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A Note on Characteristic Function for Bernstein Polynomials Involving Special Numbers and Polynomials

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Abstract. The aim of this present paper is to establish and study generating function associated with a characteristic function for the Bernstein polynomials. By this function, we derive many identities, relations and formulas relevant to moments of discrete random variable for the Bernstein polynomials (binomial distribution), Bernoulli numbers of negative order, Euler numbers of negative order and the Stirling numbers.

1. Introduction

Not only various different real world problems, but also moment generating functions, ordinary generating functions, and exponential generating functions, Fourier transforms are relevant to characteristic functions and their applications. This functions and their applications have been extensively used many different fields such as probability theory, engineering, mathematics, mathematical physics, mathematical statistics, and other related sciences (*cf.* [1]-[16]; and the references cited therein). We here mention that our paper motivation is to give generating function for characteristic functions the well-known Bernstein polynomials and also derive formulas, identities and relations.

We now give some well-known definitions and relations which are used to give results of this paper.

1.1. Some special numbers and polynomials

The Apostol-Bernoulli numbers and polynomials of order k are defined by the following generating functions, respectively:

$$F(t;\lambda,k)=\frac{t^k}{\left(\lambda\exp(t)-1\right)^k}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!} \tag{1}$$

and

$$G(t,x;\lambda,k)=F(t;\lambda,k)\exp(tx)=\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(x,\lambda)\frac{t^n}{n!},$$

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where $\exp(x) = e^{tx}$ (*cf.* [7], [11], [15]; and the references cited therein). With the aid of equation (1), an explicit formula for the Apostol-Bernoulli numbers of order k is given by:

$$\mathcal{B}_n^{(k)}(\lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j^{(k-1)}(\lambda)\, \mathcal{B}_{n-j}(\lambda). \tag{2}$$

where $\mathcal{B}_n(\lambda) = \mathcal{B}_n^{(1)}(\lambda)$ (*cf.* [7], [11], [12], [15]; and the references cited therein).

The Apostol-Euler numbers and polynomials of order k are defined by the following generating functions, respectively:

$$C(t;\lambda,k) = \frac{2^k}{(\lambda \exp(t) + 1)^k} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(\lambda) \frac{t^n}{n!} \tag{3}$$

and

$$H(t,x;\lambda,k) = C(t;\lambda)\exp(tx) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(x;\lambda) \frac{t^n}{n!}$$

(*cf.* [7], [11], [15]; and the references cited therein). With the aid of equation (3), an explicit formula for the Apostol-Euler numbers of order k is given by:

$$\mathcal{E}_n^{(k)}(\lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{E}_j^{(k-1)}(\lambda)\, \mathcal{E}_{n-j}(\lambda).$$

where $\mathcal{E}_n(\lambda) = \mathcal{E}_n^{(1)}(\lambda)$ (*cf.* [4], [7], [12], [15]; and the references cited therein).

The Stirling numbers of the second kind is defined by

$$\frac{(\exp(t) - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!} \tag{4}$$

(*cf.* [5], [4], [7], [11], [12], [15]; and the references cited therein). With the aid of equation (4), an explicit formulas for the Stirling numbers of the second kind is given by

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n$$

and $S(n,0) = \delta_{n,0}$ ($\delta_{n,0}$ denoted the Kronecker symbol) (*cf.* [5], [4], [7], [11], [15]; and the references cited therein).

Let k be a nonnegative integer and λ be a complex number. The combinatorial numbers $y_1(n,k;\lambda)$ are defined by

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!} (\lambda \exp(t) + 1)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda) \frac{t^n}{n!}. \tag{5}$$

By using the above generating function, we have

$$y_1(n,k;\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n \lambda^j$$

(*cf.* [15], [12]; and the references cited therein).

1.2. Characteristic functions and Moment generating functions

Let X be a random variable of the probability distribution $f(x)$. Let $\mathbb{E}(X)$ be expected value (mean) of the random variable X. The characteristic function of of the random variable X is defined by

$$K_x(t) = \mathbb{E}(\exp(itx)), \tag{6}$$

where $i^2 = -1$ (*cf.* [1], [10, p. 10, Eq-(1.3.2)], [14, p. 112]; and the references cited therein).

We give some properties of characteristic function is given as follows.

Let $f(x)$ be a distribution function. A characteristic function $K_x(t)$ satisfies the following Fourier transform property:

$$K_x(t) = \int_{-\infty}^{\infty} f(x) \exp(itx) dx$$

(*cf.* [1], [10]). By using (6),we have $K(0) = 1$, $|K(t)| \leq 1$ and $K_x(-t) = \overline{K_x(t)}$, where $\overline{K_x(t)}$ is a the complex conjugate of $K_x(t)$. We also mention the following well-known property: $K_x(t)$ is uniformly continuous on $\mathbb{R}$, the set of whole real numbers (*cf.* [1], [10], [14]).

Moment generating functions of the random variable X is defined by

$$M_x(t) = \mathbb{E}(\exp(tx)),$$

(*cf.* [1], [10, p. 10, Eq-(1.3.2)], [14, p. 112]; and the references cited therein).

1.3. Moment generating function for Bernstein polynomials (binomial distribution)

The Bernstein polynomials are defined by

$$B_k^n(x; a, b) = \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}, \tag{7}$$

where a and b are real numbers (*cf.* [6, Chapter 5, pp. 299-306], [8], [9], [13]). Now,assume that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$, then equation (7) reduces to binomial type distribution (*cf.* [9], [13]). We note that when $a = 0$ and $b = 1$, equation (7) reduces to the binomial distribution for $0 \leq x \leq 1$.

In [3], we studied on the following well-known moment generating function for the Bernstein polynomials:

$$M_X(t, x : n; a, b) = \sum_{k=0}^{n} \exp(kt) B_k^n(x; a, b). \tag{8}$$

and

$$M_X(t, x : n; a, b) = \left(\exp(t)\frac{x-a}{b-a} + \frac{b-x}{b-a}\right)^n. \tag{9}$$

Setting $a = 0$ and $b = 1$ in (9), we have

$$M_X(t, x : n; 0, 1) = (x \exp(t) + 1 - x)^n$$

(*cf.* [1], [10], [14, p. 100]). We [3] also studied on the following well-known characteristic functions are given by

$$K_X(t, x : n, a, b) = \sum_{k=0}^{n} \exp(ikt) B_k^n(x; a, b) \tag{10}$$

and

$$K_X(t, x : n, a, b) = \left(\frac{x-a}{b-a} \exp(it) + \frac{b-x}{b-a}\right)^n. \tag{11}$$

By using moment generating function for the Bernstein polynomials, we [3] proved the following theorems:

Theorem 1.1. *([3]) Let a and b are real numbers and n be nonegative integer. Then we have*

$$y_1\left(m,n;\frac{x-a}{b-x}\right)=\frac{1}{n!}\left(\frac{b-a}{b-x}\right)^n\sum_{k=0}^{n}k^mB_k^n(x;a,b). \tag{12}$$

Theorem 1.2. *([3]) Let a and b are real numbers and n be nonegative integer. We assume that* $0\leq\frac{x-a}{b-a}\leq 1$ *and* $0\leq\frac{b-x}{b-a}\leq 1$*. Then we have*

$$\mathbb{E}(X^m:n;a,b)=n!\frac{x-b}{b-a}S_2\left(m,n;\frac{x-a}{x-b}\right).$$

We [3] also studies some properties of the following moments:

$$\mathbb{E}(X^m;n;a,b)=\sum_{k=0}^{n}k^mB_k^n(x;a,b).$$

2. Exponential gnenerating function for the function $K_X(t,x:n,a,b)$ and their applications

In this section, we give an exponential generating function for the function $K_X(t,x:n,a,b)$. By using the function $K_X(t,x:n,a,b)$, we derive some identities and relations involving the Bernstein polynomials (binomial distribution), Bernoulli numbers of negative order, Euler numbers of negative order and the Stirling numbers.

Theorem 2.1.

$$B(z;t,x:n;a,b)=\sum_{n=0}^{\infty}K_X(t,x:n,a,b)\frac{z^n}{n!},$$

where

$$B(z;t,x:n;a,b)=\exp\left(z\left(\frac{x-a}{b-a}\exp(it)+\frac{b-x}{b-a}\right)\right)$$

Proof. Setting

$$B(z;t,x:n;a,b)=\sum_{n=0}^{\infty}K_X(t,x:n,a,b)\frac{z^n}{n!}.$$

Substituting (11) into the above equation yields

$$B(z;t,x:n;a,b)=\sum_{n=0}^{\infty}\left(\frac{x-a}{b-a}\exp(it)+\frac{b-x}{b-a}\right)^n\frac{z^n}{n!}.$$

Since

$$\exp(uf(x))=\sum_{n=0}^{\infty}(f(x))^n\frac{u^n}{n!},$$

we have

$$B(z;t,x:n;a,b)=\exp\left(z\left(\frac{x-a}{b-a}\exp(it)+\frac{b-x}{b-a}\right)\right),$$

which completes proof of theorem. □

Theorem 2.2. *Let a and b are real numbers and n be nonnegative integer. We assume that $0 \leq \frac{x-a}{b-a} \leq 1$ and $0 \leq \frac{b-x}{b-a} \leq 1$. Then we have*

$$\mathbb{E}(X^m : n; a, b) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) B_k^n(x; a, b). \tag{13}$$

Proof. Using (10), we obtain

$$K_X(t, x : n, a, b) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} k^m B_k^n(x; a, b) \right) \frac{(it)^m}{m!}. \tag{14}$$

From the above equation, we have

$$K_X(t, x : n, a, b) = \sum_{m=0}^{\infty} \mathbb{E}(X^m : n; a, b) \frac{(it)^m}{m!} \tag{15}$$

and also

$$K_X(t, x : n, a, b) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} (\exp(it) - 1)^j B_k^n(x; a, b).$$

Combining the above equation with (4), we have

$$K_X(t, x : n, a, b) = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) B_k^n(x; a, b) \frac{(it)^m}{m!} \tag{16}$$

By (15) and (16), we obtain

$$\sum_{m=0}^{\infty} \mathbb{E}(X^m : n; a, b) \frac{(it)^m}{m!} = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) B_k^n(x; a, b) \frac{(it)^m}{m!}.$$

Comparing coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we obtain the assertion of the theorem. □

We also combine equation (14) and equation (16), we obtain

$$\sum_{m=0}^{\infty} \left(\sum_{k=0}^{n} k^m B_k^n(x; a, b) \right) \frac{(it)^m}{m!} = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) B_k^n(x; a, b) \frac{(it)^m}{m!}.$$

Comparing coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we get

$$\sum_{k=0}^{n} k^m B_k^n(x; a, b) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) B_k^n(x; a, b). \tag{17}$$

Hence

$$\sum_{k=0}^{n} B_k^n(x; a, b) \left(k^m - \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) \right) = 0. \tag{18}$$

If $x = a$, $x = b$ and $k > n$, then

$$B_k^n(x; a, b) = 0.$$

We assume that $x \neq a$ and $x \neq b$. From equation (18), we have the following well-known formula for the Stirling numbers of the second kind, by different method:

Theorem 2.3.

$$k^m - \sum_{j=0}^{k} \binom{k}{j} j! S(m, j) = 0.$$

Theorem 2.4. *Let a and b are real numbers and n be nonegative integer. We assume that* $0 \leq \frac{x-a}{b-a} \leq 1$ *and* $0 \leq \frac{b-x}{b-a} \leq 1$. *Then we have*

$$\mathcal{E}_m^{(-n)}\left(\frac{x-a}{b-x}\right) = 2^{-n}\left(\frac{b-a}{b-x}\right)^n \sum_{k=0}^{n} k^m B_k^n(x; a, b). \tag{19}$$

Proof. Using (11), we get

$$K_X(t, x : n; a, b) = 2^{-n}\left(\frac{x-b}{b-a}\right)^n C\left(it; \frac{x-a}{x-b}, n\right). \tag{20}$$

Combinig the above equation with (3), we obtain

$$K_X(t, x : n; a, b) = 2^{-n}\left(\frac{x-b}{b-a}\right)^n \sum_{m=0}^{\infty} i^m \mathcal{E}_n^{(k)}\left(\frac{x-a}{x-b}\right)\frac{t^m}{m!}. \tag{21}$$

Combinig the above equation with (14), we also have

$$2^{-n}\left(\frac{x-b}{b-a}\right)^n \sum_{m=0}^{\infty} \mathcal{E}_n^{(k)}\left(\frac{x-a}{x-b}\right)\frac{(it)^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{n} k^m B_k^n(x; a, b)\right)\frac{(it)^m}{m!}.$$

Comparing coefficients of $\frac{(it)^m}{m!}$ on both sides of the above equation, we obtain the assertion of the theorem. □

By using (3) and (1), we have the following well-known relation for both the Apostol-Bernoulli numbers and the Apostol-Euler numbers:

$$\mathcal{B}_m(-\lambda) = -\frac{m}{2}\mathcal{E}_{m-1}(\lambda) \text{ (cf. [15]).}$$

Combining the above well-known relation with (19), we get the following identity:

Corollary 2.5.

$$\mathcal{B}_{m+n}^{(-n)}\left(\frac{a-x}{b-x}\right) = -2^{-n} k! \binom{m}{k}\left(\frac{b-a}{b-x}\right)^n \sum_{k=0}^{n} k^m B_k^n(x; a, b).$$

Combining (12) with (17), we derive the following theorem:

Theorem 2.6. *Let a and b are real numbers. Let m and n be nonegative integers. Then we have*

$$y_1\left(m, n; \frac{x-a}{b-x}\right) = \frac{1}{n!}\left(\frac{b-a}{b-x}\right)^n \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j} j! S(m, j) B_k^n(x; a, b).$$

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