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Notes on the Poly-Korobov Polynomials and Related Polynomials

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Abstract. In recent years, many mathematicians ([2], [7], [8], [9], [15], [16], [21]) introduced and investigated for the Korobov polynomials. They gave some identities and relations for the Korobov type polynomials. In this work, we give some relations for the first kind Korobov polynomials and Korobov type Changhee polynomials. Further, we give two relations between the poly-Changhee polynomials and the poly-Korobov polynomials. Also, we give a relation among the poly-Korobov type Changhee polynomials, the Stirling numbers of the second kind, the Euler polynomials and the Bernoulli numbers.

1. Introduction, Definitions and Notations

A usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the complex numbers. We begin by introducing the following definition and notations (see also [11]-[15], [17], [19], [20], [21]). It is well known, the Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ are defined by the following generating functions, respectively;

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},\ |t| < 2\pi \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},\ |t| < \pi \tag{2}$$

when $x = 0$, $B_n(0) = B_n$ and $E_n(0) = E_n$ are called the Bernoulli numbers and the Euler numbers.

Generating function for the Stirling numbers of the second kind are given by

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!} \tag{3}$$

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where k is nonnegative integer ([9], [21]).

The classical polylogarithm function $Li_k(z)$ is defined

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, k \in \mathbb{Z}, k > 1. \tag{4}$$

This function is convergent for $|z| < 1$, when $k = 1$, $Li_1(z) = -\log(1-z)$ in [12].

The poly-Bernoulli polynomials are defined as ([1], [6], [12])

$$\frac{Li_k(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(x) \frac{t^n}{n!}. \tag{5}$$

For $k = 1$, we have $\mathcal{B}_n^{(1)}(x) = B_n(x)$, $n \geq 0$. For $x = 0$, $\mathcal{B}_n^{(k)} := \mathcal{B}_n^{(k)}(0)$ are called poly-Bernoulli numbers.

Hamahata *et al.* in [3] defined the poly-Euler polynomials as

$$\frac{2Li_k(1-e^{-t})}{t\,(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x) \frac{t^n}{n!} \tag{6}$$

when $x = 0$, $\mathcal{E}_n^{(k)} := \mathcal{E}_n^{(k)}(0)$ are called poly-Euler numbers.

The Korobov polynomials $K_n\,(x \mid \lambda)$ of the first kind are given by the generating function

$$\frac{\lambda t}{(1+t)^{\lambda}-1}\,(1+t)^x = \sum_{n=0}^{\infty} K_n\,(x \mid \lambda)\,\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $K_n\,(\lambda) = K_n\,(0 \mid \lambda)$ are called Korobov numbers of the first kind ([2], [9], [21]).

D. S. Kim *et al.* in [7] introduced and investigated some properties of the Korobov polynomials of the third kind $K_{n,3}\,(x \mid \lambda)$ and the Korobov polynomials of the fourth kind $K_{n,4}\,(x \mid \lambda)$.

D. S. Kim *et al.* in ([9], [10]) defined the Changhee polynomials and the Korobov type Changhee polynomials, respectively:

$$\sum_{n=0}^{\infty} Ch_n\,(x)\,\frac{t^n}{n!} = \frac{2}{t+2}\,(1+t)^x \tag{8}$$

and

$$\sum_{n=0}^{\infty} Ch_n\,(x \mid \lambda)\,\frac{t^n}{n!} = \frac{2}{(1+t)^{\lambda}+1}\,(1+t)^x\,. \tag{9}$$

Srivastava [17] and Srivastava *et al.* in ([17]-[20]) gave some theorems and recurrence relations for the classical Bernoulli polynomials, the classical Euler polynomials and the classical Genocchi polynomials. Korobov ([11], [13]) introduced the Korobov polynomials. Hamahata in [2], Bayad *et al.* in [1], Imatomi *et al.* in [4] and D. Kim *et al.* in [6] defined and investigated some properties and relations for the poly-Bernoulli polynomials and poly-Euler polynomials. D. S. Kim *et al.* in [9] considered and investigated some relations for the Korobov type polynomials associated with p-adic integrals. Seo *et al.* in [16] considered the degenerate Korobov polynomials. Kruchinin in ([14], [15]) gave explicit formulas for Korobov polynomials. Yardimci *et al.* in [21] gave some identities for Korobov type polynomials.

2. Explicit Relations For The Korobov Polynomials, The Changhee polynomials and The Korobov Type Changhee polynomials and Related Polynomials

In this section, we give some relations the Korobov polynomials and the Korobov type Changhee polynomials. Also, we give a relation the poly-Korobov type Changhee polynomials and the Euler polynomials.

Theorem 2.1. *The Changhee polynomials satisfy the following relation*

$$\sum_{n=0}^{m} E_n(x) S_2(m,n) = \sum_{l=0}^{m}\sum_{n=0}^{l} Ch_n(x) S_2(l,n) S_2(m,l). \tag{10}$$

Proof. By replacing t by $e^{(e^t-1)}-1$ in (8) and by (2), we get

$$\sum_{n=0}^{\infty} Ch_n(x) \frac{\left(e^{(e^t-1)}-1\right)^n}{n!} = \sum_{n=0}^{\infty} E_n(x) \frac{(e^t-1)^n}{n!}.$$

After some calculations by (3) and using the Cauchy product in the above equation, we have

$$\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{n=0}^{l} Ch_n(x) S_2(l,n) S_2(m,l)\right)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E_n(x) S_2(m,n)\frac{t^m}{m!}.$$

Comparing the coefficients of both sides, we get results. □

Theorem 2.2. *There is the following relation between the Changhee polynomials and the Korobov polynomials*

$$K_n(x+\lambda \mid \lambda) - K_n(x \mid \lambda) = \lambda n\left(Ch_{n-1}(x) + \frac{1}{2}(n-1) Ch_{n-2}(x)\right). \tag{11}$$

Proof. By (7) and (8), we write

$$\frac{(1+t)^x (t+2)}{t+2} = (1+t)^x \frac{(1+t)^\lambda - 1}{(1+t)^\lambda - 1}$$

$$\frac{2(1+t)^x}{t+2} + \frac{t(1+t)^x}{t+2} = \frac{(1+t)^{x+\lambda}}{(1+t)^\lambda - 1} - \frac{(1+t)^x}{(1+t)^\lambda - 1}$$

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} + \frac{t}{2}\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{1}{\lambda t}\left(\sum_{n=0}^{\infty}\left(K_n(x+\lambda \mid \lambda) - K_n(x \mid \lambda)\right)\right)\frac{t^n}{n!}.$$

From here, we have

$$\lambda n\left(Ch_{n-1}(x) + \frac{1}{2}(n-1) Ch_{n-2}(x)\right) = K_n(x+\lambda \mid \lambda) - K_n(x \mid \lambda).$$

□

Theorem 2.3. *There is the following relation between the Korobov type Changhee polynomials and Korobov polynomials*

$$K_n(x+\lambda \mid \lambda) - K_n(x \mid \lambda) = \frac{\lambda n}{2}\left(Ch_{n-1}(x+\lambda \mid \lambda) + Ch_{n-1}(x \mid \lambda)\right).$$

Proof. The proof of this theorem is similiar to **Theorem 2.2**, we omit it. □

Theorem 2.4. *The following relations hold true:*

$$K_n(x+\lambda \mid \lambda) = K_n(x \mid \lambda) + \lambda n (x)_{n-1} \tag{12}$$

and

$$Ch_n(x+\lambda \mid \lambda) = 2(x)_n - Ch_n(x \mid \lambda), \text{ where } (x)_n = x(x-1)...(x-n+1) \text{ for } n > 0. \tag{13}$$

Proof. It is easy known that

$$\frac{2}{\left((1+t)^{\lambda}-1\right)(1+t)^{\lambda}}=\frac{2}{(1+t)^{\lambda}-1}-\frac{2}{(1+t)^{\lambda}}.$$

Thus, we have

$$\frac{2(1+t)^{x}}{(1+t)^{\lambda}-1}=\frac{2(1+t)^{x+\lambda}}{(1+t)^{\lambda}-1}-2(1+t)^{x}.$$

By using (7), we write

$$\frac{2}{\lambda t}\sum_{n=0}^{\infty}K_n(x\mid\lambda)\frac{t^n}{n!}=\frac{2}{\lambda t}\sum_{n=0}^{\infty}K_n(x+\lambda\mid\lambda)\frac{t^n}{n!}-2\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.$$

Thus, we have

$$\sum_{n=0}^{\infty}\left(K_n(x\mid\lambda)-K_n(x+\lambda\mid\lambda)\right)\frac{t^n}{n!}=-\lambda\sum_{n=0}^{\infty}(x)_n\frac{t^{n+1}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we get (12). The proof of the (13) is similiar to that of (12), so we omit it. □

Theorem 2.5. *There is the following relation between the Korobov polynomials and the Bernoulli polynomials:*

$$\sum_{k=0}^{\infty}\frac{B_k\left(\frac{x+1}{\lambda}\right)-B_k\left(\frac{x}{\lambda}\right)}{k!}\lambda^k\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}l^m=\sum_{l=0}^{m}\sum_{n=0}^{l}K_n(x\mid\lambda)S_2(l,n)S_2(m,l). \tag{14}$$

Proof. By replacing t by $e^{\left(e^t-1\right)}-1$ in (7),

$$\frac{\lambda\left(e^{\left(e^t-1\right)}-1\right)e^{\left(e^t-1\right)x}}{e^{\lambda\left(e^t-1\right)}-1}=\sum_{n=0}^{\infty}K_n(x\mid\lambda)\frac{\left(e^{\left(e^t-1\right)}-1\right)^n}{n!}. \tag{15}$$

The right hand of (15) is

$$\sum_{n=0}^{\infty}K_n(x\mid\lambda)\sum_{l=n}^{\infty}S_2(l,n)\frac{\left(e^t-1\right)^l}{l!}=\sum_{l=0}^{\infty}\sum_{n=0}^{l}K_n(x\mid\lambda)S_2(l,n)\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}$$

$$=\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{n=0}^{l}K_n(x\mid\lambda)S_2(l,n)S_2(m,l)\right)\frac{t^m}{m!}. \tag{16}$$

The left hand of (15) is

$$\frac{\left(e^{\left(e^t-1\right)}-1\right)\lambda}{\left(e^{\lambda\left(e^t-1\right)}-1\right)}e^{\left(e^t-1\right)x}=\frac{\lambda e^{\left(e^t-1\right)(x+1)}}{e^{\lambda\left(e^t-1\right)}-1}-\frac{\lambda e^{\left(e^t-1\right)^x}}{e^{\lambda\left(e^t-1\right)}-1}$$

$$=\frac{1}{\left(e^t-1\right)}\left(\frac{\lambda\left(e^t-1\right)e^{\lambda\left(e^t-1\right)\left(\frac{x+1}{\lambda}\right)}}{e^{\lambda\left(e^t-1\right)}-1}-\frac{\lambda\left(e^t-1\right)e^{\lambda\left(e^t-1\right)\frac{x}{\lambda}}}{e^{\lambda\left(e^t-1\right)}-1}\right)$$

$$=\sum_{k=0}^{\infty}\frac{B_k\left(\frac{x+1}{\lambda}\right)-B_k\left(\frac{x}{\lambda}\right)}{k!}\lambda^k\sum_{l=0}^{k-1}(-1)^{k-1-l}\binom{k-1}{l}\sum_{m=0}^{\infty}l^m\frac{t^m}{m!}. \tag{17}$$

From (14) and (15), comparing the coefficients of $\frac{t^m}{m!}$, we have (14). □

The relation (14) and equation (6) in [16] are similarity. But the transformation is different. Finally, the relation (14) is true.

3. The Poly-Korobov Polynomials, The Poly-Changhee polynomials and The Poly-Korobov Type Changhee polynomials

By the motification in [9], we define the following the poly-Changhee polynomials, the poly-Korobov polynomials and the poly-Korobov type Changhee polynomials, respectively

$$\sum_{n=0}^{\infty} Ch_n^{(k)}(x)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})(1+t)^x}{t(t+2)}, \tag{18}$$

$$\sum_{n=0}^{\infty} K_n^{(k)}(x \mid \lambda)\frac{t^n}{n!} = \frac{\lambda Li_k(1-e^{-t})}{(t+1)^{\lambda}-1}(1+t)^x \tag{19}$$

and

$$\sum_{n=0}^{\infty} Ch_n^{(k)}(x \mid \lambda)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t\left((t+1)^{\lambda}+1\right)}(1+t)^x. \tag{20}$$

For $k=1$, we get $Ch_n^{(1)}(x) = Ch_n(x)$, $K_n^{(1)}(x \mid \lambda) = K_n(x \mid \lambda)$ and $Ch_n^{(1)}(x \mid \lambda) = Ch_n(x \mid \lambda)$.
The following relations can be easily from (18), (19) and (20).

$$\lambda n\left((n-1)Ch_{n-2}^{(k)}(x) + nCh_{n-1}^{(k)}(x)\right) = 2\left(K_n^{(k)}(x+\lambda \mid \lambda) - K_n^{(k)}(x \mid \lambda)\right) \tag{i}$$

and

$$\lambda\left(nCh_{n-1}^{(k)}(x+\lambda \mid \lambda) + Ch_n^{(k)}(x \mid \lambda)\right) = 2\left(K_n^{(k)}(x+\lambda \mid \lambda) - K_n^{(k)}(x \mid \lambda)\right). \tag{ii}$$

Theorem 3.1. *There is the following relation between the poly-Korobov type Changhee polynomials and the Euler polynomials*

$$\begin{aligned}\sum_{m=0}^{\infty}(m+1)Ch_m^{(k)}(x \mid \lambda)S_2(n,m+1)(-1)^n &= \frac{1}{\lambda}\sum_{p=0}^{n}\binom{n}{p}\sum_{m=0}^{n-p}\binom{n-p}{m}B_{n-p-m}\\ &\times(-\lambda)^{n-p}\frac{(-1)^{m+1}m!}{(m+1)^k}\left(E_p\left(\frac{x+\lambda}{\lambda}\right) - E_p\left(\frac{x}{\lambda}\right)\right).\end{aligned} \tag{21}$$

Proof. By (20),

$$\sum_{n=0}^{\infty} Ch_n^{(k)}(x \mid \lambda)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t\left((t+1)^{\lambda}-1\right)}\left(\frac{(t+1)^{\lambda+x}}{(t+1)^{\lambda}+1} - \frac{(1+t)^x}{(t+1)^{\lambda}+1}\right). \tag{22}$$

By replacing t by $e^{-t}-1$ in the last equation, we have

$$\sum_{m=0}^{\infty} Ch_m^{(k)}(x \mid \lambda)\frac{(e^{-t}-1)^{m+1}}{m!} = -\frac{1}{t\lambda}\frac{(-t\lambda Li_k(-t))}{e^{-t\lambda}-1}\left(\frac{2e^{-t\lambda\left(\frac{x+\lambda}{\lambda}\right)}}{e^{-t\lambda}+1} - \frac{2e^{-t\lambda\left(\frac{x}{\lambda}\right)}}{e^{-t\lambda}+1}\right). \tag{23}$$

By using (1), (2), (3) and (4) in (23) and using Cauchy product and comparing the coefficients, we have (21). □

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