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Some Identities and Formulas Derived From Analysis of Distribution Functions Including Bernoulli Polynomials and Stirling Numbers

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Abstract. In this paper, we investigate applications of characteristic function of the uniform density function. Using this characteristic function, some identities and formulas associated with Bernoulli polynomials of negative order, Stirling numbers and probability distribution functions were derived.

1. Introduction

The characteristic function of uniform density function is studied in this paper. The following definitions, notations and relations are used when deriving identities and formulas summarized in this paper.

Ω is a nonempty space. $f \subset \Omega$. $\Re$ is a σ-field. A random variable which is a measurable real function X on a probability measure space $(\Omega, \mathfrak{f}, P)$. The distribution of a random variable is the probability measure μ on $(\mathbb{R}, \Re)$ defined by

$$\mu(A) = P(X \in A), \; A \in \Re$$

(*cf.* [1, p.187]). The distribution function of X is defined as follows

$$F(x) = \mu(-\infty, x] = P(X \leq x), \quad \text{for real } x.$$

If F is a increasing distribution function and right continuous function satisfying the following $\mu(A) = P(X \in A)$, these exist on some probability space a random variable X for which $F(x) = P[X \leq x]$ (*cf.* [1, p.188]).

Assuming that μ and g is a real function of a real variable, expected value of $g(X)$ defined as

$$E(g(X)] = \int_{-\infty}^{\infty} g(x)\mu(dx) = \int_{-\infty}^{\infty} g(x)dF(x)$$

(*cf.* [1], [2], [7]). If X has density function f, then we have

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

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(*cf.* [1], [7]). Choosing $g(X) = X^k$ in above definitions will give *kth* moment of X.

Moment generating functions of X is given by

$$\mu(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} dF(x) \tag{1}$$

for all t for which this to finite (the integrand is nonnegative) (*cf.* [1, p. 279]).

The characteristic function of a probability measure μ on the line is defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \mu(dx) = \int_{-\infty}^{\infty} \cos(tx)\, \mu(dx) + i \int_{-\infty}^{\infty} \sin(tx)\, \mu(dx),$$

where $i^2 = -1$ (*cf.* [1], [7]). Therefore, a random variable X with distribution μ has characteristic function

$$\varphi(t) = E\left[e^{itx}\right] = \int_{-\infty}^{\infty} e^{itx} \mu(dx). \tag{2}$$

Since e^{itx} is bounded, the characteristic function always exists. Also, $\varphi(0) = 1$, $\left|\varphi(t)\right| \leq 1$ for all t (*cf.* [1], [7]). Note that, the characteristic function is also referred as the Fourier transform in non-probalistic contexts.

Relations between characteristic function, moments and density functions are given as follows, respectively:

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E\left[x^k\right] \quad \text{and} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)\, dt$$

(*cf.* [1], [7]).

The characteristic functions of some of the well-known probability distributions are summarized in the following table:

Table 1: The characteristic functions of some of the well-known probability distributions (*cf.* [1, p. 348])

Distribution	Density	Interval	Characteristic Function
Uniform	1	$0 < x < 1$	$\frac{e^{it}-1}{it}$
Exponential	e^{-x}	$0 < x < \infty$	$\frac{1}{1-it}$
Normal	$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$	$-\infty < x < \infty$	$e^{-\frac{t^2}{2}}$
Laplace	$\frac{1}{2}e^{-\lvert x\rvert}$	$-\infty < x < \infty$	$\frac{1}{1+t^2}$
Cauchy	$\frac{1}{\pi}\frac{1}{1+x^2}$	$-\infty < x < \infty$	$e^{-\lvert t\rvert}$
Triangular	$1 - \lvert x\rvert$	$-1 < x < 1$	$2\frac{1-\cos t}{t^2}$

Next, we summarize the generating function of the Bernoulli numbers and Bernoulli polynomials, then present Stirling numbers of second type. The Bernoulli numbers of order k, $B_n^{(k)}$, are defined by the following generating function:

$$\frac{t^k}{(e^t - 1)^k} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} \tag{3}$$

(*cf.* [6], [8], [9]). Substituting $k = a + b$ into (3), we have

$$\sum_{n=0}^{\infty} B_n^{(a+b)} \frac{t^n}{n!} = \frac{t^a}{(e^t-1)^a} \frac{t^b}{(e^t-1)^b}.$$

Therefore

$$\sum_{n=0}^{\infty} B_n^{(a+b)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{(a)} \frac{t^n}{n!} \sum_{m=0}^{\infty} B_m^{(b)} \frac{t^m}{m!}.$$

By using some elementary computations, we have the following well-known computation formula for the Bernoulli numbers of order k:

$$B_n^k = B_n^{(a+b)} = \sum_{j=0}^{n} \binom{n}{j} B_j^{(a)} B_{n-j}^{(b)}. \tag{4}$$

Setting $k = 1$ in (3) and (4), we have the Bernoulli numbers $B_n = B_n^{(1)}$:

$$B_0 = 1, \;\; B_n = \sum_{j=0}^{n} \binom{n}{j} B_j, \text{ for } n > 1 \tag{5}$$

(*cf.* [5], [6], [8], [9]). By using (4), few values of $B_n^{(k)}$ are as follows:

$$B_1^{(1)} = -\frac{1}{2}, B_0^{(2)} = 1, B_1^{(2)} = -1, B_2^{(2)} = \frac{5}{6}, B_0^{(3)} = 1, B_1^{(3)} = -\frac{3}{2}, B_2^{(3)} = 2, B_3^{(3)} = -\frac{9}{4}.$$

The Bernoulli polynomials of order k, $B_n^{(k)}(x)$, are defined by the following generating function:

$$\frac{t^k}{(e^t-1)^k} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \tag{6}$$

(*cf.* [6], [8], [9]). Combining (6) and (3), the well-known formula for the Bernoulli polynomials of order k is obtained:

$$B_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(k)} x^{n-j} \tag{7}$$

(*cf.* [6], [8], [9]). By using (7), few values of the Bernoulli polynomials of order k are given as follows:

$$\begin{aligned} B_0^{(k)}(x) &= 1, B_1^{(2)}(x) = x^2 - 1, B_2^{(2)}(x) = x^2 - 2x + \frac{5}{6}, \ldots \\ B_1^{(3)}(x) &= x - \frac{3}{2}, B_2^{(3)}(x) = x^2 - 3x + 2, \ldots, B_n^{(0)}(x) = x^n \ldots \end{aligned}$$

The negative order Bernoulli numbers and polynomials are defined by the following generating functions; respectively:

$$\left(e^t - 1\right)^k = t^k \sum_{n=0}^{\infty} B_n^{(-k)} \frac{t^n}{n!} \tag{8}$$

and

$$\left(e^t - 1\right)^k e^{xt} = t^k \sum_{n=0}^{\infty} B_n^{(-k)}(x) \frac{t^n}{n!} \tag{9}$$

(*cf.* [8], [9]). When letting $x = 0$ in (9), both negative and positive order Bernoulli polynomials are reduced to negative and positive order Bernoulli numbers, respectively:

$$B_n^{(-k)} = B_n^{(-k)}(0) \text{ and } B_n^{(k)} = B_n^{(k)}(0)$$

(*cf.* [8], [9]).

The Stirling numbers of the second kind is defined by the following generating function:

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!} \tag{10}$$

(*cf.* [4], [5], [6], [8], [9]). From equation (10), the Stirling numbers of the second kind is derived:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j} j^n,$$

$$S(n,1) = S(n,n) = 1, \quad S(n,n-1) = \binom{n}{2}, \quad \text{and} \quad S(n,0) = \delta_{n,0}$$

where $\delta_{n,0}$ denoted the Kronecker symbol (*cf.* [4], [5], [6], [8], [9]).

Luo and Srivastava [8, Eq-125] gave the following formula for the polynomilas $B_n^{(-k)}(x)$ which involves Stirling numbers of the second kind:

$$B_n^{(-k)}(x) = \frac{1}{\binom{n+k}{k}}\sum_{j=0}^{n+k}\binom{n+k}{j}S(j,k)\,x^{n+k-j}. \tag{11}$$

By letting $k = n$ into (11), Luo and Srivastava [8] derived the following well-known computation formulas $B_n^{(-k)}(x)$ and the numbers $B_n^{(-n)}$, respectively:

$$B_n^{(-n)}(x) = \frac{(n!)^2}{(2n)!}\sum_{k=0}^{2n}\binom{2n}{k}S(k,n)\,x^{2n-k} \quad \text{and} \quad B_n^{(-n)} = \frac{(n!)^2}{(2n)!}S(2n,n). \tag{12}$$

By using (12), the following numbers are easily obtained

$$B_1^{(-1)} = \frac{1}{2}, B_2^{(-2)} = \frac{7}{6}, B_3^{(-3)} = \frac{9}{2}, B_4^{(-4)} = \frac{243}{10}, B_5^{(-5)} = \frac{6075}{36}, \ldots .$$

2. Main Results

In this section, by using moment generating functions, characteristic function of uniform distribution function, we derive some formulas and identities including the Bernoulli polynomials of negative order, the Stirling numbers and also combinatorial sums. The uniform distribution of random variable X over on interval $(a,b]$ is given by:

$$f_{Uniform}(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x \le b \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

Theorem 2.1.

$$\varphi_{uniform}(t) = \frac{1}{ti(b-a)}\left(e^{bit} - e^{ait}\right), \tag{14}$$

where $b > a$ and $i^2 = -1$.

Proof of (14) is given by various books and manuscript (*cf.* [1, p. 348]). We give brief proof of (14). By using definition of distributing functions with the uniform distribution function, we have the following well-known result:

$$\varphi_{uniform}(t) = \frac{1}{b-a}\int_a^b e^{itx}dx$$

(see for detail [1, p.348]).

Theorem 2.2.

$$\varphi_{uniform}(t) = \sum_{n=0}^{\infty}(b-a)^n B_n^{(-1)}\left(\frac{a}{b-a}\right)\frac{(it)^n}{n!}. \tag{15}$$

Proof. By using (14), we get

$$\varphi_{uniform}(t) = \frac{1}{(b-a)\,it}e^{ait}\left(e^{(b-a)it}-1\right).$$

Combining the above equation with (9), we get

$$\varphi_{uniform}(t) = \sum_{n=0}^{\infty}B_n^{(-1)}\left(\frac{a}{b-a}\right)(b-a)^n\frac{(it)^n}{n!}$$

Thus, proof of the theorem completed. □

Theorem 2.3.

$$\varphi_{uniform}(t) = \frac{2e^{ti\left(\frac{b+a}{2}\right)}}{t(b-a)}\sin\left(\frac{b-a}{2}t\right) \tag{16}$$

(*cf.* [10], [3]).

Proof. The proof of Theorem 2.3 is summarized briefly. By combining (14) with the Euler formula, we get

$$\begin{aligned}\varphi_{uniform}(t) &= \frac{\cos(bt)+i\sin(bt)-(\cos(at)+i\sin(at))}{(b-a)\,it}\\ &= \frac{\cos(bt)-\cos(at)+i(\sin(bt)-\sin(at))}{(b-a)\,it}.\end{aligned}$$

By combining the following well-known trigonometric identities with the above equation, with i^2-1,

$$\cos(bt)-\cos(at) = -2\sin\left(\frac{bt+at}{2}\right)\sin\left(\frac{bt-at}{2}\right)$$

and

$$\sin(bt)-\sin(at) = 2\cos\left(\frac{bt+at}{2}\right)\sin\left(\frac{bt-at}{2}\right),$$

we obtain

$$\begin{aligned}\varphi_{uniform}(t) &= \frac{2\sin\left(\frac{bt-at}{2}\right)}{t(b-a)}\left(\cos\left(\frac{bt+at}{2}\right)+i\sin\left(\frac{bt+at}{2}\right)\right)\\ &= \frac{2e^{ti\left(\frac{b+a}{2}\right)}}{t(b-a)}\sin\left(\frac{bt-at}{2}\right).\end{aligned}$$

Proof of the theorem is completed. □

Theorem 2.4.

$$B_n^{(-1)}\left(\frac{a}{b-a}\right)=\frac{1}{n+1}\frac{b^{n+1}-a^{n+1}}{(b-a)^{n+1}}. \tag{17}$$

Proof. By combining (15) with (14), we obtain

$$\frac{1}{(b-a)\,it}\left(e^{bit}-e^{ait}\right)=\sum_{n=0}^{\infty}B_n^{(-1)}\left(\frac{a}{b-a}\right)(b-a)^n\,\frac{(it)^n}{n!}.$$

Substituting Taylor series of $e^{\alpha t}$ into left sides of the above equation yields

$$\frac{1}{(b-a)\,it}\sum_{n=0}^{\infty}(b^n-a^n)\,\frac{(it)^n}{n!}=\sum_{n=0}^{\infty}B_n^{(-1)}\left(\frac{a}{b-a}\right)(b-a)^n\,\frac{(it)^n}{n!}.$$

Thus

$$\sum_{n=0}^{\infty}\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right)\frac{i^nt^n}{n!}=\sum_{n=0}^{\infty}B_n^{(-1)}\left(\frac{a}{b-a}\right)(b-a)^{n+1}\,\frac{i^nt^n}{n!}.$$

Comparing coefficients of $\frac{t^n}{n!}$ the above equation, after some elementary algebraic manipulations, we arrive at the desired result. □

Theorem 2.5. *Let $n\geq 1$ and $b>a$. Then we have*

$$\sum_{j=0}^{\left[\frac{n-1}{2}\right]}\frac{(b-a)^{2j}\,(b+a)^{n-1-2j}}{(2j+1)!\,(n-1-2j)!}=\frac{2^{n-1}}{n!}\left(\frac{b^n-a^n}{b-a}\right), \tag{18}$$

where $[x]$ denotes the greatest integer function.

Proof. Substituting the following well-known identities

$$\sin(xt)=\sum_{n=0}^{\infty}(-1)^n\,\frac{(xt)^{2n+1}}{(2n+1)!}$$

and

$$e^{xt}=\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}$$

into (16), after some elementary calculations, we obtain

$$\frac{1}{it}\sum_{n=0}^{\infty}\left(\frac{b^n-a^n}{b-a}\right)(it)^n=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{n!\,(b-a)^{2j}\,(b+a)^{n-2j}}{(2j+1)!\,(n-2j)!}\right)\frac{(it)^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\left(\frac{b^n-a^n}{b-a}\right)\frac{(it)^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\left[\frac{n-1}{2}\right]}\frac{n!\,(b-a)^{2j}\,(b+a)^{n-1-2j}}{(2j+1)!\,(n-1-2j)!}\right)\frac{(it)^n}{n!}.$$

Comparing coefficients $\frac{t^n}{n!}$ on the both sides of the above equation, we get assertion of the theorem. □

Theorem 2.6. *Let $a < b$. Then we have*

$$B_n^{(-1)}\left(\frac{a}{b-a}\right) = \frac{n!}{2^n}\sum_{j=0}^{\left[\frac{n}{2}\right]}\frac{\left(\frac{b+a}{b-a}\right)^{n-2j}}{(2j+1)!\,(n-2j)!}.$$

Proof. Combining (15) and (17) with (18), after some elementary calculations, we get assertion of the theorem. □

Combining (11) with (17), we get the following result:

Corollary 2.7. *Let $a < b$. Then we have*

$$\sum_{j=0}^{n+1}\binom{n+1}{j}\left(\frac{a}{b-a}\right)^{n+1-j} = \frac{b^{n+1}-a^{n+1}}{(b-a)^{n+1}}.$$

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