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Extended Incomplete Version of Hypergeometric Functions

Mehmet Ali Özarslan[a], Ceren Ustaoğlu[b]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey*
[b]*Department of Computer Engineering, Faculty of Engineering, Final International University,Toroslar Caddesi, No: 6, Çatalköy, Girne, TRNC, Mersin 10, Turkey.*

Abstract. Recently, the incomplete Pochhammer ratios are defined in terms of incomplete beta and gamma functions [10]. In this paper, we introduce the extended incomplete version of Pochhammer symbols in terms of the generalized incomplete gamma functions. With the help of this extended incomplete version of Pochhammer symbols we introduce the extended incomplete version of Gauss hypergeometric and Appell's functions and investigate several properties of them such as integral representations, derivative formulas, transformation formulas, Mellin transforms and log convex properties. Furthermore, we investigate incomplete fractional derivatives for extended incomplete version of some elementary functions.

1. Introduction and preliminaries

Special functions have been an active research area in recent years. Some extensions of the well-known special functions have been considered by several authors (cf.[2, 5, 6, 9, 12, 13, 16, 18, 21, 22]).

The familiar incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ are defined by

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt, \ \ Re(s) > 0;\ x \geqq 0, \ \ \Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt\,, \ \ \ x \geqq 0;\ Re(s) > 0 \ \text{ when } \ x = 0,$$

respectively. The function $\Gamma(s)$ and its incomplete versions $\gamma(s,x)$ and $\Gamma(s,x)$ are crucial in the study for analytical solutions of various problems including different branches of science and engineering (cf.[8]).

The widely used Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1, & \nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}\,, \\ \lambda(\lambda+1)\ldots(\lambda+\nu-1)\,, & \nu \in \mathbb{N};\ \lambda \in \mathbb{C}. \end{array} \right\} \tag{1}$$

In terms of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, the incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ $(\lambda;\nu \in \mathbb{C};\ x \geq 0)$ were defined as follows (cf.[16]):

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)}, \quad \lambda,\nu \in \mathbb{C};\ x \geq 0, \ \ [\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)}, \quad \lambda,\nu \in \mathbb{C};\ x \geq 0. \tag{2}$$

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Email addresses: mehmetali.ozarslan@emu.edu.tr (Mehmet Ali Özarslan), ceren.ustaoglu@final.edu.tr (Ceren Ustaoğlu)

The generalized incomplete Gauss hypergeometric functions ${}_r\gamma_s$ and ${}_r\Gamma_s$ with r numerator and s denominator parameters were defined by means of the incomplete gamma functions as follows (cf.[16]):

$$ {}_r\gamma_s\left[\begin{array}{cc}(a_1,x),a_2,...,a_r; & z\\ c_1,...,c_s; & \end{array}\right]:=\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n...(a_r)_n}{(c_1)_n(c_2)_n...(c_s)_n}\frac{z^n}{n!}, \tag{3}$$

and

$$ {}_r\Gamma_s\left[\begin{array}{cc}(a_1,x),a_2,...,a_r; & z\\ c_1,...,c_s; & \end{array}\right]:=\sum_{n=0}^{\infty}\frac{[a_1;x]_n(a_2)_n...(a_r)_n}{(c_1)_n(c_2)_n...(c_s)_n}\frac{z^n}{n!}. \tag{4}$$

The generalized representation of the incomplete gamma functions were defined by (cf.[1]):

$$\gamma(\alpha,\kappa;p):=\int_0^{\kappa}t^{\alpha-1}exp(-t-\frac{p}{t})dt,\ \ \Gamma(\alpha,\kappa;p):=\int_{\kappa}^{\infty}t^{\alpha-1}exp(-t-\frac{p}{t})dt. \tag{5}$$

It should be noted if we take $p=0$ in (5), we get $\gamma(\alpha,\kappa;0)=\gamma(\alpha,\kappa)$ and $\Gamma(\alpha,\kappa;0)=\Gamma(\alpha,\kappa)$. These generalized incomplete gamma functions satisfy the following decomposition formula:

$$\gamma(\alpha,\kappa;p)+\Gamma(\alpha,\kappa;p)=2p^{\frac{\alpha}{2}}K_\alpha\left(2\sqrt{p}\right)=\Gamma_p(\alpha). \tag{6}$$

where

$$\Gamma_p(x):=\int_0^{\infty}t^{x-1}\exp\left(-t-\frac{p}{t}\right)dt,\ \ Re(p)>0. \tag{7}$$

is the extension of gamma function was introduced by Chaudhry and Zubair [1]. Clearly, when $p=0$, (7) reduces to the familiar gamma function $\Gamma(x)$.

The incomplete beta function is defined by (cf. [10])

$$B_y(x,z):=\int_0^y t^{x-1}(1-t)^{z-1}dt, Re(x)>0, Re(z)>0,\ 0\leq y<1. \tag{8}$$

It can be seen easily when $y\to 1$ in equation (8), this equation will reduce to the well-known beta function.

In terms of the incomplete beta function $B_y(x,z)$, the incomplete Pochhammer ratios $[b,c;y]_n$ and $\{b,c;y\}_n$ were introduced as follows [10]:

$$[b,c;y]_n:=\frac{B_y(b+n,c-b)}{B(b,c-b)},\ \ \{b,c;y\}_n:=\frac{B_{1-y}(c-b,b+n)}{B(b,c-b)} \tag{9}$$

where $0\leq y<1$. With the help of these incomplete Pochhammer ratios $[b,c;y]_n$ and $\{b,c;y\}_n$, incomplete Gauss and confluent hypergeometric functions were defined as follows [10]:

$${}_2F_1(a,[b,c;y];x):=\sum_{n=0}^{\infty}(a)_n[b,c;y]_n\frac{x^n}{n!},\ \ {}_2F_1(a,\{b,c;y\};x):=\sum_{n=0}^{\infty}(a)_n\{b,c;y\}_n\frac{x^n}{n!}, \tag{10}$$

$${}_1F_1([a,b;y];x):=\sum_{n=0}^{\infty}[a,b;y]_n\frac{x^n}{n!},\ \ {}_1F_1(\{a,b;y\};x):=\sum_{n=0}^{\infty}\{a,b;y\}_n\frac{x^n}{n!} \tag{11}$$

where $0\leq y<1$.

On the other hand, the incomplete Appell's functions were defined by [10]:

$$F_1[a,b,c;d;x,z;y]:=\sum_{m,n=0}^{\infty}[a,d;y]_{m+n}(b)_m(c)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{12}$$

$$F_1\{a,b,c;d;x,z;y\} := \sum_{m,n=0}^{\infty} \{a,d;y\}_{m+n}(b)_m(c)_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{13}$$

where $\max\{|x|,|z|\} < 1$, and

$$F_2[a,b,c;d,e;x,z;y] := \sum_{m,n=0}^{\infty} (a)_{m+n}[b,d;y]_m[c,e;y]_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{14}$$

$$F_2\{a,b,c;d,e;x,z;y\} := \sum_{m,n=0}^{\infty} (a)_{m+n}\{b,d;y\}_m\{c,e;y\}_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{15}$$

where $|x| + |z| < 1$.

Furthermore, the incomplete Riemann-Liouville fractional integral operator were introduced as follows [10]:

$$D_z^{\mu}[f(z);y] := \frac{z^{-\mu}}{\Gamma(-\mu)}\int_0^y f(uz)(1-u)^{-\mu-1}du,\ \ D_z^{\mu}\{f(z);y\} := \frac{z^{-\mu}}{\Gamma(-\mu)}\int_y^1 f(uz)(1-u)^{-\mu-1}du;\ \ Re(\mu) < 0. \tag{16}$$

In this paper, in terms of the generalized incomplete gamma functions $\gamma(\alpha,\kappa;p)$ and $\Gamma(\alpha,\kappa;p)$, the extended incomplete version of Pochhammer symbols $(\alpha,\kappa;p)_n$ and $\langle\alpha,\kappa;p\rangle_n$ are defined by

$$(\alpha,\kappa;p)_n := \frac{\gamma(\alpha+n,\kappa;p)}{\Gamma(\alpha)},\ \ \langle\alpha,\kappa;p\rangle_n := \frac{\Gamma(\alpha+n,\kappa;p)}{\Gamma(\alpha)};\quad Re(\alpha) > 0. \tag{17}$$

On the other hand, the extended incomplete binomial expansions are introduced as follows:

$$(1-z;p)_{\kappa^-}^{-\alpha} := \sum_{n=0}^{\infty} \langle\alpha,\kappa;p\rangle_n \frac{z^n}{n!},\ \ (1-z;p)_{\kappa^+}^{-\alpha} := \sum_{n=0}^{\infty} (\alpha,\kappa;p)_n \frac{z^n}{n!};\quad |z| < 1. \tag{18}$$

The organization of the paper as follows:

In Section 2, extended incomplete versions of hypergeometric functions are introduced with the help of these extended incomplete versions of Pochhammer symbols and we obtain integral representations, derivative formulas, transformation formulas, Mellin transforms and log convex properties for these functions. In Section 3, we define extended incomplete versions of Appell's functions and obtain their integral representations. In Section 4, the Riemann-Liouville fractional integrals of some elementary functions are given.

2. Extended incomplete version of hypergeometric functions

In this section, we begin by introducing the extended incomplete versions of Gauss hypergeometric functions by

$${}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right) := \sum_{n=0}^{\infty} (\alpha,\kappa;p)_n [b,c;y]_n \frac{x^n}{n!}, \tag{19}$$

$${}_2F_1\left((\alpha,\kappa;p),\{b,c;y\};x\right) := \sum_{n=0}^{\infty} (\alpha,\kappa;p)_n \{b,c;y\}_n \frac{x^n}{n!}, \tag{20}$$

and

$${}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right) := \sum_{n=0}^{\infty} \langle\alpha,\kappa;p\rangle_n [b,c;y]_n \frac{x^n}{n!}, \tag{21}$$

$$ {}_2F_1\left(\langle\alpha,\kappa;p\rangle,\{b,c;y\};x\right):=\sum_{n=0}^{\infty}\langle\alpha,\kappa;p\rangle_n\{b,c;y\}_n\frac{x^n}{n!} \tag{22}$$

where $0\leq y<1$.

Remark 2.1. *Note that when $p=0$ and $\kappa=0$, (20) and (22) reduce to the corresponding versions given by (10), respectively.*

Remark 2.2. *Note that when $\kappa=0$ and $y=0$, (20) and (22) reduce to the corresponding versions given by (4), respectively.*

Remark 2.3. *In view of (6), these extended incomplete versions of Gauss hypergeometric functions satisfy the following decomposition formula:*

$$ {}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right)+{}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right)=\frac{2p^{\frac{\alpha}{2}}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}[b,c;y]_n\,p^{\frac{n}{2}}K_{\alpha+n}\left(2\sqrt{p}\right), \tag{23}$$

and

$$ {}_2F_1\left((\alpha,\kappa;p),\{b,c;y\};x\right)+{}_2F_1\left(\langle\alpha,\kappa;p\rangle,\{b,c;y\};x\right)=\frac{2p^{\frac{\alpha}{2}}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}b,c;y_n\,p^{\frac{n}{2}}K_{\alpha+n}\left(2\sqrt{p}\right). \tag{24}$$

Theorem 2.4. *The following integral representations hold true:*

$$ {}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right)=\frac{k^\alpha}{\Gamma(\alpha)}\int_0^1u^{\alpha-1}e^{-uk-\frac{p}{uk}}\,{}_1F_1([b,c;y];xuk)du,\ Re(\alpha)>0, \tag{25}$$

$$ {}_2F_1\left((\alpha,\kappa;p),\{b,c;y\};x\right)=\frac{k^\alpha}{\Gamma(\alpha)}\int_0^1u^{\alpha-1}e^{-uk-\frac{p}{uk}}\,{}_1F_1(\{b,c;y\};xuk)du, \tag{26}$$

$$ {}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right)=\frac{1}{\Gamma(\alpha)}\int_\kappa^\infty t^{\alpha-1}e^{-t-\frac{p}{t}}\,{}_1F_1([b,c;y];xt)dt, \tag{27}$$

$$ {}_2F_1\left(\langle\alpha,\kappa;p\rangle,\{b,c;y\};x\right)=\frac{1}{\Gamma(\alpha)}\int_\kappa^\infty t^{\alpha-1}e^{-t-\frac{p}{t}}\,{}_1F_1(\{b,c;y\};xt)dt. \tag{28}$$

where $Re(\alpha)>0, Re(c)>Re(b)>0$.

Proof. Replacing the extended incomplete version of Pochhammer symbol $(\alpha,\kappa;p)$ in the definition (17) by its integral representation given by (5) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis of the Theorem, we find

$$\begin{aligned} {}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right) &= \frac{1}{\Gamma(\alpha)}\int_0^k t^{\alpha-1}e^{-t-\frac{p}{t}}\sum_{n=0}^{\infty}[b,c;y]_n\frac{(xt)^n}{n!}dt,\\ &= \frac{k^\alpha}{\Gamma(\alpha)}\int_0^1u^{\alpha-1}e^{-uk-\frac{p}{uk}}\,{}_1F_1([b,c;y];xuk)du.\end{aligned}$$

Hence the proof is completed. Formula (26), (27) and (28) can be proved in a similar way. □

Remark 2.5. *The case $p=0$ in the above Theorem gives*

$$ {}_2F_1\left((\alpha,\kappa),[b,c;y];x\right)=\frac{1}{\Gamma(\alpha)}\int_0^\kappa t^{\alpha-1}e^{-t}\,{}_1F_1\left([b,c;y];xt\right)dt={}_2\gamma_1\left((\alpha,\kappa),[b,c;y];x\right), \tag{29}$$

$$ {}_2F_1\left((\alpha,\kappa),\{b,c;y\};x\right)=\frac{1}{\Gamma(\alpha)}\int_0^\kappa t^{\alpha-1}e^{-t}\,{}_1F_1(\{b,c;y\};xt)dt={}_2\gamma_1\left((\alpha,\kappa),\{b,c;y\};x\right), \tag{30}$$

$$ {}_2F_1\left(\langle\alpha,\kappa\rangle,[b,c;y];x\right)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\infty}t^{\alpha-1}e^{-t}\,{}_1F_1\left([b,c;y];xt\right)dt={}_2\Gamma_1\left((\alpha,\kappa),[b,c;y];x\right), \tag{31} $$

$$ {}_2F_1\left(\langle\alpha,\kappa\rangle,\{b,c;y\};x\right)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\infty}t^{\alpha-1}e^{-t}\,{}_1F_1(\{b,c;y\};xt)dt={}_2\Gamma_1\left((\alpha,\kappa),\{b,c;y\};x\right). \tag{32} $$

which are called double incomplete Gauss hypergeometric functions.

Theorem 2.6. *The following integral representations hold true:*

$$ \begin{aligned} {}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right) &= \frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-xuy;p)_{\kappa^+}^{-\alpha}\,du, \\ Re(c) &> Re(b)>0,\ |x|<1, \end{aligned} \tag{33} $$

$$ \begin{aligned} {}_2F_1\left((\alpha,\kappa;p),\{b,c;y\};x\right) &= \frac{(1-y)^{c-b}}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u(1-y))^{b-1}(1-x(1-u(1-y));p)_{\kappa^+}^{-\alpha}\,du, \\ Re(c) &> Re(b)>0,\ |x|<1. \end{aligned} \tag{34} $$

Proof. Replacing the incomplete Pochhammer ratio $[b,c;y]$ in the definition (9) by its integral representation given by (8) and using the extended incomplete version of $(1-z;p)_{\kappa^+}^{-\alpha}$ in the definition (18),we get the result in (33). Formula (34) can be proved in a similar way. □

Theorem 2.7. *The following integral representations hold true:*

$$ \begin{aligned} {}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right) &= \frac{y^b}{B(b,c-b)}\int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-xuy;p)_{\kappa^-}^{-\alpha}\,du, \\ Re(c) &> Re(b)>0,\ |x|<1, \end{aligned} \tag{35} $$

$$ \begin{aligned} {}_2F_1\left(\langle\alpha,\kappa;p\rangle,\{b,c;y\};x\right) &= \frac{(1-y)^{c-b}}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u(1-y))^{b-1}(1-x(1-u(1-y));p)_{\kappa^+}^{-\alpha}\,du, \\ Re(c) &> Re(b)>0,\ |x|<1. \end{aligned} \tag{36} $$

Theorem 2.8. *The following derivative formula holds true:*

$$ \frac{d^n}{dx^n}\left({}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right)\right)=\frac{(\alpha)_n(b)_n}{(c)_n}\,{}_2F_1\left((\alpha+n,\kappa;p),[b+n,c+n;y];x\right). \tag{37} $$

Theorem 2.9. *The following transformation formulas hold true:*

$$ {}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right)=(1-x)^{-\alpha}\,{}_2F_1\left((\alpha,\kappa(1-x);p(1-x)),\{c-b,c;1-y\};\frac{-x}{1-x}\right), \tag{38} $$

$$ {}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right)=(1-x)^{-\alpha}\,{}_2F_1\left(\langle\alpha,\kappa(1-x);p(1-x)\rangle,\{c-b,c;1-y\};\frac{-x}{1-x}\right). \tag{39} $$

Proof. Using transformation formula for incomplete confluent hypergeometric function (cf. [10])

$$ {}_1F_1([b,c;y];x)=e^x\,{}_1F_1\left(\{c-b,c;1-y\};-x\right) \tag{40} $$

in (27), we find that

$$ {}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\infty}t^{\alpha-1}e^{-t(1-x)-\frac{p}{t}}\,{}_1F_1\left(\{c-b,c;1-y\};-xt\right)dt. \tag{41} $$

The substitution $\tau=(1-x)t$ in (41), we get the desired result. Formula (38) can be proved in a similar way. □

In the following theorem, we give the Mellin transform representation of the function ${}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right)$ in terms of the double incomplete Gauss hypergeometric function.

Theorem 2.10. *The following Mellin transformation representation holds true:*

$$\mathfrak{M}\left\{{}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right):p\to s\right\}=\frac{\Gamma(s)\Gamma(\alpha+s)}{\Gamma(\alpha)}\,{}_2\gamma_1\left((\alpha,\kappa),[b,c;y];x\right). \tag{42}$$

Proof. Using the Mellin transform operator and we find from (20) that

$$\mathfrak{M}\left\{{}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right):p\to s\right\}=\int_0^\infty p^{s-1}\left(\frac{1}{\Gamma(\alpha)}\sum_{n=0}^\infty \gamma(\alpha+n,\kappa;p)\,[b,c;y]_n\frac{x^n}{n!}\right)dp$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty p^{s-1}\int_0^\kappa t^{\alpha-1}e^{-t-\frac{p}{t}}\sum_{n=0}^\infty [b,c;y]_n\frac{(xt)^n}{n!}dtdp. \tag{43}$$

From the uniform convergence of the integral, the order of integration in (43) can be interchanged. Therefore, we have

$$\mathfrak{M}\left\{{}_2F_1\left((\alpha,\kappa;p),[b,c;y];x\right):p\to s\right\}=\frac{1}{\Gamma(\alpha)}\int_0^\kappa t^{\alpha-1}e^{-t}\,{}_1F_1\left([b,c;y];xt\right)\left\{\int_0^\infty p^{s-1}e^{\frac{-p}{t}}dp\right\}dt$$

$$=\frac{\Gamma(s)}{\Gamma(\alpha)}\int_0^\kappa t^{\alpha+s-1}e^{-t}\,{}_1F_1\left([b,c;y];xt\right)dt. \tag{44}$$

Using (29) in (44), we have the result. □

In the following theorem, we give log convex property of the function ${}_2F_1\left(\langle\alpha,\kappa;p\rangle,[b,c;y];x\right)$.

Theorem 2.11. *Let* $1<q<\infty$ *and* $\frac{1}{q}+\frac{1}{z}=1$. *Then*

$${}_2F_1\left(\left\langle\frac{\beta}{q}+\frac{\gamma}{z},\kappa;p\right\rangle,[b,c;y];x\right)\le\left({}_2F_1\left(\langle\beta,\kappa;p\rangle,[b,c;y];x\right)\right)^{\frac{1}{q}}\left({}_2F_1\left(\langle\gamma,\kappa;p\rangle,[b,c;y];x\right)\right)^{\frac{1}{z}}. \tag{45}$$

Proof. Taking $\alpha=\frac{\beta}{q}+\frac{\gamma}{z}$ in (27), we find

$${}_2F_1\left(\left\langle\beta q^{-1}+\gamma z^{-1},\kappa;p\right\rangle,[b,c;y];x\right)=\frac{1}{\Gamma(\alpha)}\int_\kappa^\infty\left(t^{\beta-1}e^{-t-\frac{p}{t}}\,{}_1F_1([b,c;y];xt)\right)^{\frac{1}{q}}\left(t^{\gamma-1}e^{-t-\frac{p}{t}}\,{}_1F_1([b,c;y];xt)\right)^{\frac{1}{z}}dt.$$

Using the Hölder inequality, we find

$${}_2F_1\left(\left\langle\beta q^{-1}+\gamma z^{-1},\kappa;p\right\rangle,[b,c;y];x\right)\le\left(\frac{1}{\Gamma(\alpha)}\int_\kappa^\infty t^{\beta-1}e^{-t-\frac{p}{t}}\,{}_1F_1([b,c;y];xt)dt\right)^{\frac{1}{q}}\left(\frac{1}{\Gamma(\alpha)}\int_\kappa^\infty t^{\gamma-1}e^{-t-\frac{p}{t}}\,{}_1F_1([b,c;y];xt)dt\right)^{\frac{1}{z}}.$$

Whence the result. □

3. Extended incomplete version of Appell's functions

In this section, we define the extended incomplete versions of Appell's functions as follows:

$$F_1[a,(b,\kappa;p),(c,\kappa;p);d;x,z;y]:=\sum_{m,n=0}^\infty [a,d;y]_{m+n}(b,\kappa;p)_m(c,\kappa;p)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{46}$$

$$F_1\{a,(b,\kappa;p),(c,\kappa;p);d;x,z;y\}:=\sum_{m,n=0}^\infty \{a,d;y\}_{m+n}(b,\kappa;p)_m(c,\kappa;p)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{47}$$

$$F_1[a, \langle b, \kappa; p\rangle, \langle c, \kappa; p\rangle; d; x, z; y] := \sum_{m,n=0}^{\infty} [a, d; y]_{m+n} \langle b, \kappa; p\rangle_m \langle c, \kappa; p\rangle_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{48}$$

$$F_1\{a, \langle b, \kappa; p\rangle, \langle c, \kappa; p\rangle; d; x, z; y\} := \sum_{m,n=0}^{\infty} \{a, d; y\}_{m+n} \langle b, \kappa; p\rangle_m \langle c, \kappa; p\rangle_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{49}$$

where $\max\{|x|, |z|\} < 1$,

$$F_2[(a, \kappa; p), b, c; d, e; x, z; y] := \sum_{m,n=0}^{\infty} (a, \kappa; p)_{m+n} [b, d; y]_m [c, e; y]_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{50}$$

$$F_2\{(a, \kappa; p), b, c; d, e; x, z; y\} := \sum_{m,n=0}^{\infty} (a, \kappa; p)_{m+n} \{b, d; y\}_m \{c, e; y\}_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{51}$$

$$F_2[\langle a, \kappa; p\rangle, b, c; d, e; x, z; y] := \sum_{m,n=0}^{\infty} \langle a, \kappa; p\rangle_{m+n} [b, d; y]_m [c, e; y]_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{52}$$

$$F_2\{\langle a, \kappa; p\rangle, b, c; d, e; x, z; y\} := \sum_{m,n=0}^{\infty} \langle a, \kappa; p\rangle_{m+n} \{b, d; y\}_m \{c, e; y\}_n \frac{x^m}{m!}\frac{z^n}{n!}, \tag{53}$$

where $|x| + |z| < 1$.

Remark 3.1. *Note that when $\kappa = 0$ and $p = 0$, (46) and (47) reduce to the corresponding versions given by (13), respectively (similarly, when $\kappa = 0$ and $p = 0$, (50) and (51) reduce to the corresponding versions given by (15)).*

Now, we proceed by obtaining the integral representations of these functions.

Theorem 3.2. *The following integral representations holds true:*

$$F_1[a, (b, \kappa; p), (c, \kappa; p); d; x, z; y] = \frac{y^a}{B(a, d-a)} \int_0^1 u^{a-1}(1-uy)^{d-a-1} (1-xuy; p)_{\kappa^+}^{-b} (1-zuy; p)_{\kappa^+}^{-c} du, \tag{54}$$

$$F_1\{a, (b, \kappa; p), (c, \kappa; p); d; x, z; y\} = \frac{(1-y)^{d-a}}{B(a, d-a)} \int_0^1 u^{d-a-1}(1-u(1-y))^{a-1} (1-x(1-u(1-y)); p)_{\kappa^+}^{-b} \times (1-z(1-u(1-y)); p)_{\kappa^+}^{-c} du, \tag{55}$$

$$F_1[a, \langle b, \kappa; p\rangle, \langle c, \kappa; p\rangle; d; x, z; y] = \frac{y^a}{B(a, d-a)} \int_0^1 u^{a-1}(1-uy)^{d-a-1} (1-xuy; p)_{\kappa^-}^{-b} (1-zuy; p)_{\kappa^-}^{-c} du, \tag{56}$$

$$F_1\{a, \langle b, \kappa; p\rangle, \langle c, \kappa; p\rangle; d; x, z; y\} = \frac{(1-y)^{d-a}}{B(a, d-a)} \int_0^1 u^{d-a-1}(1-u(1-y))^{a-1} (1-x(1-u(1-y)); p)_{\kappa^-}^{-b} \times (1-z(1-u(1-y)); p)_{\kappa^-}^{-c} du. \tag{57}$$

Proof. Replacing the integral representation for incomplete beta function which is given by (8) and using the extended incomplete binomial expansion in the definition (18),we get the result in (54). Formula (55), (56) and (57) can be proved in a similar way. □

Theorem 3.3. *The following integral representations holds true:*

$$F_2[(a, \kappa; p), b, c; d, e; x, z; y] = \frac{y^{b+c}}{B(b, d-b) B(c, e-c)} \int_0^1 \int_0^1 u^{b-1} (1-uy)^{d-b-1} v^{c-1} (1-vy)^{e-c-1} \tag{58}$$

$$\times (1-xuy-zvy; p)_{\kappa^+}^{-a} dudv, \tag{59}$$

$$F_2\{(a,\kappa;p),b,c;d,e;x,z;y\} = \frac{(1-y)^{d-b+e-c}}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 u^{d-b-1}(1-u(1-y))^{b-1}v^{e-c-1}(1-v(1-y))^{c-1} \times(1-x(1-u(1-y))-z(1-v(1-y));p)_{\kappa^+}^{-a}\,dudv, \quad (60)$$

$$F_2[\langle a,\kappa;p\rangle,b,c;d,e;x,z;y] = \frac{y^{b+c}}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 u^{b-1}(1-uy)^{d-b-1}v^{c-1}(1-vy)^{e-c-1} \times(1-xuy-zvy;p)_{\kappa^-}^{-a}\,dudv, \quad (61)$$

$$F_2\{\langle a,\kappa;p\rangle,b,c;d,e;x,z;y\} = \frac{(1-y)^{d-b+e-c}}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 u^{d-b-1}(1-u(1-y))^{b-1}v^{e-c-1}(1-v(1-y))^{c-1} \times(1-x(1-u(1-y))-z(1-v(1-y));p)_{\kappa^-}^{-a}\,dudv. \quad (62)$$

Proof. Replacing the integral representation for incomplete beta function which is given by (8) and using the extended incomplete binomial expansion in the definition (18), we are led to the desired result (58). Formula (60), (61) and (62) can be proved in a similar way. □

4. Fractional calculus

In this section, we use the incomplete Riemann-Liouville fractional integral operators which are given by (16) and investigate the incomplete fractional derivatives for extended incomplete versions of some elementary functions.

Theorem 4.1. *Let $Re(\lambda)>0$, $Re(\alpha)>0$, $Re(\mu)<0$ and $|z|<1$. Then*

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha};y\right]=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}{}_2F_1((\alpha,\kappa;p),[\lambda,\mu;y];z), \quad (63)$$

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha};y\right\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}{}_2F_1((\alpha,\kappa;p),\{\lambda,\mu;y\};z). \quad (64)$$

Proof. Direct calculations yield

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha};y\right] = \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}(1-uz;p)_{\kappa^+}^{-\alpha}(1-u)^{\mu-\lambda-1}du.$$

By (33), we can easily obtain $D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha};y\right]$ asserted by (63). Formula (64) can be proved in a similar way. □

Theorem 4.2. *Let $Re(\lambda)>Re(\mu)>0$, $Re(\alpha)>0$, $Re(\beta)>0$; $|az|<1$ and $|bz|<1$. Then*

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-az;p)_{\kappa^+}^{-\alpha}(1-bz;p)_{\kappa^+}^{-\beta};y\right]=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1[\lambda,(\alpha,\kappa;p),(\beta,\kappa;p);\mu;az,bz;y], \quad (65)$$

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-az;p)_{\kappa^+}^{-\alpha}(1-bz;p)_{\kappa^+}^{-\beta};y\right\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1\{\lambda,(\alpha,\kappa;p),(\beta,\kappa;p);\mu;az,bz;y\}. \quad (66)$$

Proof. Considering Theorem 3.2, we have

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-az;p)_{\kappa^+}^{-\alpha}(1-bz;p)_{\kappa^+}^{-\beta};y\right]=\frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}(1-auz;p)_{\kappa^+}^{-\alpha}(1-buz;p)_{\kappa^+}^{-\beta}(1-u)^{\mu-\lambda-1}du.$$

By (54), we can write

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-az;p)_{\kappa^+}^{-\alpha}(1-bz;p)_{\kappa^+}^{-\beta};y\right] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)F_1[\lambda,(\alpha,\kappa;p),(\beta,\kappa;p);\mu;az,bz;y]$$

Hence the proof is completed. Formula (66) can be proved in a similar way. □

Theorem 4.3. *Let* $Re(\lambda) > Re(\mu) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\gamma) > 0;\ \left|\frac{t}{1-z}\right| < 1$ *and* $|t| + |z| < 1$ *we have*

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha}\ {}_2F_1\left((\alpha,\kappa;p),[\beta,\gamma;y];\frac{t}{1-z}\right);y\right] = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2\left[(\alpha,\kappa;p),\lambda,\beta;\mu,\gamma;z,t;y\right], \tag{67}$$

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha}\ {}_2F_1\left((\alpha,\kappa;p),\{\beta,\gamma;y\};\frac{t}{1-z}\right);y\right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_2\left\{(\alpha,\kappa;p),\lambda,\beta;\mu,\gamma;z,t;y\right\}. \tag{68}$$

Proof. Direct calculations yield

$$\begin{aligned}&D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha}\ {}_2F_1\left((\alpha,\kappa;p),[\beta,\gamma;y];\frac{t}{1-z}\right);y\right]\\ =\ &\frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)B(\beta,\gamma-\beta)}\int_0^y\int_0^y (uz)^{\lambda-1}(1-u)^{\mu-\lambda-1}s^{\beta-1}(1-s)^{\gamma-\beta-1}(1-uz-st;p)_{\kappa^+}^{-\alpha}\,dsdu.\end{aligned}$$

By (58), we can write

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z;p)_{\kappa^+}^{-\alpha}\ {}_2F_1\left((\alpha,\kappa;p),[\beta,\gamma;y];\frac{t}{1-z}\right);y\right] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)F_2\left[(\alpha,\kappa;p),\lambda,\beta;\mu,\gamma;z,t;y\right]$$

Whence the result. Formula (68) can be proved in a similar way. □

5. Concluding Remarks

It should be mentioned that further generalization of extended Gauss, Appell and Lauricella hypergeometric functions have been introduced and investigated in [14] and their generating relations have been obtained with the help of extended fractional derivative operator. On the other hand, families of incomplete H-functions and $\bar{H}$-functions have been defined and investigated and also potential applications of them involving probability theory have been indicated in [15].

The above mentioned papers will be the motivation of the new investigations, where the authors can introduce more general forms of the functions defined in this paper. These new defined functions will have many potential applications in different areas.

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