Shirshov's method works for also associative algebras. Hence, for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner-Shirshov basis* theory. There are some important studies on this subject related to the groups (see, for instance, [7, 11]). We may finally refer the papers [2, 3, 8, 14–17] for some other recent studies over Gröbner-Shirshov bases.

Algorithmic problems such as the *word*, *conjugacy* and *isomorphism problems* have played an important role in group theory since the work of M. Dehn in early 1900's. These problems are called *decision problems* which ask for a yes or no answer to a specific question. Among these decision problems especially the word problem has been studied widely in groups (see [1]). It is well known that the word problem for finitely presented groups is not solvable in general; that is, given any two words obtained by generators of the group, there may be no algorithm to decide whether these words represent the same element in this group. Gröbner-Shirshov basis theory, which is the main theme of this paper, is one of the most effective

2010 *Mathematics Subject Classification*. Primary 20F10; Secondary 16S15, 20F05, 20F36, 20E22, 20M05
Keywords. Gröbner-Shirshov basis, presentation, braid group, word problem
Received: 07 January 2019; Revised: 13 March 2019; Accepted: 16 March 2019
Communicated by Prof. Yilmaz Simsek
This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No (130-246-D1439). The authors, therefore, acknowledge with thanks DSR technical and financial support.
This article is dedicated to *Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary* and presented in the conference *The Mediterranean International Conference of Pure & Applied Mathematics and related areas* (MICOPAM 2018)-Antalya, Turkey
Email addresses: ahmetsinancevik@gmail.com (Ahmet Sinan Cevik), aalbarqi@kau.edu.sa (Amer Hassan Albargi)

and computational method to solve the word problem for a given algebraic structure not only for groups, monoids and semigroups but also other important algebras.

In the study [5], the authors first defined the presentations for the braid groups associated with the complex reflection groups G_{24} and G_{27} and they used VKCURVE that is a GAP package implementing Van Kampen's method to obtain these presentations. Moreover, they added some conjectures for the cases of G_{29}, G_{31}, G_{33} and G_{34}.

Shephard and Todd (1954) classified all finite complex reflection groups in [18]. Later, Cohen (1976) gave a more systematic description for these groups in terms of root systems, vector graphs and root graphs [12]. Recently, in [13], Howlett and Shi defined a simple root system (B, w) for such these groups which is analogous to the corresponding concept for a Coxeter group.

It is well known that any Coxeter group can be presented by generators and relations. A finite complex reflection group G can also be presented in a similar way (see, for example, [9]). But such a presentation is not unique for G in general. Different presentations of G may reveal various different properties of G. Then it is worth to define a congruence relation among the presentations of G (see [19]) and then to ask that question "How many congruence classes of presentations are there for any irreducible finite complex reflection group G?". In [9], the authors solve this problem for the finite primitive complex reflection groups $G = \{G_7, G_{11}, G_{15}, G_{19}, G_{27}\}$ and in [19], Shi studied the finite primitive complex reflection groups $G = \{G_{12}, G_{24}, G_{25}, G_{26}\}$ (in the notations of Shephard and Todd, 1954). So by considering the presentation of G_{12} given in [19], our aim in this paper is to find Gröbner-Shirshov bases of thise important group.

The method of Gröbner-Shirshov bases which is the main theme of this paper gives a new algorithm to get normal forms of elements of groups, and so a new algorithm for solving the word problem in these groups. By considering this fact, our aim in this paper is to find Gröbner-Shirshov bases for the braid group associated with the congruence classes of complex reflection group G_{12}.

Throughout this paper, by considering the lengths of any two words, we will use the deg-lex ordering if the lengths of these words are different or lexicographically ordering if otherwise. Additionally the notations $(i) \wedge (j)$ and $(i) \vee (j)$ will denote the intersection and inclusion compositions of relations (i) and (j), respectively.

2. Gröbner-Shirshov Bases and Composition-Diamond Lemma

Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$. Suppose that X^* is a well ordered set. Then every nonzero polynomial $f \in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Let f and g be two monic polynomials in $K\langle X\rangle$. We then have two compositions as follows:

- If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the i*ntersection composition* of f and g with respect to w. The word w is called an *ambiguity* of intersection.

- If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

If g is monic, $\overline{f} = a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then transformation $f \mapsto f - \alpha agb$ is called elimination (ELW) of the leading word of g in f.

Let $S \subseteq K\langle X\rangle$ with each $s \in S$ is monic. Then the composition $(f, g)_w$ is called trivial modulo (S, w) if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in K, a_i, b_i \in X^*, s_i \in S$ and $a_i\overline{s_i}b_i < w$. If this is the case, then we write $(f, g)_w \equiv 0 \; mod(S, w)$.

We call the set S endowed with the well ordering $<$ a *Gröbner-Shirshov basis* for $K\langle X \mid S\rangle$ if any composition $(f, g)_w$ of polynomials in S is trivial modulo S and corresponding w.

The following lemma was proved by Shirshov [20] for free Lie algebras with deg-lex ordering.

Lemma 2.1 (Composition-Diamond Lemma). *Let K be a field, $A = K\langle X \mid S\rangle = K\langle X\rangle/Id(S)$ and $<$ a monomial ordering on X^*, where $Id(S)$ is the ideal of $K\langle X\rangle$ generated by S.Then the following statements are equivalent:*

1. *S is a Gröbner-Shirshov basis.*
2. *$f \in Id(S) \Rightarrow \overline{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.*
3. *$Irr(S) = \{u \in X^* \mid u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a basis for the algebra $A = K\langle X \mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process many times (maybe infinitely), we eventually obtain a Gröbner-Shirshov basis S^{comp}. We should note that such a process is called the *Shirshov algorithm*.

3. Main Results

In this section, we would like to obtain normal form of elements of the braid group associated with the congruence classes of complex reflection group G_{12} by using Gröbner-Shirshov basis theory. In [19], the author has just obtained a presentation for G_{12} he did not give the structure of elements of the group G_{12}. On that respect the result given here is worth to study in Algebra. The presentation of the braid group associated with the congruence classes of complex reflection group G_{12} is as follows:

Theorem 3.1 ([19]). *The braid group associated with the congruence classes of complex reflection group G_{12} admits the presentation*

$$\left\langle s, t, u; s^2 = u^2 = t^2 = 1, suts = utsu, tsut = suts \right\rangle. \qquad (1)$$

Now we consider the monoid presentation of G_{12} given in (1). Since we have the relations $s^2 = u^2 = t^2 = 1$ for the generators s, u, t, it can be easily seen that the monoid presentation is the same with the group presentation of G_{12} given in (1). To obtain Gröbner-Shirshov basis of the group G_{12}, let us consider an ordering among the generators as $t > u > s$.

The main result of this paper is the following.

Theorem 3.2. *A Gröbner-Shirshov basis of the braid group associated with the congruence classes of the complex reflection group G_{12} consists of the following polynomials:*

(1) $s^2 - 1$,	(2) $u^2 - 1$,	(3) $t^2 - 1$,	(4) $utsu - suts$,
(5) $utst - stsu$,	(6) $utus - stut$,	(7) $ustu - stus$,	(8) $usuts - tsu$,
(9) $ustsu - tst$,	(10) $(ut)^2u - (st)^2s$,	(11) $(us)^2t - s(tu)^2$,	(12) $(us)^2ut - (ts)^2$,
(13) $(us)^3 - (su)^3$,	(14) $u(st)^2s - (tu)^2$,	(15) $us(ut)^2 - (su)^2tu$,	(16) $tsut - suts$,
(17) $tuts - sutu$,	(18) $tsus - usut$,	(19) $tusu - sust$,	(20) $tust - stus$,
(21) $tstu - usts$,	(22) $(ts)^2u - s(ut)^2$,	(23) $(ts)^2t - (us)^2u$,	(24) $(tu)^2t - (su)^2s$.

Proof. We need to prove that all compositions among relations (1) – (24) are trivial. We start with listing all intersections compositions among relations. Actually we have the following ambiguities w:

$(1) \wedge (1) : w = s^3$,	$(2) \wedge (2) : w = u^3$,	$(2) \wedge (4) : w = u^2tsu$,
$(2) \wedge (5) : w = u^2tst$,	$(2) \wedge (6) : w = u^2tus$,	$(2) \wedge (7) : w = u^2stu$,
$(2) \wedge (8) : w = u^2suts$,	$(2) \wedge (9) : w = u^2stsu$,	$(2) \wedge (10) : w = u^2(tu)^2$,
$(2) \wedge (11) : w = u^2sust$,	$(2) \wedge (12) : w = u^2(su)^2t$,	$(2) \wedge (13) : w = u^2(su)^2s$,
$(2) \wedge (14) : w = u^2(st)^2s$,	$(2) \wedge (15) : w = u^2s(ut)^2$,	$(3) \wedge (3) : w = t^3$,
$(3) \wedge (16) : w = t^2sut$,	$(3) \wedge (17) : w = t^2uts$,	$(3) \wedge (18) : w = t^2sus$,
$(3) \wedge (19) : w = t^2usu$,	$(3) \wedge (20) : w = t^2ust$,	$(3) \wedge (21) : w = t^2stu$,
$(3) \wedge (22) : w = t^2stsu$,	$(3) \wedge (23) : w = t^2(st)^2$,	$(3) \wedge (24) : w = t^2(ut)^2$,
$(4) \wedge (2) : w = utsu^2$,	$(4) \wedge (4) : w = utsutsu$,	$(4) \wedge (5) : w = (uts)^2t$,
$(4) \wedge (6) : w = utsutus$,	$(4) \wedge (7) : w = utsustu$,	$(4) \wedge (8) : w = utsusuts$,
$(4) \wedge (9) : w = utsustsu$,	$(4) \wedge (10) : w = uts(ut)^2u$,	$(4) \wedge (11) : w = uts(us)^2t$,

$(4) \wedge (12) : w = ut(su)^3t,$ $(4) \wedge (13) : w = ut(su)^3s,$ $(4) \wedge (14) : w = utsu(st)^2s,$
$(4) \wedge (15) : w = ut(su)^2tut,$ $(4) \wedge (16) : w = utsut,$ $(4) \wedge (18) : w = utsus,$
$(5) \wedge (3) : w = utst^2,$ $(5) \wedge (16) : w = utstsut,$ $(5) \wedge (17) : w = utstuts,$
$(5) \wedge (18) : w = utstsus,$ $(5) \wedge (19) : w = utstusu,$ $(5) \wedge (20) : w = utstust,$
$(5) \wedge (21) : w = utststu,$ $(5) \wedge (21) : w = utstu,$ $(5) \wedge (22) : w = ut(st)^2su,$
$(5) \wedge (22) : w = u(ts)^2u,$ $(5) \wedge (23) : w = ut(st)^3,$ $(5) \wedge (23) : w = u(ts)^2t,$
$(5) \wedge (24) : w = utst(ut)^2,$ $(6) \wedge (1) : w = utus^2,$ $(6) \wedge (7) : w = utustu,$
$(6) \wedge (8) : w = utusuts,$ $(6) \wedge (9) : w = utustsu,$ $(6) \wedge (11) : w = ut(us)^2t,$
$(6) \wedge (12) : w = ut(us)^2ut,$ $(6) \wedge (13) : w = ut(us)^3,$ $(6) \wedge (14) : w = utus(ts)^2,$
$(6) \wedge (15) : w = utus(ut)^2,$ $(6) \wedge (19) : w = utusu,$ $(6) \wedge (20) : w = utust,$
$(7) \wedge (2) : w = ustu^2,$ $(7) \wedge (4) : w = ustutsu,$ $(7) \wedge (5) : w = ustutst,$

$(7) \wedge (6) : w = ustutus,$ $(7) \wedge (7) : w = (ust)^2u,$ $(7) \wedge (8) : w = ustusuts,$
$(7) \wedge (9) : w = (ust)^2su,$ $(7) \wedge (10) : w = us(tu)^3,$ $(7) \wedge (11) : w = ust(us)^2t,$
$(7) \wedge (12) : w = ust(us)^2ut,$ $(7) \wedge (13) : w = ust(us)^3,$ $(7) \wedge (14) : w = ustu(st)^2s,$
$(7) \wedge (15) : w = ustus(ut)^2,$ $(7) \wedge (17) : w = ustuts,$ $(7) \wedge (19) : w = ustusu,$
$(7) \wedge (20) : w = (ust)^2,$ $(7) \wedge (24) : w = ust(ut)^2,$ $(8) \wedge (1) : w = usuts^2,$
$(8) \wedge (4) : w = usutsu,$ $(8) \wedge (5) : w = usutst,$ $(8) \wedge (16) : w = u(sut)^2,$
$(8) \wedge (18) : w = usutsus,$ $(8) \wedge (21) : w = usutstu,$ $(8) \wedge (22) : w = usu(ts)^2u,$
$(8) \wedge (23) : w = usut(st)^2,$ $(9) \wedge (2) : w = ustsu^2,$ $(9) \wedge (4) : w = ustsutsu,$
$(9) \wedge (5) : w = ustsutst,$ $(9) \wedge (6) : w = ustsutus,$ $(9) \wedge (7) : w = ustsustu,$
$(9) \wedge (8) : w = ust(su)^2ts,$ $(9) \wedge (9) : w = ustsustsu,$ $(9) \wedge (10) : w = ustsu(tu)^2,$
$(9) \wedge (11) : w = ust(su)^2st,$ $(9) \wedge (12) : w = ust(su)^3t,$ $(9) \wedge (13) : w = ust(su)^3s,$

$(9) \wedge (14) : w = ustsu(st)^2s,$ $(9) \wedge (15) : w = ustsus(ut)^2,$ $(9) \wedge (16) : w = ustsut,$
$(9) \wedge (18) : w = ustsus,$ $(10) \wedge (2) : w = (ut)^2u^2,$ $(10) \wedge (4) : w = (ut)^3su,$
$(10) \wedge (5) : w = (ut)^3st,$ $(10) \wedge (6) : w = (ut)^3us,$ $(10) \wedge (6) : w = (ut)^2us,$
$(10) \wedge (7) : w = (ut)^2ustu,$ $(10) \wedge (8) : w = (ut)^2usuts,$ $(10) \wedge (9) : w = (ut)^2ustsu,$
$(10) \wedge (10) : w = (ut)^4u,$ $(10) \wedge (10) : w = (ut)^3u,$ $(10) \wedge (11) : w = (ut)^2(us)^2t,$

$(10) \wedge (12) : w = (ut)^2(us)^2ut,$ $(10) \wedge (13) : w = (ut)^2(us)^3,$ $(10) \wedge (14) : w = (ut)^2us(ts)^2,$
$(10) \wedge (15) : w = (ut)^2us(ut)^2,$ $(10) \wedge (17) : w = (ut)^3s,$ $(10) \wedge (19) : w = (ut)^2usu,$
$(10) \wedge (20) : w = (ut)^2ust,$ $(10) \wedge (24) : w = (ut)^4,$ $(10) \wedge (24) : w = (ut)^3,$
$(11) \wedge (3) : w = (us)^2t^2,$ $(11) \wedge (7) : w = (us)^2tu,$ $(11) \wedge (9) : w = (us)^2tsu,$
$(11) \wedge (14) : w = (us)^2(ts)^2,$ $(11) \wedge (16) : w = (us)^2tsut,$ $(11) \wedge (17) : w = (us)^2tuts,$
$(11) \wedge (18) : w = (us)^2tsus,$ $(11) \wedge (19) : w = (us)^2tusu,$ $(11) \wedge (20) : w = (us)^2tust,$
$(11) \wedge (21) : w = (us)^2tstu,$ $(11) \wedge (22) : w = (us)^2(ts)^2u,$ $(11) \wedge (23) : w = (us)^2(ts)^2t,$
$(11) \wedge (24) : w = (us)^2(tu)^2t,$ $(12) \wedge (3) : w = (us)^2ut^2,$ $(12) \wedge (4) : w = (us)^2utsu,$
$(12) \wedge (5) : w = (us)^2utst,$ $(12) \wedge (6) : w = (us)^2utus,$ $(12) \wedge (8) : w = (us)^2uts,$

$(12) \wedge (10) : w = (us)^2(ut)^2u,$ $(12) \wedge (15) : w = (us)^2(ut)^2,$ $(12) \wedge (16) : w = (us)^2utsut,$
$(12) \wedge (17) : w = (us)^2(ut)^2s,$ $(12) \wedge (18) : w = (us)^2utsus,$ $(12) \wedge (19) : w = (us)^2utusu,$
$(12) \wedge (20) : w = (us)^2utust,$ $(12) \wedge (21) : w = (us)^2utstu,$ $(12) \wedge (22) : w = (us)^2utstsu,$
$(12) \wedge (23) : w = (us)^2ut(st)^2,$ $(12) \wedge (24) : w = (us)^2(ut)^3,$ $(13) \wedge (1) : w = (us)^3s,$
$(13) \wedge (7) : w = (us)^3tu,$ $(13) \wedge (8) : w = (us)^3uts,$ $(13) \wedge (9) : w = (us)^3tsu,$
$(13) \wedge (11) : w = (us)^4t,$ $(13) \wedge (11) : w = (us)^3t,$ $(13) \wedge (12) : w = (us)^4ut,$
$(13) \wedge (13) : w = (us)^5,$ $(13) \wedge (13) : w = (us)^4,$ $(13) \wedge (14) : w = (us)^3(ts)^2,$
$(13) \wedge (15) : w = (us)^3(ut)^2,$ $(14) \wedge (1) : w = u(st)^2s^2,$ $(14) \wedge (16) : w = u(st)^2sut,$

$(14) \wedge (18) : w = u(st)^2sus,$ $(14) \wedge (21) : w = u(st)^3u,$ $(14) \wedge (22) : w = u(st)^3su,$
$(14) \wedge (22) : w = u(st)^2su,$ $(14) \wedge (23) : w = u(st)^4,$ $(14) \wedge (23) : w = u(st)^3,$
$(15) \wedge (3) : w = us(ut)^2t,$ $(15) \wedge (4) : w = us(ut)^2su,$ $(15) \wedge (5) : w = us(ut)^2st,$
$(15) \wedge (6) : w = us(ut)^2us,$ $(15) \wedge (10) : w = us(ut)^3u,$ $(15) \wedge (10) : w = us(ut)^2u,$
$(15) \wedge (16) : w = us(ut)^2sut,$ $(15) \wedge (17) : w = us(ut)^3s,$ $(15) \wedge (17) : w = us(ut)^2s,$
$(15) \wedge (18) : w = us(ut)^2sus,$ $(15) \wedge (19) : w = us(ut)^2usu,$ $(15) \wedge (20) : w = us(ut)^2ust,$
$(15) \wedge (21) : w = us(ut)^2stu,$ $(15) \wedge (22) : w = us(ut)^2stsu,$ $(15) \wedge (23) : w = us(ut)^2(st)^2,$
$(15) \wedge (24) : w = us(ut)^4,$ $(15) \wedge (24) : w = us(ut)^3,$ $(16) \wedge (3) : w = tsut^2,$
$(16) \wedge (4) : w = tsutsu,$ $(16) \wedge (5) : w = tsutst,$ $(16) \wedge (6) : w = tsutus,$

$(16) \wedge (10) : w = ts(ut)^2u,$ $(16) \wedge (16) : w = (tsut)^2,$ $(16) \wedge (17) : w = ts(ut)^2s,$
$(16) \wedge (18) : w = tsutsus,$ $(16) \wedge (19) : w = tsutusu,$ $(16) \wedge (20) : w = tsutust,$
$(16) \wedge (21) : w = tsutstu,$ $(16) \wedge (22) : w = tsutstsu,$ $(16) \wedge (23) : w = tsu(ts)^2t,$
$(16) \wedge (24) : w = ts(ut)^3,$ $(17) \wedge (1) : w = tuts^2,$ $(17) \wedge (4) : w = tutsu,$
$(17) \wedge (5) : w = tutst,$ $(17) \wedge (16) : w = tutsut,$ $(17) \wedge (18) : w = tutsus,$
$(17) \wedge (21) : w = tutstu,$ $(17) \wedge (22) : w = tu(ts)^2u,$ $(17) \wedge (23) : w = tut(st)^2,$
$(18) \wedge (1) : w = tsus^2,$ $(18) \wedge (7) : w = tsustu,$ $(18) \wedge (8) : w = t(su)^2ts,$
$(18) \wedge (9) : w = tsustsu,$ $(18) \wedge (11) : w = t(su)^2st,$ $(18) \wedge (12) : w = t(su)^3t,$
$(18) \wedge (13) : w = t(su)^3s,$ $(18) \wedge (14) : w = tsu(st)^2s,$ $(18) \wedge (15) : w = t(su)^2tut,$
$(19) \wedge (2) : w = tusu^2,$ $(19) \wedge (4) : w = tusutsu,$ $(19) \wedge (5) : w = tusutst,$
$(19) \wedge (6) : w = tusutus,$ $(19) \wedge (7) : w = tusustu,$ $(19) \wedge (8) : w = tu(su)^2ts,$
$(19) \wedge (8) : w = tusuts,$ $(19) \wedge (9) : w = t(us)^2tsu,$ $(19) \wedge (10) : w = tusu(tu)^2,$
$(19) \wedge (11) : w = t(us)^3t,$ $(19) \wedge (11) : w = t(us)^2t,$ $(19) \wedge (12) : w = t(us)^3ut,$
$(19) \wedge (12) : w = t(us)^2ut,$ $(19) \wedge (13) : w = t(us)^4,$ $(19) \wedge (13) : w = t(us)^3,$
$(19) \wedge (14) : w = t(us)^2(ts)^2,$ $(19) \wedge (15) : w = t(us)^2(ut)^2,$ $(19) \wedge (15) : w = tus(ut)^2,$

$(20) \wedge (3) : w = tust^2,$ $(20) \wedge (7) : w = tustu,$ $(20) \wedge (9) : w = tustsu,$
$(20) \wedge (14) : w = tu(st)^2s,$ $(20) \wedge (16) : w = tustsut,$ $(20) \wedge (17) : w = tustuts,$
$(20) \wedge (18) : w = tustsus,$ $(20) \wedge (19) : w = (tus)^2u,$ $(20) \wedge (20) : w = t(ust)^2,$
$(20) \wedge (21) : w = tu(st)^2u,$ $(20) \wedge (22) : w = tu(st)^2su,$ $(20) \wedge (23) : w = tu(st)^3,$
$(20) \wedge (24) : w = tus(tu)^2t,$ $(21) \wedge (2) : w = tstu^2,$ $(21) \wedge (4) : w = tstutsu,$
$(21) \wedge (5) : w = tstutst,$ $(21) \wedge (6) : w = ts(tu)^2s,$ $(21) \wedge (7) : w = tstustu,$
$(21) \wedge (8) : w = tstusuts,$ $(21) \wedge (9) : w = tstustsu,$ $(21) \wedge (10) : w = ts(tu)^3,$
$(21) \wedge (11) : w = tst(us)^2t,$ $(21) \wedge (12) : w = tst(us)^2ut,$ $(21) \wedge (13) : w = tst(us)^3,$
$(21) \wedge (14) : w = tstu(st)^2s,$ $(21) \wedge (15) : w = tstus(ut)^2,$ $(21) \wedge (17) : w = tstuts,$
$(21) \wedge (20) : w = tstust,$ $(21) \wedge (24) : w = tst(ut)^2,$ $(22) \wedge (2) : w = (ts)^2u^2,$

$(22) \wedge (4) : w = (ts)^2utsu,$ $(22) \wedge (5) : w = (ts)^2utst,$
$(22) \wedge (6) : w = (ts)^2utus,$ $(22) \wedge (7) : w = (ts)^2ustu,$
$(22) \wedge (8) : w = (ts)^2usuts,$ $(22) \wedge (9) : w = (ts)^2ustsu,$
$(22) \wedge (10) : w = (ts)^2(ut)^2u,$ $(22) \wedge (11) : w = (ts)^2(us)^2t,$
$(22) \wedge (12) : w = (ts)^2(us)^2ut,$ $(22) \wedge (13) : w = (ts)^2(us)^3,$
$(22) \wedge (14) : w = (ts)^2us(ts)^2,$ $(22) \wedge (15) : w = (ts)^2us(ut)^2,$
$(22) \wedge (16) : w = (ts)^2ut,$ $(22) \wedge (18) : w = (ts)^2us,$
$(23) \wedge (3) : w = (ts)^2t^2,$ $(23) \wedge (16) : w = (ts)^3ut,$
$(23) \wedge (17) : w = (ts)^2tuts,$ $(23) \wedge (18) : w = (ts)^3us,$
$(23) \wedge (19) : w = (ts)^2tusu,$ $(23) \wedge (20) : w = (ts)^2tust,$
$(23) \wedge (21) : w = (ts)^3tu,$

$(23) \wedge (22)\ :\ w = (ts)^4u,$	$(23) \wedge (23)\ :\ w = (ts)^4t,$	$(23) \wedge (23)\ :\ w = (ts)^3t,$
$(23) \wedge (24)\ :\ w = (ts)^2(tu)^2t,$	$(23) \wedge (21)\ :\ w = (ts)^2tu,$	$(23) \wedge (22)\ :\ w = (ts)^3u,$
$(24) \wedge (3)\ :\ w = (tu)^2t^2,$	$(24) \wedge (4)\ :\ w = (tu)^2tsu,$	$(24) \wedge (5)\ :\ w = (tu)^2tst,$
$(24) \wedge (6)\ :\ w = (tu)^3s,$	$(24) \wedge (10)\ :\ w = (tu)^4,$	$(24) \wedge (10)\ :\ w = (tu)^3,$
$(24) \wedge (16)\ :\ w = (tu)^2tsut,$	$(24) \wedge (17)\ :\ w = (tu)^3ts,$	$(24) \wedge (17)\ :\ w = (tu)^2ts,$
$(24) \wedge (18)\ :\ w = (tu)^2tsus,$	$(24) \wedge (19)\ :\ w = (tu)^3su,$	$(24) \wedge (20)\ :\ w = (tu)^3st,$
$(24) \wedge (21)\ :\ w = (tu)^2tstu,$	$(24) \wedge (22)\ :\ w = (tu)^2(ts)^2u,$	$(24) \wedge (23)\ :\ w = (tu)^2(ts)^2t,$
$(24) \wedge (24)\ :\ w = (tu)^4t,$	$(24) \wedge (24)\ :\ w = (tu)^3t.$	

All above intersection compositions are trivial. Let us check some of them as follows:

$$\begin{aligned}
(13) \wedge (1) : w &= (us)^3s,\\
(f,g)_w &= ((us)^3 - (su)^3)s - (us)^2(s^2-1)\\
&= (us)^3s - (su)^3s - (us)^3s + (us)^2u\\
&= (us)^2u - (su)^3s\\
&\equiv (us)^2u - s^2(us)^2u \equiv (us)^2u - (us)^2u \equiv 0.
\end{aligned}$$

$$\begin{aligned}
(16) \wedge (18) : w &= tsutsus,\\
(f,g)_w &= (tsut - suts)sus - tsu(tsus - usut)\\
&= tsutsus - suts^2us - tsutsus + tsu^2sut\\
&\equiv tut - sutus \equiv tut - s^2tut \equiv tut - tut \equiv 0.
\end{aligned}$$

$$\begin{aligned}
(24) \wedge (10) : w &= (tu)^3,\\
(f,g)_w &= ((tu)^2t - (su)^2s)u - t((ut)^2u - (st)^2s)\\
&= (tu)^3 - (su)^3 - (tu)^3 + (ts)^3\\
&= (ts)^3 - (su)^3 \equiv (us)^3 - (su)^3 \equiv (su)^3 - (su)^3 \equiv 0.
\end{aligned}$$

$$\begin{aligned}
(24) \wedge (4) : w &= (ts)^2utsu,\\
(f,g)_w &= ((ts)^2u - s(ut)^2)tsu - (ts)^2(utsu - suts)\\
&= (ts)^2utsu - sutut^2su - (ts)^2utsu + tsts^2uts\\
&\equiv tstuts - sutusu \equiv u(st)^2s - s^2(tu)^2 \equiv (tu)^2 - (tu)^2 \equiv 0.
\end{aligned}$$

It remains to check including compositions of relations (1) − (24). But it is seen that there are no any compositions of this type.

Hence the result follows. □

Now let R_{12} be the set of relations (1) − (24) and $C(u)$ be a normal form of a word $u \in G_{12}$. By using the Composition-Diamond Lemma 2.1 and Theorem 3.2, the normal form for the braid group associated with the congruence classes of complex reflection group G_{12} can be given as follows:

Corollary 3.3. *$C(u)$ has a form*

$$ws^{\epsilon_1}w'u^{\epsilon_2}w''t^{\epsilon_3}w''' \quad (0 \leq \epsilon_1, \epsilon_2, \epsilon_3 < 2),$$

where w, w', w'' and w''' are R_{12}−reduced words in G_{12}.

By considering Corollary 3.3, we have the following other main result of this paper.

Theorem 3.4. *The word problem for the braid group associated with the congruence classes of complex reflection group G_{12} is solvable.*

Remark 3.5. *The Gröbner-Shirshov basis is one of the best methods to obtain the solvability of the word problem since it can be adapted to some software based algorithms. Specially it has a good advantage in which the old algorithms cannot be applied to obtain solvable groups, monoids or some other algebraic structures.*

Acknowledgements 1. *This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary.*

Acknowledgements 2. *This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No (130-246-D1439). The authors, therefore, acknowledge with thanks DSR technical and financial support.*

Acknowledgements 3. *The authors are thankful to all referees for their valuable comments.*

References

[1] S. I. Adian, V. G. Durnev, Decision problems for groups and semigroups, Russian Math. Surveys 55-2 (2000) 207-296.
[2] F.Ates, A. S. Cevik, E. G. Karpuz, Gröbner-Shirshov basis for the singular part of the Brauer semigroup, Turkish Journal of Mathematics 42 (2018) 1338-1347.
[3] F. Ates, E. G. Karpuz, C. Kocapinar, A. S. Cevik, Gröbner-Shirshov bases of some monoids, Discrete Mathematics 311 (2011) 1064-1071.
[4] G. M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178-218.
[5] D. Bessis, J. Michel, Explicit presentations for exceptional Braid groups, Experimental Mathematics 13-3 (2004) 257-266.
[6] L. A. Bokut, Imbedding into simple associative algebras, Algebra and Logic 15 (1976) 117-142.
[7] L. A. Bokut, A. Vesnin, Gröbner-Shirshov bases for some Braid groups, Journal of Symbolic Computation 41 (2006) 357-371.
[8] L. A. Bokut, Gröbner-Shirshov basis for the Braid group in the Birman-Ko-Lee generators, Journal of Algebra 321 (2009) 361-376.
[9] M. Broue, G. Malle, R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, Journal für die reine und angewandte Mathematik 500 (1998) 127-190.
[10] B. Buchberger, An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Ideal, Ph.D. Thesis, University of Innsbruck, 1965.
[11] Y. Chen, C. Zhong, Gröbner-Shirshov bases for HNN extentions of groups and for the Alternating group, Communications in Algebra 36 (2008) 94-103.
[12] A. M. Cohen, Finite complex reflection groups, Annales scientifiques de l.E.N.S l'É.N.S. 4^e Serie 9-3 (1976) 379-436.
[13] R. B. Howlett, J. Y. Shi, On regularity of finite reflection groups, Manuscripta Mathematica 102-3 (2000) 325-333.
[14] E. G. Karpuz, Gröbner-Shirshov bases of some semigroup constructions, Algebra Colloquium 22-1 (2015) 35-46.
[15] E. G. Karpuz, F. Ates, A. S. Cevik, Gröbner-Shirshov bases of some Weyl groups, Rocky Mountain Journal of Mathematics 45-4 (2015) 1165-1175.
[16] E. G. Karpuz, F. Ates, N. Urlu, A. S. Cevik, I. N. Cangul, A note on the Gröbner-Shirshov bases over ad-hoc extensions of groups, Filomat 30-4 (2016) 1037-1043.
[17] C. Kocapinar, E. G. Karpuz, F. Ates, A. S. Cevik, Gröbner-Shirshov bases of the generalized Bruck-Reilly $*$-extension, Algebra Colloquium 19-1 (2012) 813-820.
[18] G. C. Shephard, J. A. Todd, Finite unitary reflection groups, Canad. J. Math. 6 (1954) 274-304.
[19] J. Y. Shi, Simple root systems and presentations for certain complex reflection groups, Communications in Algebra 33 (2005) 1765-1783.
[20] A. I. Shirshov, Some algorithmic problems for Lie algebras, Siberian Math. J. 3 (1962) 292-296.