Filomat 34:2 (2020), 515–520
https://doi.org/10.2298/FIL2002515B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Generalized q-Poly-Bernoulli Numbers and Polynomials

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Abstract. Many mathematicians in ([1], [2], [5], [14], [20]) introduced and investigated the generalized q-Bernoulli numbers and polynomials and the generalized q-Euler numbers and polynomials and the generalized q-Gennochi numbers and polynomials.

Mahmudov ([15], [16]) considered and investigated the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α and the q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α. In this work, we define generalized q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α. We give new relations between the generalized q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α and the generalized q-poly-Euler polynomials $\mathcal{E}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α and the q-Stirling numbers of the second kind $S_{2,q}(n,k)$.

1. Introduction, Definitions and Notations

As usual, throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

In the usual notations, let $B_n(x)$ and $E_n(x)$ denote respectively, the classical Bernoulli polynomials and the classical Euler polynomials in x defined by the generating functions, respectively

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt},\ |t| < 2\pi. \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},\ |t| < \pi. \tag{2}$$

Also, let

$$B_n(0) := B_n \text{ and } E_n(0) := E_n$$

2010 *Mathematics Subject Classification*. 11B68; 11B73, 11S80.
Keywords. Bernoulli numbers and polynomials; Euler numbers and polynomials; q-Bernoulli polinomials; q-Euler polynomials; Polylogarithm function; Stirling numbers of the second kind; poly-Bernoulli polynomials; poly-Genocchi polynomials.
Received: 17 January 2019; Revised: 02 May 2019; Revised: 09 July 2019; Accepted: 30 July 2019
Communicated by Yilmaz Simsek
This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary.
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where B_n and E_n are respectively, the Bernoulli numbers and the Euler numbers.

$k \in \mathbb{Z}$ and $k \geq 1$, then k-th polylogarithm is defined by ([3], [7], [13]) as

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}. \tag{3}$$

This function is convergent for $|z| < 1$, when $k = 1$,

$$Li_1(z) = -\log(1-z) \tag{4}$$

[15]. The q-numbers and q-factorial are defined by

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q}, q \neq 1 \\ [n]_q! &= [n]_q [n-1]_q [n-2]_q \dots [1]_q, n \in \mathbb{N}, q \in \mathbb{Z} \end{aligned}$$

respectively, where $[0]_q! = 1$. The q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q}{[k]_q! [n-k]_q!}, \ 0 \leq k \leq n$$

The q-power basis is defined by

$$(x+y)_q^n = \begin{cases} (x+y)(x+qy)\dots(x+q^{n-1}y), & n = 1, 2, \dots \\ 1, & n = 0 \end{cases}$$

From above equality, we get

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

The q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \ 0 < |q| < 1, \ |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \ 0 < |q| < 1, z \in \mathbb{C}.$$

From here, we easily see that $e_q(z) E_q(-z) = 1$. The above q-notation can be found in ([8], [13]). Mahmudov in ([15], [16]) defined the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x, y)$ in x, y of order α and the q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x, y)$ in x, y of order α, respectively

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(tx) E_q(ty), \ |t| < 2\pi \tag{5}$$

and

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \ |t| < \pi \tag{6}$$

where $q \in \mathbb{C}$, $\alpha \in \mathbb{N}_0$, $0 < \left|q\right| < 1$. It is obvious that

$$\mathcal{B}_{n,q}^{(\alpha)} \quad : \quad = \mathcal{B}_{n,q}^{(\alpha)}(0,0), \ \lim_{q\to 1^-} \mathcal{B}_{n,q}^{(\alpha)}(x,y) = B_n^{(\alpha)}(x+y), \ \lim_{q\to 1^-} \mathcal{B}_{n,q}^{(\alpha)} = B_n^{(\alpha)}$$

$$\mathcal{E}_{n,q}^{(\alpha)} \quad : \quad = \mathcal{E}_{n,q}^{(\alpha)}(0,0), \ \lim_{q\to 1^-} \mathcal{E}_{n,q}^{(\alpha)}(x,y) = E_n^{(\alpha)}(x+y), \ \lim_{q\to 1^-} \mathcal{E}_{n,q}^{(\alpha)} = E_n^{(\alpha)}$$

Carlitz defined in [6] the q-Stirling numbers of the second kind $S_{2,q}(n,k)$ as

$$\sum_{m=0}^{\infty} S_{2,q}(m,k) \frac{t^m}{[m]_q!} = \frac{\left(e_q(t)-1\right)^k}{[k]_q!} \tag{7}$$

[15]. D. Kim *et al.* in [11] and Bayad *et al.* in [3] defined the poly-Bernoulli polynomials by the following generating function

$$\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{Li_k\left(1-e^{-t}\right)}{e^t-1} e^{xt}. \tag{8}$$

Hamahata in [7] defined the poly-Euler polynomials by,

$$\sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{2Li_k\left(1-e^{-t}\right)}{t\left(e^t+1\right)} e^{xt}. \tag{9}$$

For $k = 1$, from (4). We get $B_n^{(1)}(x) = B_n(x)$ and $E_n^{(1)}(x) = E_n(x)$.

By this motivation, we define the generalized q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α and the generalized q-poly-Euler polynomials $\mathcal{E}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α as the following generating functions, respectively

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{[k,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{Li_k\left(1-e^{-t}\right)}{e_q(t)-1}\right)^{\alpha} e_q(xt) E_q(ty) \tag{10}$$

and

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{[k,\alpha]}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2Li_k\left(1-e^{-t}\right)}{t\left(e_q(t)+1\right)}\right)^{\alpha} e_q(xt) E_q(ty). \tag{11}$$

For $k = 1$, from $Li_1(x) = -\log(1-x)$. The equations (10) and (11) reduces to (5) and (6) respectively.

Srivastava in [20] and Srivastava *et al.* in [21] gave basic knowledge the Bernoulli polynomials, the Euler polynomials and q-Bernoulli polynomials and q-Euler polynomials.

Kim *et al.* in [11] introduced the poly-Bernoullli polynomials, Luo in [14] and Sadjang in [17] and Simsek in [18] considered and gave some relations the q-Bernoulli polynomials and the Stirling numbers of the second kind.

Carlitz in [5] gave some properties of q-Bernoulli polynomials. Mahmudov in ([15], [16]) considered and investigated some recurrences relations between q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α and q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α.

Firstly, Kaneko in [9] defined poly-Bernoulli numbers. Bayat *et al.* in [3] and Hamahata in [7] gave some identities for the poly-Bernoulli polynomials and the poly-Euler polynomials. Kim *et al.* in [10] and Kurt in [12] gave some relations and identities for the q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x, y of order α.

2. Explicit Relations for The Generalized q-Poly-Bernoulli Polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α

In this section, we give some identities and relations for the generalized q-poly-Bernoullli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α. Also, we prove the closed theorem between the generalized q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ and the q-Stirling numbers of the second kind $S_{2,q}(n,k)$.

Theorem 2.1. *The generalized q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ in x, y of order α satisfy the following relations.*

$$\begin{aligned}
\mathcal{B}_{n,q}^{[k,\alpha]}(x,y) &= \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q (x+y)_q^l\ \mathcal{B}_{n-l,q}^{[k,\alpha]}. \\
\mathcal{B}_{n,q}^{[k,\alpha]}(x,y) &= \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{B}_{n-l,q}^{[k,\alpha]}(x,0)\, q^{\binom{l}{2}} y^l. \\
\mathcal{B}_{n,q}^{[k,\alpha]}(x,y) &= \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathcal{B}_{n-l,q}^{[k,\alpha]}(0,y)\, x^l.
\end{aligned}$$

Proof. We can see easily from (10). □

Theorem 2.2. *There is the following relation between the q-poly-Bernoulli polynomials $\mathcal{B}_{n,q}^{[k,\alpha]}(x,y)$ and the q-Stirling numbers of the second kind $S_{2,q}(n,k)$*

$$\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}\mathcal{B}_{n-l}^{[k,1]}(x+y) - \mathcal{B}_{n}^{[k,1]}(x+y) \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+n-l}(m+1)!}{(m+1)^k} S_2(n-l,m+1).
\end{aligned} \tag{12}$$

Proof. By (7) and (10), for $\alpha = 1$ and $q \to 1^-$, we have (12). □

Theorem 2.3. *The following relation holds true*

$$n\mathcal{B}_{n-1}^{[k,1]}(x+y) = \sum_{m=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\mathcal{B}_l(x+y)\frac{(-1)^{m+n-l}}{(m+1)^k}(m+1)!S_2(n-l,m+1). \tag{13}$$

Proof. By (10) for $\alpha = 1$, by using (7), we write as

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,q}^{[k,1]}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{t}\frac{te_q(xt)E_q(ty)}{e_q(t)-1}Li_k\left(1-e^{-t}\right) \\
\sum_{n=0}^{\infty}[n]_q\mathcal{B}_{n-1,q}^{[k,1]}(x,y)\frac{t^n}{[n]_q!} &= \left\{\sum_{l=0}^{\infty}\mathcal{B}_{l,q}(x,y)\frac{t^l}{[l]_q!}\right. \\
&\quad \left.\sum_{m=0}^{\infty}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}S_{2,q}(p,m+1)(-1)^{p+1}\frac{t^p}{p!}\right\}
\end{aligned}$$

We take to limit $q \to 1^-$ both sides and by using the Cauchy product, we have (13). □

Theorem 2.4 (Closed Formula)**.** *The following relation holds true*

$$\mathcal{B}_{n}^{[-k,1]}(x+y) = \sum_{j=0}^{\min(n,k)}(j!)^2 S_2(n,j,x+y)S_2(k,j,1). \tag{14}$$

Proof. By replacing k by $-k$ in (10) , for $\alpha = 1$, we get

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{[-k,1]}(x,y)\frac{t^n}{[n]_q!} = \sum_{m=0}^{\infty}(m+1)^k\left(1-e^{-t}\right)^{m+1}\frac{e_q(tx)E_q(ty)}{e_q(t)-1}$$

we take to limit $q \to 1^-$ in both sides, we have

$$\sum_{n=0}^{\infty} \mathcal{B}_{n}^{[-k,1]}(x+y)\frac{t^n}{n!} = \sum_{m=0}^{\infty}(m+1)^k\left(1-e^{-t}\right)^{m+1}\frac{e^{xt+ty}}{e^t-1},$$

From here, we write as

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{B}_{n}^{[-k,1]}(x+y)\frac{t^n}{n!}\frac{u^k}{k!} &= \sum_{k=0}^{\infty}\frac{1}{e^t-1}\sum_{m=0}^{\infty}(m+1)^k\left(1-e^{-t}\right)^{m+1}e^{xt+ty}\frac{u^k}{k!}\\ &= \frac{1}{e^t-1}\sum_{m=0}^{\infty}\left(1-e^{-t}\right)^{m+1}e^{xt+ty}e^{(m+1)u}\\ &= \frac{e^{xt+ty}\left(1-e^{-t}\right)e^u}{e^t-1}\sum_{m=0}^{\infty}\left(\left(1-e^{-t}\right)e^u\right)^m \end{aligned} \tag{15}$$

Carlitz et al in [6] defined the weighted Stirling numbers of the second is defined kind as

$$\frac{e^{xt}\left(e^t-1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k,x)\frac{t^n}{n!} \tag{16}$$

[18]. By using (15) and (16) ,we get

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{B}_{n}^{[-k,1]}(x+y)\frac{t^n}{n!}\frac{u^k}{k!} &= \frac{e^{(x+y)t}e^u}{1-(e^t-1)(e^u-1)}\\ &= \sum_{j=0}^{\infty} e^{(x+y)t}\left(e^t-1\right)^j e^u\left(e^u-1\right)^j\\ &= \sum_{j=0}^{\infty}[j!\frac{e^{(x+y)t}\left(e^t-1\right)^j}{j!}][\frac{j!e^u\left(e^u-1\right)^j}{j!}]\\ &= \sum_{j=0}^{\infty} j!\sum_{n=0}^{\infty} S_2(n,j,x+y)\frac{t^n}{n!}.j!\sum_{k=0}^{\infty} S_2(k,j,1)\frac{u^k}{k!}\end{aligned}$$

By using Cauchy product and comparing the coefficients of both sides , we have (14). □

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