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Some New Identities and Formulas for Higher-Order Combinatorial-Type Numbers and Polynomials

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Abstract. The main purpose of this paper is to provide various identities and formulas for higher-order combinatorial-type numbers and polynomials with the help of generating functions and their both functional equations and derivative formulas. The results of this paper comprise some special numbers and polynomials such as the Stirling numbers of the first kind, the Cauchy numbers, the Changhee numbers, the Simsek numbers, the Peters poynomials, the Boole polynomials, the Simsek polynomials. Finally, remarks and observations on our results are given.

1. Introduction

The aim of this paper is to provide some new identities and relations for a family of generating functions constructed via inspiring by the following equations given by Simsek [9, p. 567]:

$$\int_{\mathbb{X}} \lambda^x (1+\lambda t)^x \chi(x) d\mu_{-q}(x) = \frac{1+q}{(\lambda q)^d (1+\lambda t)^d - 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j (1+\lambda t)^j$$

and

$$H(t;\lambda,q,\chi) = \frac{(1+q)\sum_{j=0}^{d-1}(-1)^j \chi(j)(\lambda q)^j (1+\lambda t)^j}{(\lambda q)^d (1+\lambda t)^d - 1} = \sum_{n=0}^{\infty} Y_{n,\chi}(\lambda,q)\frac{t^n}{n!},$$

where

$$\mu_q(x) = \mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]},$$

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1, \end{cases}$$

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$$\mathbb{X} = \mathbb{X}_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad \mathbb{X}_1 = \mathbb{Z}_p \text{ (the set of } p\text{-adic integers)},$$

and d is an even positive integer, $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 1$ and χ denotes the Dirichlet character with even conductor d.

Here, we note that for any integer d, we give some fundamental properties of the numbers $I_{n,d}(\lambda, q)$ (of higher-order) and the polynomials $I_{n,d}(x; \lambda, q)$ (of higher-order) which are defined by means of the following generating functions, respectively:

$$F_d(t; \lambda, q) = \frac{\log(1+\lambda t)}{(\lambda q)^d (1+\lambda t)^d - 1} = \sum_{n=0}^{\infty} I_{n,d}(\lambda, q) \frac{t^n}{n!}, \tag{1}$$

and

$$G_d(t, x; \lambda, q) = (1+\lambda t)^x F_d(t; \lambda, q) = \sum_{n=0}^{\infty} I_{n,d}(x; \lambda, q) \frac{t^n}{n!}, \tag{2}$$

also generating functions of their higher-order are given by

$$\mathcal{F}_d(t; \lambda, q, k) = \left(\frac{\log(1+\lambda t)}{(\lambda q)^d (1+\lambda t)^d - 1} \right)^k = \sum_{n=0}^{\infty} I_{n,d}^{(k)}(\lambda, q) \frac{t^n}{n!}, \tag{3}$$

and

$$\mathcal{G}_d(t, x; \lambda, q, k) = (1+\lambda t)^x \mathcal{F}_d(t; \lambda, q, k) = \sum_{n=0}^{\infty} I_{n,d}^{(k)}(x; \lambda, q) \frac{t^n}{n!}, \tag{4}$$

where k is a nonnegative integer (*cf.* [5], [6]).

Theorem 1.1 (*cf.* [5]). *Let n be a nonnegative integer. Then we have*

$$I_{n,d}^{(k)}(x; \lambda, q) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} (x)_{n-j} I_{j,d}^{(k)}(\lambda, q) \tag{5}$$

where $(x)_n = x(x-1)\dots(x-n+1)$.

By (5), one can easily see that $I_{0,d}^{(k)}(x; \lambda, q) = I_{0,d}^{(k)}(\lambda, q)$. Setting $k = 0$ into (5) yields $I_{n,d}^{(0)}(x; \lambda, q) = \lambda^n (x)_n$.

Higher-order Simsek numbers $Y_n^{(k)}(\lambda)$ and higher-order Simsek polynomials $Y_n^{(k)}(x; \lambda)$ are defined by means of the following generating functions, respectively (*cf.* [4], [9]):

$$\mathcal{F}(t, k; \lambda) = \left(\frac{2}{\lambda(1+\lambda t) - 1} \right)^k = \sum_{n=0}^{\infty} Y_n^{(k)}(\lambda) \frac{t^n}{n!}, \tag{6}$$

and

$$\mathcal{F}(t, x, k; \lambda) = (1+\lambda t)^x \mathcal{F}(t, k; \lambda) = \sum_{n=0}^{\infty} Y_n^{(k)}(x; \lambda) \frac{t^n}{n!} \tag{7}$$

where k is a nonnegative integer and λ is a real or complex number. It should be noted that $Y_n(\lambda) = Y_n^{(1)}(\lambda)$, $Y_n(x; \lambda) = Y_n^{(1)}(x; \lambda)$ and $Y_n^{(k)}(\lambda) = Y_n^{(k)}(0; \lambda)$ (*cf.* [4], [9], [11]).

In order to obtain our results, we also need the following generating functions for well-known special numbers and polynomials:

The Stirling numbers of the first kind $S_1(n,k)$ are given by

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{8}$$

(*cf.* [7]; see also the references cited therein).

The generating function for the Bernoulli numbers $b_n(0)$ of the second kind, which are also called the Cauchy numbers, are given by (*cf.* [7, p. 116]):

$$F_C(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0)\frac{t^n}{n!} \tag{9}$$

and these numbers are calculated by the following definite integral (*cf.* [7, p. 114]):

$$b_n(0) = \int_0^1 (x)_n\, dx. \tag{10}$$

The generating function for the Peters polynomials $s_n(x;\lambda,\mu)$, which is a member of the family of Sheffer sequences, is given by

$$F_{\mathcal{P}}(t,x;\lambda,\mu) = \frac{1}{\left(1+(1+t)^{\lambda}\right)^{\mu}}(1+t)^x = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}, \tag{11}$$

(*cf.* [1], [3], [7], [8]). Besides, in their special case when $\mu = 1$, the Peters polynomials $s_n(x;\lambda,\mu)$ are reduced to the Boole polynomials $\xi_n(x;\lambda)$ (*cf.* [3]):

$$\xi_n(x;\lambda) = s_n(x;\lambda,1)$$

which, for $x = 0$ and $\lambda = 1$, yields the Changhee numbers $Ch_n = 2\xi_n(0;1)$ (*cf.* [2]).

2. Identities for the numbers $I_{n,d}^{(k)}(\lambda,q)$ and the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$

In this section, we present some identities and relations involving the numbers $I_{n,d}^{(k)}(\lambda,q)$ and the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$.

By making use of (3), we have the following functional equation:

$$\mathcal{F}_d(t;\lambda,q,k+v) = \mathcal{F}_d(t;\lambda,q,k)\,\mathcal{F}_d(t;\lambda,q,v).$$

Using the Cauchy product in the above equation yields

$$\sum_{n=0}^{\infty} I_{n,d}^{(k+v)}(\lambda,q)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} I_{j,d}^{(k)}(\lambda,q)\, I_{n-j,d}^{(v)}(\lambda,q)\right)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields a computation formula for the numbers $I_{n,d}^{(k+v)}(\lambda,q)$ by the following theorem:

Theorem 2.1.

$$I_{n,d}^{(k+v)}(\lambda,q) = \sum_{j=0}^{n}\binom{n}{j} I_{j,d}^{(k)}(\lambda,q)\, I_{n-j,d}^{(v)}(\lambda,q).$$

With the help of Theorem 2.1, some numerical values of the numbers $I_{n,d}^{(k)}(\lambda,q)$ are computed as follows:

$$I_{0,d}^{(2)}(\lambda,q)=I_{1,d}^{(2)}(\lambda,q)=0,\quad I_{2,d}^{(2)}(\lambda,q)=\frac{2\lambda^2}{\left((\lambda q)^d-1\right)^2},\quad I_{3,d}^{(2)}(\lambda,q)=\frac{6\lambda^3\left(1-(1+2d)(\lambda q)^d\right)}{\left((\lambda q)^d-1\right)^3},\ldots$$

By (3), we obtain

$$(\log(1+\lambda t))^k=\left((\lambda q)^d(1+\lambda t)^d-1\right)^k\sum_{n=0}^{\infty}I_{n,d}^{(k)}(\lambda,q)\frac{t^n}{n!}.$$

Making use of the Binomial theorem in the above equation together with combining (8), we get

$$k!\sum_{n=0}^{\infty}\lambda^n S_1(n,k)\frac{t^n}{n!}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda q)^{dj}\left(\sum_{n=0}^{\infty}(dj)_n\lambda^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}I_{n,d}^{(k)}(\lambda,q)\frac{t^n}{n!}\right).$$

Using the Cauchy product and equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.2.

$$S_1(n,k)=\frac{1}{k!\lambda^n}\sum_{j=0}^{k}\sum_{m=0}^{n}(-1)^{k-j}\binom{k}{j}\binom{n}{m}(\lambda q)^{dj}(dj)_m\lambda^m I_{n-m,d}^{(k)}(\lambda,q).$$

Substituting $x=d$ into (5) and $k=1$ into Theorem 2.2 with the help of following well-known identity

$$S_1(n,1)=(-1)^{n-1}(n-1)!, \tag{12}$$

we also arrive at the following corollary:

Corollary 2.3.

$$\lambda^n(-1)^{n-1}(n-1)!=(\lambda q)^d I_{n,d}(d;\lambda,q)-I_{n,d}(\lambda,q). \tag{13}$$

By (3) and (11), in the special case when d is odd integer, we have the following functional equation:

$$\mathcal{F}_d\left(t;\lambda,-\frac{1}{\lambda},k\right)=(-1)^k k!F_{S_1}(\lambda t,k)F_{\mathcal{P}}(\lambda t,0;d,k).$$

which yields

$$\sum_{n=0}^{\infty}I_{n,d}^{(k)}\left(\lambda,-\frac{1}{\lambda}\right)\frac{t^n}{n!}=(-1)^k k!\left(\sum_{n=0}^{\infty}\lambda^n S_1(n,k)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\lambda^n s_n(0;d,k)\frac{t^n}{n!}\right).$$

Using the Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the following theorem:

Theorem 2.4. *Let d be odd integer. Then we have*

$$I_{n,d}^{(k)}\left(\lambda,-\frac{1}{\lambda}\right)=(-1)^k k!\lambda^n\sum_{j=0}^{n}\binom{n}{j}S_1(j,k)s_{n-j}(0;d,k).$$

By setting $k = 1$ into Theorem 2.4 and using (12), we arrive at the following corollary:

Corollary 2.5. *Let d be odd integer. Then we have*

$$I_{n,d}\left(\lambda, -\frac{1}{\lambda}\right) = \lambda^n \sum_{j=1}^{n} (-1)^j \frac{(n)_j \, \xi_{n-j}(0; d)}{j}.$$

By setting $d = 1$ into Corollary 2.5, we also arrive at the following corollary:

Corollary 2.6.

$$I_{n,1}\left(\lambda, -\frac{1}{\lambda}\right) = \lambda^n \sum_{j=1}^{n} (-1)^j \frac{(n)_j \, Ch_{n-j}}{2j}. \tag{14}$$

Since

$$Ch_n = \frac{(-1)^n \, n!}{2^n}$$

(*cf.* [2]), equation (14) is also written as follows:

$$I_{n,1}\left(\lambda, -\frac{1}{\lambda}\right) = \frac{(-1)^n \, \lambda^n n!}{2^{n+1}} \sum_{j=1}^{n} \frac{2^j}{j}. \tag{15}$$

Combining the above equation with the following well-known identity,

$$\frac{2^n}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} = \sum_{j=1}^{n} \frac{2^j}{j}$$

for $n \geq 1$ (*cf.* [10]), we get the following corollary:

Corollary 2.7.

$$I_{n,1}\left(\lambda, -\frac{1}{\lambda}\right) = \frac{(-1)^n \, \lambda^n \, (n-1)!}{2} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}.$$

By (3), (6) and (8), we have the following functional equation:

$$\mathcal{F}_1(t; \lambda, 1, k) = \frac{k!}{2^k} \mathcal{F}(t, k; \lambda) \, F_{S_1}(\lambda t, k)$$

which yields

$$\sum_{n=0}^{\infty} I_{n,1}^{(k)}(\lambda, 1) \frac{t^n}{n!} = \frac{k!}{2^k} \left(\sum_{n=0}^{\infty} Y_n^{(k)}(\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \lambda^n S_1(n, k) \frac{t^n}{n!} \right).$$

Using the Cauchy product in the above equation and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the final equation, we arrive at the following theorem:

Theorem 2.8.

$$I_{n,1}^{(k)}(\lambda, 1) = \frac{k!}{2^k} \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} Y_j^{(k)}(\lambda) \, S_1(n-j, k). \tag{16}$$

Remark 2.9. *Since*

$$Y_n^{(k)}(\lambda) = (-1)^n \binom{n+k-1}{n} \frac{2^k n! \lambda^{2n}}{(\lambda-1)^{k+n}}.$$

(cf. [4]*), equation* (14) *is also written as follows:*

$$I_{n,1}^{(k)}(\lambda, 1) = \frac{k\lambda^n}{(\lambda-1)^k} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (j+k-1)! \left(\frac{\lambda}{\lambda-1}\right)^j S_1(n-j, k). \tag{17}$$

2.1. Riemann integral representation of the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$

Here, we provide integral representation of the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$.

Integrating both sides of (5) from 0 to 1 with respect to x yields

$$\int_0^1 I_{n,d}^{(k)}(x;\lambda,q)\,dx = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} I_{j,d}^{(k)}(\lambda,q) \int_0^1 (x)_{n-j}\,dx.$$

Combining the above equation with (10) yields the Riemann integral representation of the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$ by the following theorem:

Theorem 2.10.

$$\int_0^1 I_{n,d}^{(k)}(x;\lambda,q)\,dx = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} I_{j,d}^{(k)}(\lambda,q)\, b_{n-j}(0).$$

3. Further identities arising from partial derivative formulas for the function $\mathcal{G}_d(t,x;\lambda,q,k)$

The aim of this section is to present a few partial derivative formulas including the generating function $\mathcal{G}_d(t,x;\lambda,q,k)$. By making use of these derivative formulas, we derive some identities in association with the Cauchy numbers and the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$.

Taking the derivative of (4) with respect to the parameter q yields the following partial differential equation

$$\frac{\partial}{\partial q}\{\mathcal{G}_d(t,x;\lambda,q,k)\} = -\frac{dk(\lambda q)^d}{q\log(1+\lambda t)} \mathcal{G}_d(t,x+d;\lambda,q,k+1). \tag{18}$$

Upon rearranging right-hand side of the equation (18) and making use of the generating function for the Cauchy numbers in (9), we get the following functional equation:

$$\frac{\partial}{\partial q}\{\mathcal{G}_d(t,x;\lambda,q,k)\} = -\frac{dk(\lambda q)^{d-1}}{t} F_C(\lambda t)\, \mathcal{G}_d(t,x+d;\lambda,q,k+1). \tag{19}$$

which readily yields

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial q}\{I_{n,d}^{(k)}(x;\lambda,q)\} \frac{t^n}{n!} = -\frac{dk(\lambda q)^{d-1}}{t} \left(\sum_{n=0}^{\infty} \lambda^n b_n(0) \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} I_{n,d}^{(k+1)}(x+d;\lambda,q) \frac{t^n}{n!}\right).$$

By using the Cauchy product in the above equation and after some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial q}\{I_{n,d}^{(k)}(x;\lambda,q)\} \frac{t^n}{n!} = -dk(\lambda q)^{d-1} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} \lambda^j b_j(0)\, I_{n+1-j,d}^{(k+1)}(x+d;\lambda,q)\right) \frac{t^n}{n!}.$$

Upon comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at a derivative formula for the polynomials $I_{n,d}^{(k)}(x;\lambda,q)$ by the following theorem:

Theorem 3.1.

$$\frac{\partial}{\partial q}\{I_{n,d}^{(k)}(x;\lambda,q)\} = \frac{-dk(\lambda q)^{d-1}}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}\lambda^j b_j(0)\, I_{n+1-j,d}^{(k+1)}(x+d;\lambda,q).$$

On the other hand, it follows from (4) and (18) that

$$\sum_{n=0}^{\infty} I_{n,d}^{(k+1)}(x+d;\lambda,q)\frac{t^n}{n!} = -\frac{q\log(1+\lambda t)}{dk(\lambda q)^d}\sum_{n=0}^{\infty}\frac{\partial}{\partial q}\{I_{n,d}^{(k)}(x;\lambda,q)\}\frac{t^n}{n!}.$$

Combining the above equation with the Taylor series for the function $\log(1+\lambda t)$ yields

$$\sum_{n=0}^{\infty} I_{n,d}^{(k+1)}(x+d;\lambda,q)\frac{t^n}{n!} = -\frac{q}{dk(\lambda q)^d}\sum_{n=0}^{\infty}\left(n\sum_{j=0}^{n-1}(-1)^j\frac{(n-1)!\lambda^{j+1}}{(n-1-j)!(j+1)}\frac{\partial}{\partial q}\{I_{n-1-j,d}^{(k)}(x;\lambda,q)\}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3.2.

$$I_{n,d}^{(k+1)}(x+d;\lambda,q) = -\frac{n!}{dk(\lambda q)^{d-1}}\sum_{j=0}^{n-1}(-1)^j\frac{\lambda^j}{(j+1)(n-1-j)!}\frac{\partial}{\partial q}\{I_{n-1-j,d}^{(k)}(x;\lambda,q)\}. \tag{20}$$

Also, taking the derivative of (4) with respect to the parameter x yields the following partial differential equation

$$\frac{\partial}{\partial x}\{\mathcal{G}_d(t,x;\lambda,q,k)\} = \log(1+\lambda t)\,\mathcal{G}_d(t,x;\lambda,q,k). \tag{21}$$

If we combine equation (21) with equation (4) and the Taylor series for the function $\log(1+\lambda t)$, then we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\{I_{n,d}^{(k)}(x;\lambda,q)\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(n\sum_{j=0}^{n-1}(-1)^j\frac{(n-1)!\lambda^{j+1}}{(n-1-j)!(j+1)}I_{n-1-j,d}^{(k)}(x;\lambda,q)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3.3.

$$\frac{\partial}{\partial x}\{I_{n,d}^{(k)}(x;\lambda,q)\} = \lambda n!\sum_{j=0}^{n-1}(-1)^j\frac{\lambda^j}{(n-1-j)!(j+1)}I_{n-1-j,d}^{(k)}(x;\lambda,q). \tag{22}$$

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