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On Submodules of Modules over Group Rings

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Abstract. In this paper, we find some connections between submodules of a module over a group ring RG and subgroups of a group G. Also, we prove that there is a direct connection between conjugate elements of G and RG-submodules of M. Finally, we show that there is a correspondence between the associative powers $\triangle_M^i(G)$ of $\triangle_M(G)$ and ith dimension subgroups $\nabla(\triangle_R^i(G))$ of G over R.

1. Introduction

The history of group rings dates back to long time and since then, many survey articles have appeared ([3], [6], [12], [13], [17]). But modules over group rings are one of the subjects studied in recent years by a lot of authors interested in algebra ([4], [5], [7], [8], [16]). As distinct from the definition of group module over group rings as defined in [7], we gave a definition of group module over group rings with the help of a group homomorphism from G to $End(M)$ in [16]. As a continuous study of our previous paper, in here by this previous definition, we give some characterizations for modules over group rings in this paper.

Throughout this paper, all rings are commutative with identity and all modules are unital group modules over group rings unless stated otherwise.

Let R be a commutative ring with identity and G a group. The group ring of G over R is denoted by RG, which is the set of all formal expressions of the form $\sum\limits_{g\in G} r_g g$ where $r_g \in R$ and $r_g = 0$ for almost every $g \in G$. For elements $r = \sum\limits_{g\in G} r_g g$, $s = \sum\limits_{g\in G} s_g g \in RG$, by writing $r = s$, we mean $r_g = s_g$ for all $g \in G$.

In fact, RG is a ring with the following sum and multiplication

$$r + s = \sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G} (r_g + s_g) g$$

and

$$ra = \sum_{g,h\in G} \left(r_g a_h\right)(gh) = \sum_{g\in G}\sum_{h\in G} \left(r_g a_{h^{-1}g}\right) g$$

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where $a = \sum_{h \in G} a_h h \in RG$.

The augmentation map of RG is a ring homomorphism from RG to R given by $\sum_{g \in G} r_g g \to \sum_{g \in G} r_g$ and its kernel denoted by $\triangle_R(G)$ is called the augmentation ideal of RG. In other words, the ideal $\triangle_R(G)$ of RG is defined as the following set:

$$\left\{ \sum_{g \in G} r_g(g-1) : g \in G, g \neq 1, r_g \in R \right\}.$$

For a left ideal I of RG, $\triangledown(I)$, which is a subgroup of G, is defined as the following set:

$$\{g \in G : g - 1 \in I\} = G \cap (1 + I).$$

One can observe that $\triangledown(\triangle_R(G, H)) = H$ for a subgroup H of G.

Based on the definition and structure of a group ring, a group module over a group ring is defined as follows:

Let τ be a group homomorphism from G to $End(M)$. For all $g \in G, m \in M$, the multiplication mg is defined as

$$mg = \tau(g)(m).$$

In here, M is an RG-module with this multiplication and the group homomorphism τ is a representation of G for M over R ([16]).

Alkan generalized the augmentation mapping of RG to the group module in [2] and proved the following property:

$$\triangle_M(H) = \left\{ \sum_{h \in H} \alpha_h \, (h-1) \mid \alpha_h \in M \right\}$$

is an RG-submodule of M and

$$\triangle_M(H) = M.\triangle_R(G, H),$$

where H is a normal submodule of G.

Besides the relations among RG-submodules of a group module MG with regard to normal subgroups and elements of G, there are other relations among subgroups of G and $\triangle_R(G)$, its associative powers $\triangle_R^i(G)$ and $\triangle_M(G)$, its associative powers $\triangle_M^i(G)$ which are defined below.

Firstly recall that associative powers

$$\triangle_R^i(G) = \left\{ \sum_{g_1, g_2, \ldots, g_i \in H} r_{g_1, g_2, \ldots, g_i} \, (g_1 - 1) \, (g_2 - 1) \ldots (g_i - 1) \mid r_{g_1, g_2, \ldots, g_i} \in R \right\}$$

of $\triangle_R(G)$ are ideals of RG. So, $\triangle_R^i(G)$ is generated as an R-module by the elements $(g_1 - 1)(g_2 - 1)...(g_i - 1)$ where $g_1, g_2, ..., g_i \in G$. Since all $\triangle_R^i(G)$s are ideals of RG, we have $G \cap (1 + \triangle_R^i(G)) = \triangledown(\triangle_R^i(G))$, which is a normal subgroup of G. These normal subgroups $\triangledown(\triangle_R^i(G))$ of G for $i \geq 1$ are called ith dimension subgroups of G over R. And, $G = \triangledown(\triangle_R(G)) \geq \triangledown(\triangle_R^2(G)) \geq ... \geq \triangledown(\triangle_R^i(G)) \geq ...$ is a filtration of G.

Moreover, the decreasing series $\triangle_R(G) \geq \triangle_R^2(G) \geq ... \geq \triangle_R^i(G) \geq ...$ is a filtration of the augmentation ideal $\triangle_R(G)$. This filtration has the property that $\triangle_R^i(G).\triangle_R^j(G) \subseteq \triangle_R^{i+j}(G)$.

In this paper, we find some connections between RG-submodules of M over RG and subgroups of G and a correspondence between associative powers $\triangle_M^i(G)$s of $\triangle_M(G)$ and ith dimension subgroups $\triangledown(\triangle_R^i(G))$ of G.

In Section 2, we firstly deal with the structure of RG-submodules related to subgroups of G in Lemma 2.1 and Proposition 2.2. After giving a relation for a normal group of G related to an RG-submodule in Lemma 2.4, we prove that

$$\triangle_M(G, \triangledown(N)) = \sum_{i=1}^{k}(\triangle_M(G, < x_i >))$$

if $\triangledown(N) =< x_1, x_2, ..., x_k >$, where $x_i \in G$ and N is an RG-submodule of an RG-module M, in Theorem 2.5. Before Theorem 2.6, we deal with an RG-submodule of M to find a correspondence between conjugate elements of G and RG-submodules of M. Thus we close Section 2 by the following result: If

$$H_1 = gH_2g^{-1}$$

holds, then we have

$$\triangle_M(G, H_1) = \triangle_M(G, H_2),$$

where $g \in G$ and H_i is a subgroup of G.

In Section 3, we firstly define associative powers of $\triangle_M(G)$ in Definition 3.1. After showing that $\triangle^i_M(G)$ is an RG-submodule of MG for $i \geq 1$ in Lemma 3.2, we give a characterization for $\triangle^i_M(G)$ in Lemma 3.3. Finally, in Theorem 3.5, we prove that there is a correspondence between the descending module filtration

$$\triangle_M(G) \geq \triangle^2_M(G) \geq ... \geq \triangle^i_M(G) \geq ...$$

and the filtration

$$G = \triangledown(\triangle_R(G)) \geq \triangledown(\triangle^2_R(G)) \geq ... \geq \triangledown(\triangle^i_R(G)) \geq ...$$

of G.

2. Submodules related to normal subgroups

In this section, we examine the connections between RG-submodules of M over RG and subgroups of G.

Lemma 2.1. *Let H_1 and H_2 be normal subgroups of a group G and M be an RG-module. Then*
i) $\triangle_M(G, < H_1 \cup H_2 >) = \triangle_M(G, H_1) + \triangle_M(G, H_2)$, where $< H_1 \cup H_2 >$ is the set generated by H_1 and H_2.
ii) $\triangle_M(G, H_1 \cap H_2) \subseteq \triangle_M(G, H_1) \cap \triangle_M(G, H_2)$.

Proof. *i*) It is clear that $\triangle_M(G, H_1) + \triangle_M(G, H_2) \subseteq \triangle_M(G, < H_1 \cup H_2 >)$.

Take an element x in $\triangle_M(G, < H_1 \cup H_2 >)$. Then $x = \sum\limits_{g\in< H_1\cup H_2>} m_g(g-1)$, where $m_g \in M$ and $g \in< H_1 \cup H_2 >$. Since $g \in< H_1 \cup H_2 >$, we get $x \in \triangle_M(G, H_1) + \triangle_M(G, H_2)$. Thus we have $\triangle_M(G, < H_1 \cup H_2 >) = \triangle_M(G, H_1) + \triangle_M(G, H_2)$.

ii) Take an element x in $\triangle_M(G, H_1 \cap H_2)$. Then $x = \sum\limits_{g\in H_1\cap H_2} m_g(g-1)$, where $m_g \in M$ and $g \in H_1 \cap H_2$. Since $g \in H_1 \cap H_2$, it follows that $g \in H_1$ and $g \in H_2$. Thus $x \in \triangle_M(G, H_1) \cap \triangle_M(G, H_2)$. □

Proposition 2.2. *Let N be an RG-submodule of an RG-module M. Then*
i) $\triangle_N(H) = \left\{\sum\limits_{h\in H} n_h(h-1) : n_h \in N\right\} \subseteq \triangle_M(G)$ is an R-submodule of M.
ii) $\triangle_{\sum N_i}(H) = \sum \triangle_{N_i}(H)$ for submodule N_i of M.

Proof. *i*) It is clear that the sum of any two elements in $\triangle_N(H)$ is in $\triangle_N(H)$. Let $r \in R$ and $x \in \triangle_N(H)$. Then $r.x = r\left(\sum_{y\in H} n_y(y-1)\right) = \sum_{y\in H} rn_y(y-1) \in \triangle_N(H)$, where $rn_y \in N$.

ii) Let x be in $\triangle_{\sum N_i}(H)$. Then $x = \sum_{h\in H} n_h(h-1)$ with $n_h \in \sum N_i$. Since $n_h \in \sum N_i$, we have $n_h = t_{1_h} + t_{2_h} + ... + t_{k_h}$ with $t_{i_h} \in N_i$, and so

$$\begin{aligned} x &= \sum_{h\in H} n_h(h-1) = \sum_{h\in H}(t_{1_h} + t_{2_h} + ...)(h-1) \\ &= \sum_{h\in H} t_{1_h}(h-1) + \sum_{h\in H} t_{2_h}(h-1) + ... \in \sum \triangle_{N_i}(H). \end{aligned}$$

Let x be in $\sum \triangle_{N_i}(H)$. Then $x = \sum_{h\in H} n_{1_h}(h-1) + \sum_{h\in H} n_{2_h}(h-1) + ...$ and $n_{i_h} \in N_i$. Since $\sum n_{i_h} \in \sum N_i$, it follows that

$$x = \sum_{h\in H}\left(\sum n_{i_h}\right)(h-1) \in \triangle_{\sum N_i}(H)$$

This completes the proof. □

Corollary 2.3. *Let N be an RG-submodule of an RG-module M. If N is generated by the set S, then we have*

$$\triangle_N(H) = \left\{\sum_{h\in H} s_h r_h(h-1) : s_h \in S, r_h \in R\right\}.$$

Proof. The proof of this result is similar to the proof of (*ii*) in Proposition 2.2. □

Let N be an RG-submodule of M. Then a subgroup $\nabla(N)$ of G is defined in [2] as

$$\begin{aligned} \nabla(N) &= G \cap (1 + (N :_{RG} M)) \\ &= \{g \in G : g - 1 \in (N :_{RG} M)\}. \end{aligned}$$

Then we give the following lemma.

Lemma 2.4. *Let N_i be an RG-submodule of an RG-module M. Then the followings hold:*

i) $\bigcup_{i=1}^{n} \nabla(N_i) \subseteq \nabla\left(\sum_{i=1}^{n} N_i\right)$, *where n is a positive integer.*

ii) $\nabla[\cap \triangle_M(G, H_i)] = \cap H_i$, *where H_i is a normal subgroup of a group G.*

Proof. *i*) Using the distributive law, we have the following:

$$\begin{aligned} \bigcup_{i=1}^{n} \nabla(N_i) &= \bigcup_{i=1}^{n}(G \cap (1 + (N_i :_{RG} M))) \\ &= G \cap \left(\bigcup_{i=1}^{n}(1 + (N_i :_{RG} M)\right) \\ &\subseteq G \cap \left(1 + \left(\sum_{i=1}^{n} N_i :_{RG} M\right)\right) \\ &= \nabla\left(\sum_{i=1}^{n} N_i\right). \end{aligned}$$

ii) Let g be in $G \cap (1 + \cap \triangle_M(G, H_i))$. Then $g - 1 \in \cap \triangle_M(G, H_i) \subseteq \triangle_M(G, H_i)$. Thus $g \in G \cap (1 + \triangle_M(G, H_i)) = \nabla(\triangle_M(G, H_i)) = H_i$ and so $g \in \cap H_i$.

Let g be in $\cap H_i$. Then $g \in H_i$ for each i. Since $H_i = \nabla(\triangle_M(G, H_i))$, it follows that $g \in \nabla(\triangle_M(G, H_i))$. □

Theorem 2.5. *Let $x_i \in G$ and N be an RG-submodule of an RG-module M. If $\triangledown(N) =< x_1, x_2, ..., x_k >$, then we have*

$$\triangle_M(G, \triangledown(N)) = \sum_{i=1}^{k}(\triangle_M(G, < x_i >)).$$

Proof. Since $x_i \in \triangledown(N)$ for all i, it follows that $\sum_{i=1}^{k}(\triangle_M(G, < x_i >)) \subseteq \triangle_M(G, \triangledown(N))$.

Let $\triangledown(N) =< x_1, x_2, ..., x_k >$ and let x be in $\triangle_M(G, \triangledown(N))$. Then $x = \sum_{h\in\triangledown(N)} m_h(h-1)$ with $m_h \in M$. Since $\triangledown(N) =< x_1, x_2, ..., x_k >$, for each $x_i \in \triangledown(N)$, there exist some positive numbers $y_{1_h}, y_{2_h}, ..., y_{k_h}$ such that $h = x_1^{y_{1_h}} x_2^{y_{2_h}} ... x_k^{y_{k_h}}$.

$$\begin{aligned} x &= \sum_{h\in\triangledown(N)} m_h\,(h-1) = \sum_{h\in\triangledown(N)} m_h \left(x_1^{y_{1_h}} x_2^{y_{2_h}} ... x_k^{y_{k_h}} - 1\right) \\ &= \sum_{h\in\triangledown(N)} m_h \begin{pmatrix} x_1^{y_{1_h}} x_2^{y_{2_h}} ... x_{k-1}^{y_{k-1_h}} \left(x_k^{y_{k_h}} - 1\right) \\ +x_1^{y_{1_h}} x_2^{y_{2_h}} ... x_{k-2}^{y_{k-2_h}} \left(x_{k-1}^{y_{k-1_h}} - 1\right) \\ +... + x_1^{y_{1_h}} (x_2^{y_{2_h}} - 1) + (x_1^{y_{1_h}} - 1) \end{pmatrix} \in \sum_{i=1}^{k} \triangle_M(G, < x_i >) \end{aligned}$$

□

We recall a well-known definition from [9]. Let g and h be two elements of a group G. The element ghg^{-1} is called the conjugate of h by g.

We are now ready to prove the relation between conjugate subgroups of G and RG-submodules of M.

Theorem 2.6. *Let H_i be a subgroup of a group G and $g \in G$. If $H_1 = gH_2g^{-1}$, then we have*

$$\triangle_M(G, H_1) = \triangle_M(G, H_2)$$

Proof. Suppose that $H_1 = gH_2g^{-1}$ and let x be in $\triangle_M(G, H_1)$. Then $x = \sum_{h\in H_1} m_h(h-1)$ with $h \in H_1$ and $m_h \in M$. Since $h \in H_1 = gH_2g^{-1}$, we have $h = gh_2g^{-1}$, where $h_2 \in H_2$. Thus

$$x = \sum_{h\in H} m_h(gh_2g^{-1} - g1g^{-1}) = g\left(\sum_{h_2\in H_2} m_{h_2}(h_2 - 1)\right)g^{-1} \in g\,\triangle_M\,(G, H_2)g^{-1}.$$

Thus we have $\triangle_M(G, H_1) \subseteq g\,\triangle_M\,(G, H_2)g^{-1}$. For the converse inclusion, let x be in $g\,\triangle_M\,(G, H_2)g^{-1}$. Then $x = g\left(\sum_{h\in H_2} m_h(h-1)\right)g^{-1}$, where $h \in H_2$ and $m_h \in M$. For each $h \in H_2$, there exists an element $t_h \in H_1$ such that $h = g^{-1}t_hg$. Thus $x = g\left(\sum_{h\in H_2} m_h(g^{-1}t_hg - 1)\right)g^{-1} = \sum_{t_h\in H_1} m_{t_h}(t_h - 1) \in \triangle_M(G, H_1)$. Hence we have $g\,\triangle_M\,(G, H_2)g^{-1} \subseteq \triangle_M(G, H_1)$. □

Theorem 2.7. *Let H and K be subgroups of G such that $G = HK$ and let M be an RG-module. Then the followings hold:*

i) $\triangle_M(G, H).\,\triangle_R\,(G, K) = M.(K-1)(H-1)$,

ii) $[\triangle_M(G, H).\,\triangle_R\,(G, K)] \cap M(H \cap K) = M.((H \cap K) - 1)^2$.

Proof. *i*) Let $y = \sum_{h\in H} m_h (h-1) \in \triangle_M(G,H)$ and $x = \sum_{k\in K} r_k (k-1) \in \triangle_R(G,K)$, where $r_k \in RG$ and $m_h \in M$. Then we have

$$\begin{aligned} xy &= \left(\sum_{h\in H} m_h (h-1)\right)\left(\sum_{k\in K} r_k (k-1)\right) \\ &= \sum_{h\in H}\sum_{k\in K} m_h r_k (h-1)(k-1). \end{aligned}$$

Since $m_h r_k \in M$, it follows $xy \in M.(K-1)(H-1)$.

To prove its converse, the same method can be used.

ii) It is enough to show $M.(K-1)(H-1) \cap M(H\cap K) = M.((H\cap K)-1)^2$ with (*i*).

Let $x \in M.((H\cap K)-1)^2$. Then there exist $m \in M$ and $t \in RG((H\cap K)-1)^2$ such that $x = mt$. Also there exist $a_i, b_i \in (H\cap K)$ and $r_i \in RG$ such that $t = \sum r_i(a_i-1)(b_i-1)$. Thus we have

$$x = m\left[\sum r_i(a_i-1)(b_i-1)\right] \in M.(K-1)(H-1).$$

Since $x = \sum mr_ia_ib_i - \sum mr_ia_i - \sum mr_ib_i - \sum mr_i1_G \in M(H\cap K)$, it follows $x \in M.(K-1)(H-1) \cap M(H\cap K)$.

To prove its converse, the same method can be used. □

3. The dimension subgroups $\nabla(\triangle_R^i(G))$ of G and the associative powers $\triangle_M^i(G)$ of MG

In this section, we introduce the associative powers deal with associative powers $\triangle_M^i(G)$ of $\triangle_M(G)$, and also give some relations about $\triangle_M^i(G)$ and the dimension subgroups $\nabla(\triangle_R^i(G))$.

Definition 3.1. *Let G be a group, R a ring and M an R-module. The associative powers of* $\triangle_M(G)$ *for* $i \geq 1$ *is the set defined as*

$$\triangle_M^i(G) = \left\{ \sum_{g_1,g_2,\ldots,g_i\in G} m_{g_1,g_2,\ldots,g_i} (g_1-1)(g_2-1)\ldots(g_i-1) \mid m_{g_1,g_2,\ldots,g_i} \in M \right\}.$$

Lemma 3.2. $\triangle_M^i(G)$ *is an RG-submodule of MG for* $i \geq 1$. *Moreover,* $\triangle_M^{i+1}(G)$ *is an RG-submodule of* $\triangle_M^i(G)$ *for* $i \geq 1$.

Proof. We know that $\triangle_M(G)$ is an RG–submodule of MG. Let $\delta = \sum_{g\in G} r_g g \in RG$ and $\mu = \sum_{g_1,g_2\in G} m_{g_1,g_2}(g_1-1)(g_2-1) \in \triangle_M^2(G)$. Since

$$g(g_1-1)(g_2-1) = (gg_1-1)(g_2-1) - (g-1)(g_2-1),$$

we have $\delta\mu \in \triangle_M^2(G)$. The rest of the proof is by induction. Similarly, we get $\triangle_M^{i+1}(G)$ is an RG–submodule of $\triangle_M^i(G)$. □

Lemma 3.3. *Let G be a group and M an R-module. Then we have* $\triangle_M^i(G) = M.\triangle_R^i(G)$.

Proof. Since for any $\sum_{g_1,g_2,\ldots,g_i\in G} m_{g_1,g_2,\ldots,g_i}(g_1-1)(g_2-1)\ldots(g_i-1) \in \triangle_M^i(G)$, we can write

$$\begin{aligned} &\sum_{g_1,g_2,\ldots,g_i\in G} m_{g_1,g_2,\ldots,g_i}(g_1-1)(g_2-1)\ldots(g_i-1) \\ =& \sum_{g_1,g_2,\ldots,g_i\in G} m.r_{g_1,g_2,\ldots,g_i}(g_1-1)(g_2-1)\ldots(g_i-1) \\ =& \; m.\sum_{g_1,g_2,\ldots,g_i\in G} r_{g_1,g_2,\ldots,g_i}(g_1-1)(g_2-1)\ldots(g_i-1) \end{aligned}$$

for some $m \in M$ and $\sum\limits_{g_1,g_2,\dots,g_i \in G} r_{g_1,g_2,\dots,g_i}(g_1-1)(g_2-1)\dots(g_i-1) \in RG$ by the definition of $\triangle_R^i(G)$. Hence, $\triangle_M^i(G) = M.\triangle_R^i(G)$. $\square$

Theorem 3.4. *Let G be a group and M an R-module. Then $\triangle_M(G) \geq \triangle_M^2(G) \geq \dots \geq \triangle_M^i(G) \geq \dots$ is a descending module filtration of $\triangle_M(G)$.*

Proof. For the set of additive subgroups $\triangle_M^i(G)$ of $\triangle_M(G)$ for $i \geq 1$, we have $\triangle_M^i(G) \subseteq \triangle_M^{i+1}(G)$. Moreover $\triangle_R^i(G).\triangle_M^j(G) \subseteq \triangle_M^{i+j}(G)$ for all $i, j \geq 1$, obviously. $\square$

Theorem 3.5. *Let G be a group and M an R-module. There is a correspondence between the descending module filtration*

$$\triangle_M(G) \geq \triangle_M^2(G) \geq \dots \geq \triangle_M^i(G) \geq \dots$$

and the filtration

$$G = \nabla(\triangle_R(G)) \geq \nabla(\triangle_R^2(G)) \geq \dots \geq \nabla(\triangle_R^i(G)) \geq \dots$$

of G. In other words, there is a correspondence between the associative powers $\triangle_M^i(G)$ of $\triangle_M(G)$ and ith dimension subgroups $\nabla(\triangle_R^i(G))$ of G over R.

Proof. We have a correspondence between $\triangle_M^i(G)$ and $\triangle_R^i(G)$ by the equality $\triangle_M^i(G) = M.\triangle_R^i(G)$ given above. For all $\triangle_R^i(G)$ are ideals of RG, we have another correspondence between the associative powers $\triangle_R^i(G)$ of $\triangle_R(G)$ and the ith dimension subgroups of G over R via $G \cap (1 + \triangle_R^i(G)) = \nabla(\triangle_R^i(G))$. Consequently, we have the correspondence between $\triangle_M^i(G)$ and $\nabla(\triangle_R^i(G))$. Hence, we get the following diagram which shows the desired correspondence between the descending module filtration of $\triangle_M(G)$ and the filtration of G by the ith dimension subgroups of G over R.

$$\begin{array}{ccccccc}
\nabla(\triangle_R(G)) & \geq \nabla(\triangle_R^2(G)) & \geq \dots & \geq \nabla(\triangle_R^i(G)) & \geq \dots \\
\uparrow & \uparrow & & \uparrow & \\
\triangle_R(G) & \geq \triangle_R^2(G) & \geq \dots & \geq \triangle_R^i(G) & \geq \dots \\
\uparrow & \uparrow & & \uparrow & \\
\triangle_M(G) & \geq \triangle_M^2(G) & \geq \dots & \geq \triangle_M^i(G) & \geq \dots
\end{array}$$

$\square$

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