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Commutators of the B-Maximal Operator and B-Maximal Commutators

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Abstract. In this paper we consider the commutator of the B-maximal operator and the B-maximal commutator associated with the Laplace-Bessel differential operator. The boundedness of the commutator of the B-maximal operator with BMO symbols on weighted Lebesque space and weak-type inequality for the commutator of the B-maximal operator are proved.

1. Introduction

The Laplace-Bessel differential operator

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + (\frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}), \quad \nu > 0$$

is known as an important operator in Fourier-Bessel harmonic analysis and applications. This operator, associated with the Bessel differential operator

$$B_\nu = \frac{d^2}{dt^2} + \frac{2\nu}{t}\frac{d}{dt}, \quad \nu > 0$$

has been studied many mathematicians.[2–7, 14–17, 23–27, 29, 31, 32]

Given a linear operator T acting on functions and given a function b, the commutator $[T, b]$ formally defined as

$$[T, b]f = T(bf) - bT(f).$$

The first result on commutators was obtained by Coifman, Rochberg, Weiss [12]. They showed that if T is a classical singular integral operator and $b \in BMO$, then the commutator $[T, b]$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Later, Chanillo [11] proved a similar result when singular integral operators are replaced by the fractional integral operators.

Coifman and Meyer [13] observed that the L_p boundedness for the commutator $[T, b]$ could be obtained from the weighted L_p estimate for T with the weight function class of Muckenhoupt A_p. Later, Alvarez, Bagby, Kurtz, Perez [9] extended the idea of Coifman and Meyer and Perez [30] obtained a weak-type inequality for the commutator $[T, b]$.

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In [28], Milman and Schonbek proved that the commutator of the classical Hardy-Littlewood maximal function $[M,b]$, defined by

$$[M,b]f(x) = M(bf)(x) - b(x)Mf(x), \quad x \in \mathbb{R}^n$$

is bounded on L_p, $1 < p < \infty$ when b is in $BMO(\mathbb{R}^n)$. Moreover, the classical maximal commutator associated with the classical translation is defined by

$$M_b(f)(x) = \sup_{Q:\, x\in Q} \frac{1}{|Q|} \int\limits_Q |b(x) - b(y)||f(y)|dy; \quad f \in L_p(\mathbb{R}^n).$$

These operators play an important role in studying the commutators of singular integral operator with BMO symbols. Alphonse [8] obtained weak type inequality for maximal commutators, and pointwise estimates of the maximal commutator and the commutator of the maximal function are proved by Agcayazı, Gogatishvili, Koca, Mustafayev [1]. Commutators have been research area many mathematicians such as Guliyev, Hasanov, Hu, Lin, Yang, Janson [18, 20, 21] and others.

In this paper, we consider the commutator $[M_B, b]$ of the Hardy-Littlewood maximal operator M_B and the B-maximal commutator associated with the Laplace-Bessel differential operator. The paper is organized as follow. Section 2 contains some basic definitions and results which are needed in this paper. Main results and its proofs are in the Section 3.

2. Preliminaries and Notations

Let $\mathbb{R}^n_+ = \{x : x = (x_1, ...x_n) \in \mathbb{R}^n, x_n \geq 0\}$ and $B(x,r) = \{y \in \mathbb{R}^n_+ : |x-y| < r\}$. For a measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\nu = \int\limits_E x_n^{2\nu} dx,\ \nu > 0.$

Denote by T^y $(y \in \mathbb{R}^n_+)$, generalized translation operator acting according to the law:

$$T^y f(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int\limits_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right) \sin^{2\nu-1}\alpha\, d\alpha,$$

where $x = (x', x_n)$, $y = (y', y_n)$ and $x', y' \in \mathbb{R}^{n-1}$. We remark that T^y is closely connected with Bessel differential operator B_ν , see [23, 24] for details.

The weighted space $L_{p,\nu} \equiv L_{p,\nu}(\mathbb{R}^n_+)$, $1 \leq p < \infty$ consists of measurable functions on $\mathbb{R}^n_+$ with the norm given by

$$\left\|f\right\|_{L_{p,\nu}} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x_n^{2\nu} dx\right)^{1/p}.$$

In the case $p = \infty$, the space $L_{\infty,\nu}$ is defined by means of the usual modification $\left\|f\right\|_{L_\infty} = ess \sup |f(x)|$, $x \in \mathbb{R}^n_+$.

We denote by $L^{loc}_{1,\nu}(\mathbb{R}^n_+)$, locally integrable with respect to the measure $x_n^{2\nu}dx$ functions defined on $\mathbb{R}^n_+$.

Let $1 < p < \infty$. A weight function w is said to be of Muckenhoupt class $A_{p,\nu}$ if $[w]_{A_{p,\nu}}$is finite, where $[w]_{A_{p,\nu}}$ is defined by

$$[w]_{A_{p,\nu}} = \sup_{x\in\mathbb{R}^n_+, r>0} \left(\frac{1}{|B(x,r)|_\nu} \int\limits_{B(x,r)} w(y) y_n^{2\nu} dy\right) \left(\frac{1}{|B(x,r)|_\nu} \int\limits_{B(x,r)} w(y)^{-1/p-1} y_n^{2\nu} dy\right)^{p-1}.$$

The Hardy-Littlewood maximal function generated by generalized translation operator, called the B-maximal function $M_B f$, is defined by

$$M_B f(x) = \sup_{r>o} \frac{1}{|B(0,r)|_\nu} \int\limits_{B(0,r)} T^y |f(x)| y_n^{2\nu} dy, \quad x \in \mathbb{R}^n_+.$$

The operator $M_B : f \to M_B f$ is called the B-maximal operator. The boundedness of the B-maximal operator M_B on $L_{p,\nu}$ is proved by V.Guliyev, [16].

The space of functions of bounded mean oscillation associated with Laplace-Bessel differential operator is denoted by $BMO_B = BMO_B(\mathbb{R}^n_+)$ and defined by the following finite norm

$$\left\|f\right\|_{BMO_B} = \sup_{x\in\mathbb{R}^n_+,\ r>0} \frac{1}{|B(0,r)|_\nu} \int\limits_{B(0,r)} |T^y f(x) - f_{B(0,r)}(x)|\ y_n^{2\nu} dy$$

where $f_{B(0,r)}(x) = \frac{1}{|B(0,r)|_\nu} \int\limits_{B(0,r)} T^y f(x) y_n^{2\nu} dy$.

The classical BMO space plays an important role in Fourier harmonic analysis and applications, introduced by John and Nirenberg [22] in 1961. It is easy to see that $L_\infty \subsetneq BMO$. A famous example is $log|x| \in BMO(\mathbb{R}^n_+) \backslash L_\infty(\mathbb{R}^n_+)$. BMO space turned out to be the "right" space to study instead of L_∞. Many of the operators which are ill-behaved on L_∞, are bounded on BMO.

Definitions of the commutator of the B-maximal operator and the B-maximal commutator associated with the Laplace-Bessel differential operator are given below.

Definition 2.1. *Let b be a measurable function defined on $\mathbb{R}^n_+$. The commutator $[M_B, b]$ of the B-maximal operator M_B is defined by*

$$[M_B, b]f(x) = M_B(bf)(x) - b(x)M_B f(x),\ x \in \mathbb{R}^n_+.$$

Definition 2.2. *Let $b \in L^{loc}_{1,\nu}(\mathbb{R}^n_+)$. The B-maximal commutator $M_{B,b}$ is defined by*

$$M_{B,b} f(x) = \sup_{r>0} \frac{1}{|B(0,r)|_\nu} \int\limits_{B(0,r)} T^y |\left(b(x) - b(y)\right) f(x)|\ y_n^{2\nu} dy,\ x \in \mathbb{R}^n_+.$$

3. Main Results

In classical theory, if w and w^{-1} belong to the Muckenhoupt class A_p, then the Hardy-Littlewood maximal operator M is bounded on $L_p\left(w^{\pm1}dx\right)$. So, Milman and Schonbek [28] prove that if $b \in BMO,\ b \geq 0$, then the commutator $[M, b]$ of the Hardy-Littlewood maximal operator is bounded on L_p, $1 < p < \infty$.

In Fourier-Bessel harmonic analysis, the boundedness of the Hardy-Littlewood maximal function generated by the Laplace-Bessel differential operator such that w belongs to the suitable Muckenhoupt class on weighted Lebesque space was proved by Guliyev [19]. This result is given in the next theorem.

Theorem 3.1. *a) If $f \in L_{p,\nu}\left(w, \mathbb{R}^n_+\right),\ w \in A_{p,\nu}\left(\mathbb{R}^n_+\right),\ 1 < p < \infty,$ then*

$$\left\|M_B f\right\|_{L_{p,\nu}\left(w,\mathbb{R}^n_+\right)} \leq C \left\|f\right\|_{L_{p,\nu}\left(w,\mathbb{R}^n_+\right)}$$

where the constant C depends on p, w, ν, n.

b) If $f \in L_{1,\nu}\left(w, \mathbb{R}^n_+\right),\ w \in A_{1,\nu}\left(\mathbb{R}^n_+\right),\ 1 < p < \infty,$ then

$$\left\|M_B f\right\|_{WL_{1,\nu}\left(w,\mathbb{R}^n_+\right)} \leq C \left\|f\right\|_{L_{1,\nu}\left(w,\mathbb{R}^n_+\right)}$$

where the constant C depends on w, ν, n. Here $WL_{1,\nu}(w, \mathbb{R}^n_+)$ denotes the weak-$L_{1,\nu}(w, \mathbb{R}^n_+)$ space.

By using similar arguments in ([28], Theorem4.4), we get the following theorem from the Theorem3.1.

Theorem 3.2. *Let $f \in L_{p,\nu}\left(\mathbb{R}^n_+\right),\ 1 < p < \infty$ and $b \in BMO_B,\ b \geq 0$. Then the commutator of the B-maximal operator $[M_B, b]$ is bounded on $L_{p,\nu}\mathbb{R}^n_+)$, that is,*

$$\left\|[M_B, b]f\right\|_{L_{p,\nu}} \leq \|b\|_{BMO_B} \left\|f\right\|_{L_{p,\nu}}.$$

The commutator of the B-maximal operator $[M_B, b]$ and the B-maximal commutator $M_{B,b}$ are essentially different from each other. However, if b satisfies some conditions, then the operator $M_{B,b}$ controls $[M_B, b]$.

Lemma 3.3. *Let b is any non-negative locally integrable function defined on $\mathbb{R}^n_+$. Then*

$$\left|[M_B, b]f(x)\right| \leq M_{B,b}f(x)$$

for all $f \in L^{loc}_{1,\nu}(\mathbb{R}^n_+)$.

Proof. Since b is non-negative

$$\begin{aligned} T^y\left|b(x)f(x)\right| - b(x)T^y\left|f(x)\right| &= c_\nu \int_0^\pi \left|(bf)\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right)\right| \sin^{2\nu-1}\alpha\, d\alpha \\ &- b(x)c_\nu \int_0^\pi \left|f\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right)\right| \sin^{2\nu-1}\alpha\, d\alpha \end{aligned}$$

$$= c_\nu \int_0^\pi \left(\left|(bf)\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right)\right| - |b(x)|\left|f\left(x'-y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha + y_n^2}\right)\right|\right) \sin^{2\nu-1}\alpha\, d\alpha$$

and we have

$$\left|T^y\left|b(x)f(x)\right| - b(x)T^y\left|f(x)\right|\right| \leq T^y\left|\left((b(.) - b(x))\, f\right)(.)\right|.$$

Since by making use of the following inequality

$$\left|\sup_{r>0} u(r) - \sup_{r>0} v(r)\right| \leq \sup_{r>0} |u(r) - v(r)|, \quad u(r), v(r) > 0$$

we have

$$\begin{aligned} \left|[M_B, b]f(x)\right| &= \left|M_B(bf)(x) - b(x)M_Bf(x)\right| \\ &= \left|\sup_{r>0} \frac{1}{|B(0,r)|_\nu} \int_{B(0,r)} T^y|b(x)f(x)|y^{2\nu}dy - b(x)\sup_{r>0} \frac{1}{|B(0,r)|_\nu} \int_{B(0,r)} T^y|f(x)|y^{2\nu}dy\right| \\ &\leq \sup_{r>0} \frac{1}{|B(0,r)|_\nu} \int_{B(0,r)} \left|T^y|b(x)f(x)| - b(x)T^y|f(x)|\right| y^{2\nu}dy \\ &= M_B\left((b(.) - b(x))f(.)\right)(x) \\ &= M_{B,b}f(x). \end{aligned}$$

□

Lemma 3.4. . *Let $b \in L^{loc}_{1,\nu}(\mathbb{R}^n_+)$ Then*

$$\left|[M_B, b]f(x)\right| \leq M_{B,b}f(x) + 2b^-(x)M_Bf(x)$$

for all $f \in L^{loc}_{1,\nu}(\mathbb{R}^n_+)$ where $b^-(x) = \max\{-b(x), 0\}$.

Proof. Since

$$\begin{aligned}\left|[M_B,b]f(x)-[M_B,|b|]f(x)\right| &= \left|M_B(bf)(x)-b(x)M_Bf(x)-M_B(|b|f)(x)+|b(x)|M_Bf(x)\right|\\ &= \left|(|b(x)|-b(x))M_Bf(x)\right|\\ &\leq 2b^-(x)M_Bf(x)\end{aligned}$$

we have

$$\left|[M_B,b]f(x)\right|\leq\left|[M_B,|b|]f(x)\right|+2b^-(x)M_Bf(x)$$

and by using Lemma 3.3, we get

$$\left|[M_B,b]f(x)\right|\leq M_{B,|b|}f(x)+2b^-(x)M_Bf(x)$$

□

The weak-type inequality for the commutator of the B-maximal operator is obtained using Lemma 3.4 and the weak type $(1,1)$ inequality for the B-maximal function. This result is the following.

Theorem 3.5. *Let $b\in L_\infty$. Then there exist a positive constant c_1,c_2 such that*

$$\left|\{x\in\mathbb{R}^n_+:\ |[M_B,b]f(x)|>\lambda\}\right|_\nu\leq c_1\|b\|_{L_\infty}\left\|f\right\|_{L_{1,\nu}}+\left(\frac{c_2\|b\|_{L_\infty}}{\lambda}\right)^q\|f\|^q_{L_{q,\nu}}$$

for all $f\in L_{1,\nu}\cap L_{q,\nu}, 1<q<\infty$ and for all $\lambda>0$.

Proof. For $\lambda>0$, by using Lemma 3.4, we have

$$\begin{aligned}\left|\{x\in\mathbb{R}^n_+:\ |[M_B,b]f(x)|>\lambda\}\right|_\nu &\leq \left|\left\{x\in\mathbb{R}^n_+:\ M_{B,|b|}f(x)>\frac{\lambda}{2}\right\}\right|_\nu+\left|\left\{x\in\mathbb{R}^n_+:\ 2b^-(x)M_Bf(x)>\frac{\lambda}{2}\right\}\right|_\nu\\ &\leq \left|\left\{x\in\mathbb{R}^n_+:\ M_{B,|b|}f(x)>\frac{\lambda}{2}\right\}\right|_\nu+\left|\left\{x\in\mathbb{R}^n_+:\ 2\|b\|_{L_\infty}M_Bf(x)>\frac{\lambda}{2}\right\}\right|_\nu\\ &= I_1+I_2.\end{aligned}$$

Since the B-maximal operator is a weak type $(1,1)$ we have

$$I_2=\left|\left\{x\in\mathbb{R}^n_+:\ 2\|b\|_{L_\infty}M_Bf(x)>\frac{\lambda}{2}\right\}\right|_\nu\leq C_1\|b\|_{L_\infty}\int_{\mathbb{R}^n_+}|f(x)|x_n^{2\nu}dx=C_1\|b\|_{L_\infty}\left\|f\right\|_{L_{1,\nu}}.$$

Let us estimate I_1. By using Hölder inequality

$$\begin{aligned}
M_{B,|b|}f(x) &= \sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)} T^y|\,(|b(x)|-|b(y)|)\,f(x)|\;y_n^{2\nu}dy\\
&\le \sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)} T^y(|b(x)-b(y)||f(x)|)\;y_n^{2\nu}dy\\
&= \sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)} c_\nu\int\limits_0^\pi |b(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2})-b(y)|\\
&\times\; |f(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2})|\sin^{2\nu-1}\alpha\,d\alpha\;y_n^{2\nu}dy\\
&\le \sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)}\left(c_\nu\int\limits_0^\pi |b(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2})-b(y)|^p\sin^{2\nu-1}\alpha\,d\alpha\right)^{1/p}\\
&\times\left(c_\nu\int\limits_0^\pi |f(x'-y',\sqrt{x_n^2-2x_ny_n\cos\alpha+y_n^2})|^q\sin^{2\nu-1}\alpha\,d\alpha\right)^{1/q} y_n^{2\nu}dy,\;\;\frac{1}{p}+\frac{1}{q}=1\\
&\le \sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)}(T^y|b(x)-b(y)|^p)^{1/p}(T^y|f(x)|^q)^{1/q}y_n^{2\nu}dy\\
&\le \left(\sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)}T^y|b(x)-b(y)|^p y_n^{2\nu}dy\right)^{1/p}\left(\sup_{r>0}\frac{1}{|B(0,r)|_\nu}\int\limits_{B(0,r)}T^y|f(x)|^q y_n^{2\nu}dy\right)^{1/q}\\
&\le c_1\|b\|_{L_\infty}(M_B|f|^q)^{1/q}(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
I_1=\left|\left\{x\in\mathbb{R}_+^n:\;M_{B,|b|}f(x)>\frac{\lambda}{2}\right\}\right|_\nu &= \left|\left\{x\in\mathbb{R}_+^n:\;M_{B,b}f(x)>\frac{\lambda}{2}\right\}\right|_\nu \le \left|\left\{x\in\mathbb{R}_+^n:\;c_1\|b\|_{L_\infty}(M_B|f|^q)^{1/q}(x)>\frac{\lambda}{2}\right\}\right|_\nu\\
&\le \left(\frac{c_2\|b\|_{L_\infty}}{\lambda}\right)^q\|f\|_{L_{q,\nu}}^q,\;\;1<q<\infty.
\end{aligned}$$

Finally the desired result follows from I_1 and I_2. □

4. Conclusions

This paper presents the boundedness of the commutator of the B-maximal operator with BMO symbols and weak-type inequality for the commutator of the B-maximal operator on weighted Lebesque space.

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References

[1] M. Agcayazı, A. Gogatishvili, K. Koca, R. Mustafayev, A note on maximal commutators and commutators of maximal functions, J. Math. Soc. Japan, 67 2 2015 581–593.

[2] I.A. Aliev, Riesz transforms generated by a generalized shift operator, Izvestiya Akad. Nauk Azerbaijan Rep. (Ser. Fiz. Techn. Mat. Nauk),(In Russian), 1 1987 7–13 .

[3] I.A. Aliev, S. Bayrakci, On inversion of B-elliptic potentials associated with the Laplace-Bessel differential operator, Fract. Calc. Appl. Anal. 4 1998 365–384.

[4] I.A. Aliev and S. Bayrakci, On inversion of Bessel potentials associated with the Laplace-Bessel differential operator, Acta Math. Hungar, 95 2002 1-2 125–145.

[5] I.A. Aliev and S. Bayrakci, Square-like functions generated by a composite wavelet transform, Mediterr. J. Math., 8 2011 553–561.

[6] I.A. Aliev and A.D. Gadjiev, Weighted estimates of multidimensional singular integrals generated by the generalized shift operator, Russian Acad. Sci. Sb. Math., 77 1994 1 37-55.

[7] I.A. Aliev and E. Saglik, Generalized Riesz potential spaces and their characterization via wavelet-type transform, FILOMAT, 30 2016 10 2809-2823.

[8] A.M. Alphonse, An end point estimates for maximal commutators, J. Fourier Anal. Appl. 6 4 2000 449–456.

[9] J. Alvarez, R.J. Bagby, D.S. Kurtz, C. Perez, Weighted estimates for commutators of linear operators, Studia Math. 104 2 1993 195–209.

[10] J. Bastero, M. Milman, F. J. Ruız, Commutators for the maximal and sharp functions, Proc. Amer. Math. Soc. 128 11 2000 3329–3334.

[11] S.C. Chanillo, A note on commutators, Indiana Univ. Math. J., 31 1982 7-16.

[12] R.R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, Annals of Math. 103 1976 611-635.

[13] R.R. Coifman, Y. Meyer, Au dela des operaturs pseudo-differentiels, Asterisque 57, Societe Mathematique de France, Paris 1978, 185pp.

[14] J. Delsarte, Sur une extension de la formule de Taylor, J. Math. Pure Appl, 17 1938 213-231.

[15] A.D. Gadjiev, I.A. Aliev, On classes of operators of potential types, generated by a generalized shift, Reports of enlarged Session of the Seminars of I.N.Vekua Inst. of Applied Mathematics, Tbilisi. 3 1988 21–24.

[16] V.S. Guliev, Sobolev theorems for the Riesz B- potentials, Dokl. RAN, 358 4 1998 450–451. 1999 155–156.

[17] V.S. Guliev, J.J. Hasanov, Sobolev-Morrey type inequality for Riesz potentilas, associated with the Laplace-Bessel differential operator,Frac. Calc. Appl. Anal. 9 1 2006 17–32.

[18] V.S. Guliev, J.J. Hasanov, Maximal and singular integral operators and their commutators on generalized weighted Morrey spaces with variable exponent, Math. Inequal. Appl., 21 1 2018 41-61.

[19] E.V. Guliyev Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator, Trans. of NAS of Azerbaijan, 26 1 2006, 71-80.

[20] G.Hu, H.Lin, D.Yang, Commutators of the Hardy-Littlewood Maximal operator with BMO symbols on spaces of homogenous type, Abstr. Appl. Anal. 2008 Art. ID 237937.

[21] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Mat. 16 1978 263–270.

[22] F. John, L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 1961 415–426.

[23] B.M. Levitan, Bessel function expansions in series and Fourier integrals, Uspekhi Mat. Nauk., 6 1951 102–143 .

[24] I.A. Kipriyanov, Singular elliptic boundary value problems, (In Russian), Nauka, Moscow 1997.

[25] M.I. Klyuchantsev, On singular integrals generated by the generalized shift operator, I, Sibirsk. Mat. Zh., 1970 11 810–821; translation in Siberian Math. J., 11 1970 612–620.

[26] J. Lofstrom, J. Peetre, Approximation Theorems connected with generalized translation, Math. Ann. 181 1969 255–268.

[27] L.N.Lyakhov, On classes of spherical functions and singular psedodifferential operators, Dokl. Akad. Nauk, 272 1983 781–784.

[28] M. Milman, T. Schonbek, Second order estimates in ınterpolation theory and applications, Proc. Amer. Math. Soc. 110 4 1990 961–969.

[29] B. Muckenhoupt, R Wheeden, Weighted norm inequalities for fractional integrals, Trans. Amer. Math. Soc. 192 1974 261–274.

[30] C. Perez, Endpoint estimates for commutators of singular integral operators, J. Func. Anal. 128 1995 163–185.

[31] K. Stempak, The Littlewood-Paley theory for the Fourier-Bessel transform, Mathematical Institute University of Wroclaw (Poland), Preprint No. 45 1985.

[32] K. Trimeche, Inversion of the Lions transmutation operators using generalized wavelets, Appl. Comput. Harmon. Anal. 4 1997 97–112.