Filomat 34:3 (2020), 843–860
https://doi.org/10.2298/FIL2003843A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Classes of Condensing Operators and Application to Solvability of Singular Integral Equations

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Abstract. In this paper, we introduce the notion of Krasnoselskii and Dugundji-Granas condensing operators in Banach spaces. In order to pave the way for a study the solvability of some classes of singular integral equations in the Banach algebra $C[a, b]$, we provide some results for the existence of fixed points for such condensing operators. An example is presented to show the applicability of the results.

1. Introduction and Preliminaries

The significance of fixed point theory and its applications in different branches of mathematics is well known. One of the most important theorems in fixed point theory, is the widely-used Schauder Theorem, which is stated as follows:

Theorem 1.1 ([1]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $F : \Omega \to \Omega$ has at least one fixed point in Ω.*

We know compactness is an essential condition in this theorem. If we want to imagine a condition which is weaker than compactness, we first look at Darbo's result which uses the concept of measures of noncompactness. We refer the reader to [2, 9, 12, 20, 22] for a review of some applications of measure of noncompactness to differential and integral equations. In this paper, the authors apply the concept of measures of noncompactness in the axiomatic form. Other methods where authors use important measures of noncompactness in different Banach spaces is also of interest (see [8, 13, 29] and the references therein).

Assume that E is a given real Banach space with the norm $\|.\|$ and X is a subset of E. The symbols $\overline{X}$, $ConvX$ represent the closure and convex closure of X, respectively. Further, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. The algebraic operations on sets will be denoted by $X + Y$ and $\lambda X (\lambda \in \mathbb{R})$. Moreover, we denote the norm of a bounded set X by $\|X\|$, i.e., $\|X\| = \sup\{\|x\| : x \in X\}$. In what follows we will use the following definition of the concept of a measure of noncompactness [15].

Definition 1.2. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness on E if it satisfies the following conditions:*

2010 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10
Keywords. Measure of noncompactness, Darbo's fixed point theorem, singular integral equation, condensing operators
Received: 22 July 2019; Revised: 24 September 2019; Accepted: 04 November 2019
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1 *the family* $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $ker\mu \subseteq \mathfrak{N}_E$*;*

2 $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$*;*

3 $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$*;*

4 $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$*;*

5 *if* (X_n) *is a sequence of closed sets from* $\mathfrak{M}_E$ *such that* $X_{n+1} \subseteq X_n$ *for* $n = 1, 2, \ldots$ *and if* $\lim_{n\to\infty} \mu(X_n) = 0$*, then the set* $X_\infty = \bigcap_{n=0}^{\infty} X_n$ *is nonempty.*

It can be shown that the set X_∞ from axiom 5 is a member of the $ker\mu$. This fact will be useful in our further considerations. Now, we turn our attention to the Darbo's fixed point theorem, that is formulated below under the concept of measure of noncompactness.

Theorem 1.3 ([21]). *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space E and let* $T : \Omega \to \Omega$ *be a continuous mapping. Assume that there exists a constant* $k \in [0, 1)$ *such that*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty subset X of Ω*, where* μ *is a measure of noncompactness defined in E. Then T has a fixed point in* Ω*.*

Recently in [3–5, 7, 12, 18, 22, 23] the authors obtained some new generalizations of Darbo fixed point theorem. In [27], Krasnoselskii investigated a class of operators T on a complete metric space (X, d) that satisfy the condition:

$$d(Tx, Ty) \leq q(\alpha, \beta)d(x, y), \quad \alpha \leq d(x, y) \leq \beta, \quad x, y \in X,$$

where $q(\alpha, \beta) < 1$ for $\beta \geq \alpha > 0$, and showed that such operators have a fixed point in X. Also, in [25] Dugundji and Granas have studied a class of mappings T on a complete metric space (X, d) that satisfy the following condition: there exist a $\theta : X \times X \to \mathbb{R}_+$ with

$$\inf\{\theta(x, y) : a \leq d(x, y) \leq b\} > 0, \quad \text{for all intervals} \quad [a, b] \subseteq \mathbb{R}_+ \setminus \{0\},$$

such that

$$d(Tx, Ty) \leq d(x, y) - \theta(x, y),$$

for all $x, y \in X$, and proved that such operators have a fixed point in X. In this paper, first we introduce the notion of a Krasnoselskii condensing operator and Dugundji-Granas condensing operator in Banach spaces and provide some results regarding the existence of fixed points for such operators. Further, we present a result on the existence of coupled fixed points for a class of condensing operators in Banach spaces. Finally, the application of our results to the problem of existence of solutions of a large class of integral equations in the Banach algebra $C[a, b]$ is discussed. We note that the solvability of the following integral equations

$$x(t) = ((Tx)(t))f\Big(t, \int_{h(t)}^{H(t)} x(s)ds, x(g(t))\Big), \quad t \in [0, a],$$

$$x(t) = \Big(p_1(t) + f_1(t, x(t)) \int_0^t \nu(t, s, x(s))ds\Big)\Big(p_2(t) + f_2(t, x(t)) \int_0^\infty \nu(t, s, x(s))ds\Big), \quad t \in [0, +\infty),$$

$$x(t) = \Big(p_1(t) + f_1(t, x(t)) \int_0^t g_1(t, s)h_1(s, x(s))ds\Big)\Big(p_2(t) + f_2(t, x(t)) \int_0^t g_2(t, s)h_2(s, x(s))ds\Big), \quad t \in [0, +\infty),$$

$$x(t) = \Big(m_1(t) + f_1(t, x(t)) \int_0^t \frac{v_1(t, s, x(s))}{(t-s)^{\alpha_1}}\Big)\Big(m_2(t) + f_2(t, x(t)) \int_0^t \frac{v_2(t, s, x(s))}{(t-s)^{\alpha_2}}\Big), \quad t \in [0, +\infty),$$

were investigated in [14, 16, 18], respectively. Also

$$x(t) = \Big(f\Big(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}ds\Big)\Big).\Big(g(t, x(\gamma(t)), x(t)\int_0^1 \nu(t,s,x(s))ds\Big), \quad t \in [0,1],$$

was discussed in [24]; for more examples see [10, 23]. In the last section of this paper, we study the solvability of the following integral equations

$$x(t) = f(t, x(t))\Big(q(t) + \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t)} \frac{\xi(t,s,x(\gamma(s)))}{(\rho(t)-s)^{1-\alpha}}ds\Big)$$

by establishing some results on the existence of fixed points for the product of two operators each of which satisfies a special conditions in a Banach algebra, using the technique of measure of noncompactness.

2. Main results

We begin by defining the notion of Krasnoselskii and Dugundji-Granas condensing operators in Banach spaces. Then using the technique of measure of noncompactness, we provide some fixed point results for such operators.

Definition 2.1. *Let Ω be a nonempty, bounded subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. We call $T : \Omega \to \Omega$ a Krasnoselskii condensing operator if there exists a mapping $\eta : \mathfrak{M}_E \to \mathbb{R}_+$ with*

$$\nu(a,b) := \sup\{\eta(X) : \quad a \le \mu(X) \le b\} < 1 \quad \textit{for all intervals} \quad [a,b] \subseteq \mathbb{R}_+ \setminus \{0\},$$

such that

$$\mu(TX) \le \eta(X)\mu(X) \tag{1}$$

for all $X \subseteq \Omega$.

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, μ be a measure of noncompactness on E and $T : \Omega \to \Omega$ a Krasnoselskii condensing and continuous operator. Then T has at least one fixed point in Ω.*

Proof. Consider the sequence $\{\Omega_n\}$ as follows

$$\begin{cases} \Omega_0 = \Omega, \\ \Omega_n = \overline{Conv} T\Omega_{n-1}, \qquad \text{n} \ge 1. \end{cases}$$

If there exists a natural number n_0 such that $\mu(\Omega_{n_0}) = 0$, then Ω_{n_0} is compact and using the Schauder fixed point theorem, we deduce that T has at least one fixed point in Ω. So without loss of generality, we assume for every $n \geqslant 1$, $\mu(\Omega_n) > 0$. The sequence $\{\mu(\Omega_n)\}$ is a positive nonincreasing sequence, and therefore convergent to some $a \geqslant 0$. We must have $a = 0$, otherwise $\mu(\Omega_n) \in [a, a+1]$ for all large n, and we could then choose such an n and use $c = \nu(a, a+1)$ to get, by induction

$$\begin{aligned} a \le \mu(\Omega_{n+k}) &= \mu(\overline{Conv} T\Omega_{n+k-1}) \\ &= \mu(T\Omega_{n+k-1}) \\ &\le c(\mu(\Omega_{n+k-1})) \\ &\le \dots \\ &\le c^k \mu(\Omega_n) \\ &\le c^k(a+1) \end{aligned}$$

for all $k > 0$, which because $c < 1$, is a contradiction; therefore $a = 0$. Thus when $n \to \infty$, $\mu(\Omega_n) \to 0$. As $\{\Omega_n\}$ is a nested sequence, using axiom 5 of measures of noncompactness, we find that Ω_∞ is nonempty and according to the property of Ω_∞, it is a member of $\ker\mu$. Also we know that Ω_∞ is closed, bounded and convex and is invariant under T. Therefore $T : \Omega_\infty \to \Omega_\infty$ satisfies the required conditions of Schauder's Theorem. As a result T has a fixed point in Ω_∞. Since $\Omega_\infty \subseteq \Omega$, the proof is complete. □

Theorem 2.3. *Let Ω be a nonempty, bounded subset of a Banach space E and μ be a measure of noncompactness on E. Then, $T : \Omega \to \Omega$ is a Krasnoselskii condensing operator if and only if for any $M > 0$ there exist $m = m(M) \in (0, M)$ and $k_M < 1$ such that $\limsup_{M\to M_0^+} \frac{m(M)}{M} < 1$ for all $M_0 > 0$ and*

$$\mu(TX) \le k_M \mu(X)$$

for all $X \subseteq \Omega$ with $m \le \mu(X) \le M$.

Proof. Let T be a Krasnoselskii condensing operator with the mapping $\eta : \mathfrak{M}_E \to \mathbb{R}_+$ given in Definition 2.1. We put

$$\begin{cases} m(M) = \frac{M}{2}, & \text{for all} \quad M > 0, \\ k_M = \sup\{\eta(X) : \frac{M}{2} \le \mu(X) \le M\}. \end{cases}$$

Therefore it is obvious that $\limsup_{M\to M_0^+} \frac{m(M)}{M} < 1$ for all $M_0 > 0$ and $k_M = \sup\{\eta(X) : \frac{M}{2} \le \mu(X) \le M\} < 1$. Conversely, we define $G : \mathbb{R}_+ \to [0, 1)$ and $\eta : \mathfrak{M}_E \to \mathbb{R}_+$ as follows

$$\begin{cases} G(M) = k_M, \\ \eta(X) = G(\mu(X)). \end{cases}$$

Then we have $\mu(TX) \le \eta(X)\mu(X)$, by assumption. Now, let $[a, b]$ be an arbitrary interval with $0 < a \le b$. We prove that the collection $\{\big(m(M), M\big), M > 0\}$ of open sets covers the interval $[a, b]$. On the contrary, we can assume that $y \in [a, b]$ and $y \notin \big(m(M), M\big)$ for all $M > 0$. Without loss of generality we may suppose that $y \notin (m(M), M)$ for all $M > y$. Consequently, $y < m(M)$ and $\lim_{M\to y^+} \frac{m(M)}{M} = 1$ which is in contrast with the assumption. Now, since $[a, b]$ is compact, we can find a finite cover $\big(m(M_i), M_i\big), i = 1, 2, 3, ..., n$ of $[a, b]$ and we have

$$\sup\{\eta(X) : \quad a \le \mu(X) \le b\} \le \max\{k_{M_i} : i = 1, 2, ..., n\} < 1,$$

and the proof is complete. □

Corollary 2.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $T : \Omega \to \Omega$ satisfies the following condition: for any $\varepsilon > 0$, there exists $l_\varepsilon < 1$ such that*

$$\mu(TX) \le l_\varepsilon \mu(X), \quad \textit{for all} \quad X \subseteq \Omega \quad \textit{with} \quad \mu(X) \ge \varepsilon.$$

Then T has at least one fixed point in Ω.

Proof. We put $m(M) = \frac{M}{2}$ and $k_M = l_{\frac{M}{2}}$ for $M > 0$ in Theorem 2.3. Now applying Theorem 2.2 we can conclude that T has at least one fixed point in Ω. □

Now we present a common fixed point theorem for commuting mappings. First, we introduce the notion of an affine mapping.

Definition 2.5. *A mapping T on a convex set M is affine if it satisfies the identity*

$$T(kx + (1 - k)y) = kT(x) + (1 - k)T(y)$$

whenever $0 < k < 1, x, y \in M$.

Theorem 2.6. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E with a measure of noncompactness μ. Also, let I be a set of indices, and $\{T_i\}_{i\in I}$, S be continuous self-maps on Ω such that for any $i \in I$, T_i commutes with S and $T_i(\overline{Conv}(A)) \subset \overline{Conv}(T_i(A))$ for any $A \subset \Omega$ and $i \in I$. If there exists a mapping $\eta : \mathfrak{M}_E \to \mathbb{R}_+$ with*

$$\nu(a, b) := \sup\{\eta(X) : \quad a \leq \mu(X) \leq b\} < 1 \quad \text{for all intervals} \quad [a, b] \subseteq \mathbb{R}_+ \setminus \{0\},$$

such that

$$\mu(SA) \leq \eta(A)\mu(T_iA)$$

for all $A \subseteq \Omega$, then S and T_i ($i \in I$) have fixed points. Moreover, if T_i is affine for all $i \in I$ then T_i and S have a common fixed point in Ω.

Proof. To prove the theorem, we consider the sequence $\{\Omega_n\}$ defined as $\Omega_0 = \Omega$ and $\Omega_n = \overline{Conv}(S(\Omega_{n-1}))$ for $n = 1, 2, 3, \ldots$. Then, we show that

$$\Omega_n \subset \Omega_{n-1} \quad , \quad T_i(\Omega_n) \subset \Omega_n \tag{2}$$

for every $n = 1, 2, 3, \ldots$ and $i \in I$.

It is clear that $\Omega_1 \subset \Omega_0$ and

$$\begin{aligned} T_i(\Omega_1) &\subset \overline{Conv}(S(T_i(\Omega_0))) \\ &\subset \overline{Conv}(S(\Omega_0)) \\ &= \Omega_1. \end{aligned}$$

Thus (2) holds for $n = 1$. Assume now that (2) is true for some $n \geq 1$ and $i \in I$. Then

$$\begin{aligned} \Omega_{n+1} &= \overline{Conv}(S(\Omega_n)) \\ &\subset \overline{Conv}(S(\Omega_{n-1})) \\ &= \Omega_n \end{aligned}$$

and

$$\begin{aligned} T_i(\Omega_{n+1}) &= T_i(\overline{Conv}(S(\Omega_n))) \\ &\subset \overline{Conv}(S(T_i\Omega_n)) \\ &\subset \overline{Conv}(S(\Omega_n)) \\ &= \Omega_{n+1} \end{aligned}$$

for any $i \in I$. Hence, (2) is true by induction.

As before, we can assume that $\mu(\Omega_n) > 0$ for all $n = 1, 2, \ldots\ldots$ Therefore, the sequence $\{\mu(\Omega_n)\}$ is a positive nonincreasing sequence. Thus, this sequence is convergent to a number say a, $a \geq 0$. We now show that $a = 0$. Suppose $a > 0$. Then for all large n, we have $\mu(\Omega_n) \in [a, a + 1]$. For an adequately large n and $c = \nu(a, a + 1)$, by employing inductive reasoning we conclude that

$$a \leq \mu(\Omega_{n+k}) \leq c^k \mu(T_i(\Omega_n)) \leq c^k \mu(\Omega_n) \leq c^k (a + 1)$$

for all $k > 0$, which because $c < 1$, is a contradiction; therefore $a = 0$. Thus, we have $\mu(\Omega_n) \to 0$ as $n \to \infty$. Since the sequence $\{\Omega_n\}$ is nested, in view of axiom 5 of Definition 1.2, $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of Ω. Hence Ω_∞ is the member of $\ker\mu$. Thus, Ω_∞ is compact. Next, keeping in mind that S maps Ω_∞ into itself and taking into account the Schauder fixed point theorem, we infer that the operator S has a fixed point x in the set Ω_∞. Obviously $x \in \Omega$. Further, the set $F = \{x \in \Omega : Sx = x\}$ is closed by the continuity of S. On the other hand, since T_i commutes with S for any $i \in I$, we see that $T_i x$ is a fixed point of S for any $x \in F$. Thus, $T_i(F) \subseteq F$. Now assume that $\mu(F) > 0$, and then we have

$$\begin{aligned}\mu(F) &\leq \mu(S(F)) \\ &\leq \eta(F)\sup_{i\in I}(T_i(F)) \\ &\leq \eta(F)\mu(F)\end{aligned}$$

and as a consequence, we obtain $1 \leq \eta(F)$, a contradiction. Then $\mu(F) = 0$, so F is compact. Thus T_i has a fixed point in F and $F_i = \{x \in \Omega : T_i x = x\}$ is closed by the continuity of T_i. Also, Sx is a fixed point of T_i for each $x \in F_i$ since T_i commutes with S. Therefore, $S(F_i) \subseteq F_i$. For every $i \in I$, F_i is convex since T_i is an affine map. Moreover, we have $T_j(F_i) \subseteq F_i$ for every $j \in I$ and F_i is convex, closed and bounded. Consider now the restriction $S : F_i \to F_i$ of S. For any $A \subseteq F_i$, we have $\mu(SA) \leq \eta(A)\mu(T_i A)$. Then, S has a fixed point in F_i by a similar way to that of employed before. Therefore S and T_i have a common fixed point in Ω. □

Definition 2.7. *We call $\phi : \mathfrak{M}_E \to \mathbb{R}_+$ a compactly positive mapping on Banach space E if*

$$\lambda(a,b) := \inf\{\phi(X) : \quad a \leq \mu(X) \leq b\} > 0 \quad \textit{for all intervals} \quad [a,b] \subseteq \mathbb{R}_+ \setminus \{0\}.$$

Definition 2.8. *Let Ω be a nonempty, bounded subset of a Banach space E with a measure of noncompactness μ. We call $T : \Omega \to \Omega$ a Dugundji-Granas condensing operator, if there exists a compactly positive mapping ϕ on E such that*

$$\mu(TX) \leq \mu(X) - \phi(X),$$

for all $X \in \mathfrak{M}_E$.

Theorem 2.9. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $T : \Omega \to \Omega$ be a Dugundji-Granas condensing and continuous operator. Then T has at least one fixed point in Ω.*

Proof. Consider the sequence $\{\Omega_n\}$ as follows

$$\begin{cases} \Omega_0 = \Omega, \\ \Omega_n = \overline{Conv}T\Omega_{n-1}, \qquad \text{n} \geq 1. \end{cases}$$

As in the proof of Theorem 2.2, without loss of generality, we can suppose that for all $n \geqslant 1$, we have $\mu(\Omega_n) > 0$. Sequence $\{\mu(\Omega_n)\}$ is a positive nonincreasing sequence. Therefore this sequence is convergent, say to a, $a \geq 0$. We show that $a = 0$. Suppose $a > 0$ and take a $b > a$. Then for all large n, we have $\mu(\Omega_n) \in [a,b]$. For an adequately large n and $c = \lambda(a,b)$, by inductive reasoning, we have

$$\begin{aligned} a \leq \mu(\Omega_{n+k}) &= \mu(\overline{Conv}T\Omega_{n+k-1}) \\ &= \mu(T\Omega_{n+k-1}) \\ &\leq \mu(\Omega_{n+k-1}) - \phi(\Omega_{n+k-1}) \\ &\leq \ldots \\ &\leq \mu(\Omega_n) - \sum_{i=0}^{k} \phi(\Omega_{n+k-i}) \\ &\leq \mu(\Omega_n) - kc \\ &\leq b - kc \end{aligned}$$

for all $k > 0$, a contradiction with regard to that $c > 0$, therefore $a = 0$. Thus, when $n \to \infty$, $\mu(\Omega_n) \to 0$. As $\{\Omega_n\}$ is a nested sequence, using axiom 5 of measures of noncompactness, we find that Ω_∞ is nonempty and according to the property of Ω_∞, it is a member of $\ker\mu$. Also we know that Ω_∞ is closed, bounded and convex and is invariant under T. Therefore Theorem 1.1 completes the proof. □

Remark 2.10. *Comparing the inequalities and assumptions of Theorems 2.2 and 2.9, we can conclude that they are equivalent. Indeed, the inequality $\mu(TX) \leq \mu(X) - \phi(X)$ takes the form $\mu(TX) \leq \eta(X)\mu(X)$ if we put*

$$\eta(X) = \begin{cases} 1 - \dfrac{\phi(X)}{\mu(X)} & \mu(X) \neq 0 \\ 0, & \mu(X) = 0. \end{cases}$$

It is not hard to see that η satisfies the conditions in Definition 2.1. The converse can be seen similarly.

We now show that some recent generalizations of Darbo's fixed point theorem studied in [3–5] are in fact particular cases of Theorem 2.2 and 2.9.

Lemma 2.11. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing, upper semicontinuous function with $\varphi(r) < r$ for $r > 0$. Then $\lambda(a, b) := \inf\{r - \varphi(r) : \quad a \leq r \leq b\} > 0$ for all finite interval $[a, b] \subset \mathbb{R}_+ \setminus \{0\}$.*

Proof. Suppose that $\lambda(a, b) = 0$ for some $b > a > 0$. Then there exists a sequence $\{r_i\} \subset [a, b]$ such that $r_i - \varphi(r_i) \to 0$. Without loss of generality, we can assume that $\{r_i\}$ is an increasing sequence. Let $r_i \to \rho \geq a$ when $i \to \infty$. Since φ is nondecreasing we get $\lim \varphi(r_i) = \overline{\lim}\varphi(r_i)$. Then the upper semicontinuity of ϕ gives us

$$\rho = \lim r_i = \lim \varphi(r_i) = \overline{\lim}\varphi(r_i) \leq \varphi(\rho),$$

a contradiction. □

Corollary 2.12. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing, upper semicontinuous function and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. If μ is an arbitrary measure of noncompactness on Banach space E then $\mu - \varphi o \mu$ is compactly positive mapping on E.*

Proof. According to Lemma 2.1 in [4], we know that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, upper semicontinuous function, then the following two conditions are equivalent:

(1) $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$.

(2) $\psi(t) < t$ for any $t > 0$.

Now using the previous lemma we get the desired result. □

Corollary 2.13 (Theorem 2.2 in [4]). *Let Ω be a nonempty, bounded, closed and convex subset of Banach space E and let $T : \Omega \to \Omega$ be a continuous operator satisfying the inequality*

$$\mu(TX) \leq \varphi(\mu(X))$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Then T has at least one fixed point in Ω.

Proof. We rewrite the inequality $\mu(TX) \leq \varphi(\mu(X))$ as

$$\mu(TX) \leq \mu(X) - (\mu(X) - \varphi(\mu(X))). \tag{3}$$

Now if we put $\phi(X) = \mu(X) - \varphi(\mu(X))$, then inequality (3) transforms into $\mu(TX) \leq \mu(X) - \phi(X)$ in which $\phi(X)$ according to the previous corollary, is a compactly positive mapping. Now Theorem 2.9 is applicable. □

Definition 2.14 ([5]). *Let C be a nonempty subset of a Banach space E and μ an arbitrary measure of noncompactness on E. We say that an operator $T : C \to C$ is a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \Longrightarrow \mu((TX)) < \varepsilon,$$

for any bounded subset X of C.

Corollary 2.15 (Theorem 2.2 in [5]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point in C.*

Proof. In view of Theorem 2.6 in [5] and Corollary 2.13, we can conclude that T has at least one fixed point in C. □

Corollary 2.16 (Theorem 2.1 in [3]). *Let Ω be a nonempty, bounded, closed and convex subset of Banach space E with an arbitrary measure of noncompactness μ. Also, let $T : \Omega \to \Omega$ be a continuous operator satisfying the inequality*

$$\mu(TX) \leq \beta(\mu(X))\mu(X)$$

for any nonempty subset X of Ω, where $\beta : \mathbb{R}_+ \to [0, 1)$ satisfies the condition: $\beta(t_n) \to 1$ implies $t_n \to 0$. Then T has at least one fixed point in Ω.

Proof. First we show that

$$\nu(a, b) := \sup\{\beta(\mu(X)) \quad a \leq \mu(X) \leq b\} < 1 \quad \text{for all intervals} \quad [a, b] \subseteq \mathbb{R}_+ \setminus \{0\}.$$

On the contrary, let us assume that a finite interval $[a_1, b_1] \subseteq \mathbb{R}_+ \setminus \{0\}$ exists with $\nu(a_1, b_1) = 1$. Therefore, there must be a sequence $\{X_n\}$ of bounded subsets of Ω with $\lim_{n\to\infty} \beta(r_n) \to 1$, where $r_n := \mu(X_n) \in [a, b]$. But then, $r_n \nrightarrow 0$ which is in contrast with the assumption. Now taking $\eta := \beta o \mu$, then Theorem 2.2 completes the proof. □

Next, we consider the definition of a coupled fixed point for a bivariate mapping and recall a useful theorem about the construction of a measure of noncompactness on a finite product space.

Definition 2.17 ([28]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $G : X \times X \to X$ if $G(x, y) = x$ and $G(y, x) = y$.*

Theorem 2.18 ([8]). *Suppose $\mu_1, \mu_2, ..., \mu_n$ are measures of noncompactness in Banach spaces $E_1, E_2, ..., E_n$, respectively. Moreover, assume that the function $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, 3, ..., n$. Then*

$$\widehat{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), ..., \mu_n(X_n)),$$

defines a measure of noncompactness in $E_1 \times E_2 \times E_3 \times ... \times E_n$ where X_i denotes the natural projection of X into E_i, for $i = 1, 2, ..., n$.

Now, as a consequence of Theorem 2.18, we have the following example (see [6]).

Example 2.19. *Let μ be a measure of noncompactness on a Banach space E, considering $F(x, y) = max\{x, y\}$ for any $(x, y) \in [0, \infty)^2$, then we see that F is convex and $F(x, y) = 0$ if and only if $x = y = 0$, hence all the conditions of Theorem 2.18 are satisfied. Therefore, $\widehat{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$ defines a measure of noncompactness in the space $E \times E$ where $X_i, i = 1, 2$ denote the natural projections of X. Similarly, by letting $F(x, y) = x + y$ for any $(x, y) \in [0, \infty)^2$, we conclude that $\widehat{\mu}(X) = \mu(X_1) + \mu(X_2)$ defines a measure of noncompactness in the space $E \times E$ where $X_i, i = 1, 2$ denote the natural projections of X.*

Now, we introduce the notion of a bivariate Dugundji-Granas condensing operators and then provide a coupled fixed point theorem for such operators.

Definition 2.20. *Let Ω be a nonempty and bounded subset of a Banach space E and μ an arbitrary measure of noncompactness on E. We say that $T : \Omega \times \Omega \to \Omega$ is a Dugundji-Granas condensing operator if there exists a compactly positive mapping $\phi : \mathfrak{M}_E \to \mathbb{R}_+$ on E such that*

$$\mu(G(X_1 \times X_2)) \leq \frac{\mu(X_1) + \mu(X_2)}{2} - \frac{\phi(X)}{2}$$

for all $X_1, X_2 \subset \Omega$.

Theorem 2.21. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and μ an arbitrary measure of noncompactness on E. If $T : \Omega \times \Omega \to \Omega$ is a continuous Dugundji-Granas condensing operator, then T has at least one coupled fixed point in $\Omega \times \Omega$.*

Proof. By Example 2.19, we conclude that $\widehat{\mu}(X) = \mu(X_1) + \mu(X_2)$ is a measure of noncompactness in the space $E_1 \times E_2$ where X_i, $(i = 1, 2)$ denote the natural projections of X. Now, we consider the map $\widehat{G} : \Omega\times\Omega \to \Omega\times\Omega$ defined by the formula

$$\widehat{G}(x, y) = (G(x, y), G(y, x)),$$

which is continuous on $\Omega \times \Omega$. We show that $\widehat{G}$ satisfies all the conditions of Theorem 2.9. For this purpose, let $X \subset \Omega \times \Omega$ be a nonempty subset. Then, we have

$$\begin{aligned}\widehat{\mu}\Big(\widehat{G}(X)) &\leq \widehat{\mu}(G(X_1 \times X_2) \times G(X_2 \times X_1)\Big)\\ &= \mu(G(X_1 \times X_2)) + \mu(G(X_2 \times X_1))\\ &\leq \frac{\mu(X_1) + \mu(X_2)}{2} - \frac{\phi(X)}{2} + \frac{\mu(X_2) + \mu(X_1)}{2} - \frac{\phi(X)}{2}\\ &= \widehat{\mu}(X) - \phi(X).\end{aligned}$$

As a result, we get

$$\widehat{\mu}(\widehat{G}(X)) \leq \widehat{\mu}(X) - \phi(X).$$

Therefore, all the conditions of Theorem 2.9 are satisfied then G has a coupled fixed point in $\Omega \times \Omega$. □

Now we employ the Dugundji-Granas condensing operators to provide some results for the existence of fixed points in Banach algebras. For a Banach space E and given subsets X and Y of E, let

$$XY = \{xy : x \in X, y \in Y\}.$$

Definition 2.22 ([16]). *We say that a measure of noncompactness μ on a Banach algebra E satisfies the condition (m), if for arbitrary sets $X, Y \in \mathfrak{M}_E$ we have*

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X),$$

where $\|X\| := \sup_{x\in X} \|x\|$.

Let us consider the Banach space $C[a, b]$ consisting of all real functions defined and continuous on interval $[a, b]$. This space is endowed with the standard norm $\|x\| = \sup\{\|x(t)\| : \quad t \in [a, b]\}$. Obviously $C[a, b]$ has also the structure of a Banach algebra with the standard multiplication of functions. Moreover, fix a set $X \in \mathfrak{M}_{C[a,b]}$ and an arbitrary $\varepsilon > 0$. For an arbitrary function $x \in X$ let us denote by $\omega(x, \varepsilon)$ the modulus of continuity of x , i.e.

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [a, b], |t - s| \leq \varepsilon\}.$$

Moreover,

$$\begin{aligned}&\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},\\ &\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X, \varepsilon).\end{aligned}$$

Now, we introduce a measure of noncompactness in the Banach algebra $C[a, b]$ which satisfies condition (m) on some subfamily of the family $\mathfrak{M}_{C[a;b]}$. To do this, let us take a set $X \in \mathfrak{M}_{C[a;b]}$ and for $x \in X$ consider the following quantities:

$$d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in [a, b]; t, s \in [a, b], t \leq s\}.$$

The quantity $d(x)$ represents the degree of decrease of the function x. In addition, d(x) = 0 if and only if x is nondecreasing on $[a,b]$. Moreover, let us put $d(X) = \sup\{d(x) : x \in X\}$ and denote $\mu_d(X) = \omega_0(X) + d(X)$. It can be shown that μ_d is a measure of noncompactness in the space $C[a,b]$ and satisfies condition (m) on the subfamily of the family $\mathfrak{M}_{C[a,b]}$, consisting of sets of functions being nonnegative on the interval $[a,b]$. (cf. [18]).

Remark 2.23. *It is noteworthy to mention that if $X \in ker\mu_d$ then X is equicontinuous and every $x \in X$ is a nondecreasing function on $[a,b]$.*

Theorem 2.24. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach algebra E and operators P and T continuously transform Ω into E such that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, we assume that the operator $S = P.T$ transforms Ω into itself. Assume P and T on Ω satisfy the conditions*

$$\begin{cases} \mu(PX) \leq \mu(X) - \phi_1(X), \\ \mu(TX) \leq \mu(X) - \phi_2(X), \end{cases}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness satisfying condition (m), $\phi_1, \phi_2 : \mathfrak{M}_E \to \mathbb{R}_+$ and $\|T(\Omega)\|\phi_1 + \|P(\Omega)\|\phi_2$ is a compactly positive mapping on E. If $\|P(\Omega)\| + \|T(\Omega)\| \leq 1$, then S has at least fixed point in Ω.

Proof. Let us take an arbitrary nonempty subset X of the set Ω. Then in view of the assumption that μ satisfies condition (m) we obtain

$$\begin{aligned} \mu(S(X)) &= \mu(P(X).T(X)) \\ &\leq \|P(X)\|\mu(T(X)) + \|T(X)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\mu(T(X)) + \|T(\Omega)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\Big(\mu(X) - \phi_2(X)\Big) + \|T(\Omega)\|\Big(\mu(X) - \phi_1(X)\Big) \\ &= \Big(\|P(\Omega)\| + \|T(\Omega)\|\Big)\mu(X) - \Big(\|P(\Omega)\|\phi_2(X) + \|T(\Omega)\|\phi_1(X)\Big) \\ &\leq \mu(X) - \Big(\|P(\Omega)\|\phi_2(X) + \|T(\Omega)\|\phi_1(X)\Big). \end{aligned} \tag{4}$$

Now, letting $\theta(X) = \|T(\Omega)\|\phi_1(X) + \|P(\Omega)\|\phi_2(X)$, then from (4), we have $\mu(S(X)) \leq \mu(X) - \theta(X)$. Now, with regard to the fact that θ is compactly positive, we can apply Theorem 2.9, to get the desired result. □

Remark 2.25. *The set of all fixed points of the operator S on the set Ω is a member of the $ker\mu$.*

Corollary 2.26. *Let E, μ, Ω, P, T, S be as in Theorem 2.24. Assume P and T satisfy the conditions*

$$\begin{cases} \mu(P(X)) \leq \psi_1(\mu(X)), \\ \mu(T(X)) \leq \psi_2(\mu(X)), \end{cases}$$

for any nonempty subset X of Ω, where $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions such that

$$\begin{cases} \lim_{n\to\infty} \psi_1^n(t) = 0, \\ \lim_{n\to\infty} \psi_2^n(t) = 0, \end{cases}$$

for any $t \geq 0$. If $\|T(\Omega)\| + \|P(\Omega)\| < 1$, then S has at least fixed point in Ω.

Proof. We let

$$\begin{cases} \phi_1(X) = \mu(X) - \psi_1(\mu(X)), \\ \phi_2(X) = \mu(X) - \psi_2(\mu(X)). \end{cases}$$

In view of Lemma 2.1 in [4], we have

$$\begin{cases} \|T(\Omega)\|\psi_1(t) < \|T(\Omega)\|t, \\ \|P(\Omega)\|\psi_2(t) < \|P(\Omega)\|t. \end{cases}$$

Therefore, $\|T(\Omega)\|\psi_1(t) + \|P(\Omega)\|\psi_2(t) < \Big(\|T(\Omega)\| + \|P(\Omega)\|\Big)t < t$ and by Lemma 2.11, we can conclude $\|T(\Omega)\|\psi_1(t) + \|P(\Omega)\|\psi_2(t)$ is compactly positive mapping on E and Theorem 2.24 completes the proof. □

3. Application

In this section, we consider the Banach algebra $C(I)$, where I is a bounded and closed interval. For the sake of simplicity we presume $I = [0, 1]$, and employ the measure of noncompactness μ_d defined in section 2. We investigate the following class of singular integral equations:

$$x(t) = f(t, x(t))\Big(q(t) + \frac{1}{\Gamma(\alpha)} \int_0^{\rho(t)} \frac{\xi(t, s, x(\gamma(s))}{(\rho(t) - s)^{1-\alpha}} \mathrm{ds}\Big) \tag{5}$$

where $t \in I$, $\alpha \in (0, 1)$ and $\Gamma(\alpha)$ symbolizes the gamma function. By Φ we denote the family of all nondecreasing and continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$. Notice that (5) can be rewritten in the form $x(t) = (Fx)(t).(Vx)(t)$, where F is the so-called superposition operator which is defined by the formula

$$(Fx)(t) = f(t, x(t)),$$

and V is the Volterra integral operator of fractional order

$$(Vx)(t) = q(t) + \frac{1}{\Gamma(\alpha)} \int_0^{\rho(t)} \frac{\xi(t, s, x(\gamma(s)))}{(\rho(t) - s)^{1-\alpha}} ds.$$

Integral and differential equations of fractional order are important for application in many problems in physics, mechanics and other fields (for example in the theory of neutron transport, the theory of radioactive transfer, the kinetic theory of gases [26], traffic theory, etc).
For our purposes, we will need the following Lemma [11]. In what follows denote by X_J the subset of $C(I)$ consisting of all functions $x : I \to J$.

Lemma 3.1. *Assume that J is an arbitrary real interval and $f : I \times J \to \mathbb{R}$ is a given function continuous on the set $I \times J$. Then the superposition operator generated by the function f maps continuously the set X_J into the space $C(I)$. Moreover, if the function $t \to f(t, x)$ is nondecreasing on I for any fixed $x \in J$ and the function $x \to f(t, x)$ is nondecreasing on J for any fixed $t \in I$, then the operator F transforms every nondecreasing function from the set X_J into a function of the same type belonging to $C(I)$.*

We will investigate (5) assuming that the following conditions are satisfied:

(I) $q \in C(I)$ and q is a nondecreasing nonnegative function on the interval I and $\rho : I \to \mathbb{R}_+$ is a nondecreasing continuous function such that $\rho(t) \le L$ for all $t \in I$, where L is a positive constant and $\gamma : [0, L] \to I$ is a continuous function.

(II) The function $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous with $f(I \times \mathbb{R}_+) \subseteq \mathbb{R}_+$. Also, the function $t \to f(t, x)$ is nondecreasing on I for any fixed $x \in \mathbb{R}_+$ and the function $x \to f(t, x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $t \in I$.

(III) There exists a function $\varphi \in \Phi$ such that for any $t \in I$ and for all $x, y \in \mathbb{R}$ we have

$$|f(t, x) - f(t, y)| \le \varphi(|x - y|).$$

Moreover, we presume that φ is superadditive i.e., $\varphi(t) + \varphi(s) \le \varphi(t + s)$ for all $t, s \in \mathbb{R}_+$.

(IV) $\xi : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\xi : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\xi(t, s, x)$ is nondecreasing with respect to each variable t, s and x, separately.

(V) There exists a continuous and nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\xi(t, s, x) \leq \Psi(|x|)$ for $t, s \in I$ and for all $x \in R$.

(VI) There exists a positive solution r_0 of the inequality

$$(\varphi(r) + \overline{F})\Big(\|q\| + \frac{\Psi(r)}{\Gamma(\alpha+1)}L^{\alpha}\Big) \leq r,$$

where $\overline{F} = \max\{f(t, 0), t \in I\}$. Moreover, the number r_0 is such that $(\varphi(r_0) + \overline{F}) + \Big(\|q\| + \frac{\Psi(r_0)}{\Gamma(\alpha+1)}L^{\alpha}\Big) < 1$.

First, we prove the following theorem that we will need in establishing our main result in this section.

Theorem 3.2. *Assume that the hypotheses (II) and (III) are satisfied and $x \in X_J$. Then*

$$d(Fx) \leq \varphi(d(x)).$$

Proof. Let I_e be the subset of $I \times I$ defined as follows

$$I_e = \{(t, s) \in I \times I : t < s \quad \text{and} \quad x(t) = x(s)\}.$$

For $(t, s) \in I_e$, we have

$$\begin{aligned} |(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] &= |f(s, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(t))] \\ &= |f(s, x(t)) - f(t, x(t))| - [f(s, x(t)) - f(t, x(t))] \\ &= 0. \end{aligned}$$

Now, assume that $t, s \in I$, $t < s$ and $(t, s) \notin I_e$, i.e $x(t) \neq x(s)$. Then we have

$$\begin{aligned} &|(Fx)(s) - (Fx)(t)| - [(Fx)(s) - (Fx)(t)] = |f(s, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(t))] \\ &\leq |f(s, x(s)) - f(t, x(s))| + |f(t, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(s))] - [f(t, x(s)) - f(t, x(t))] \\ &= [f(s, x(s)) - f(t, x(s))] + |f(t, x(s)) - f(t, x(t))| - [f(s, x(s)) - f(t, x(s))] - [f(t, x(s)) - f(t, x(t))] \\ &= |f(t, x(s)) - f(t, x(t))| - [f(t, x(s)) - f(t, x(t))]. \end{aligned}$$

If $x(t) \leq x(s)$, the expression $[f(t, x(s)) - f(t, x(t))]$ is nonnegative and hence $|f(t, x(s)) - f(t, x(t))| - [f(t, x(s)) - f(t, x(t))] = 0$. If $x(t) \geq x(s)$, the expression $[f(t, x(s)) - f(t, x(t))]$ is negative and $[f(t, x(s)) - f(t, x(t))] = -|f(t, x(s)) - f(t, x(t))|$. Hence, we have

$$\begin{aligned} |f(t, x(s)) - f(t, x(t))| - [f(t, x(s)) - f(t, x(t))] &= |f(t, x(s)) - f(t, x(t))| + |f(t, x(s)) - f(t, x(t))| \\ &\leq \varphi(|x(s) - x(t)|) + \varphi(|x(s) - x(t)|) \\ &\leq \varphi(|x(s) - x(t)|) + \varphi([x(t) - x(s)]) \\ &\leq \varphi(|x(s) - x(t)| + [x(t) - x(s)]) \\ &= \varphi(|x(s) - x(t)| - [x(s) - x(t)]). \end{aligned}$$

Let us mention that in the above calculations we used the fact that φ is superadditive. As a consequence, we infer

$$d(FX) \leq \varphi(d(X)).$$

□

Theorem 3.3. *Under assumptions $(I) - (VI)$ equation 5 has at least one solution $x(t) = x \in C(I)$ which is nonnegative and nondecreasing on I.*

Proof. First observe that with regard to assumption (II) and by Lemma 3.1, F transforms $C(I)$ into itself and is continuous . We show that the operator V has also the same properties. To do so, let us $\varepsilon > 0$ and take arbitrarily $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may presume that $t_1 < t_2$. Then, for arbitrarily fixed $x \in C(I)$, we have

$$\begin{aligned}
|(Vx)(t_2) - (Vx)(t_1)| &\leq |q(t_2) - q(t_1)| \\
&+ \frac{1}{\Gamma(\alpha)}\left|\int_0^{\rho(t_2)} \frac{\xi(t_2, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds - \int_0^{\rho(t_2)} \frac{\xi(t_1, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left|\int_0^{\rho(t_2)} \frac{\xi(t_1, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds - \int_0^{\rho(t_1)} \frac{\xi(t_1, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left|\int_0^{\rho(t_1)} \frac{\xi(t_1, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds - \int_0^{\rho(t_1)} \frac{\xi(t_1, s, x(\gamma(s)))}{(\rho(t_1) - s)^{1-\alpha}}ds\right| \\
&\leq \omega(q, \varepsilon) + \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t_2)} \frac{\xi(t_2, s, x(\gamma(s))) - \xi(t_1, s, x(\gamma(s)))}{(\rho(t_2) - s)^{1-\alpha}}ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\rho(t_1)}^{\rho(t_2)} \frac{|\xi(t_1, s, x(\gamma(s)))|}{(\rho(t_2) - s)^{1-\alpha}}ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t_1)} |\xi(t_1, s, x(\gamma(s)))|\left|\frac{1}{(\rho(t_2) - s)^{1-\alpha}} - \frac{1}{(\rho(t_1) - s)^{1-\alpha}}\right|ds \\
&\leq \omega(q, \varepsilon) + \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t_2)} \frac{\omega_{\|x\|}(\xi, \varepsilon)}{(\rho(t_2) - s)^{(1-\alpha)}}ds + \frac{1}{\Gamma(\alpha)}\int_{\rho(t_1)}^{\rho(t_2)} \frac{\Psi(\|x\|)}{(\rho(t_2) - s)^{(1-\alpha)}}ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t_1)} \Psi(\|x\|)\left[\frac{1}{(\rho(t_2) - s)^{1-\alpha}} - \frac{1}{(\rho(t_1) - s)^{1-\alpha}}\right]ds \\
&\leq \omega(q, \varepsilon) + \frac{\omega_{\|x\|}(\xi, \varepsilon)}{\alpha\Gamma(\alpha)}[\rho(t_2)]^\alpha + \frac{\Psi(\|x\|)}{\alpha\Gamma(\alpha)}(\rho(t_2) - \rho(t_1))^\alpha \\
&+ \frac{\Psi(\|x\|)}{\alpha\Gamma(\alpha)}[[\rho(t_1)]^\alpha - [\rho(t_2)]^\alpha + (\rho(t_2) - \rho(t_1))^\alpha] \\
&\leq \omega(q, \varepsilon) + \frac{\omega_{\|x\|}(\xi, \varepsilon)}{\Gamma(\alpha + 1)}L^\alpha + \frac{2\Psi(\|x\|)}{\Gamma(\alpha + 1)}[\omega(\rho, \varepsilon)]^\alpha,
\end{aligned}$$

where

$$\omega_d(\xi, \varepsilon) = \sup\{|\xi(t_2, s, y) - \xi(t_1, s, y)| : t_2, t_1 \in I, s \in [0, L], |t_2 - t_1| \leq \varepsilon, y \in [-d, d]\},$$

and

$$\omega(\rho, \varepsilon) = \sup\{|\rho(t_2) - \rho(t_1)|, t_2, t_1 \in I, |t_2 - t_1| \leq \varepsilon\}.$$

Therefore, considering the uniform continuity of the function $\xi(t, s, x)$ on the compact set $I \times I \times [-\|x\|, \|x\|]$ we conclude that the function Vx is continuous on I. Thus, V transforms $C(I)$ into itself. For a fixed $x \in C(I)$ and $t \in I$ we have

$$\begin{aligned}
|(Fx)(t)| &\leq |f(t, x) - f(t, 0)| + |f(t, 0)| \\
&\leq \varphi(|x(t)|) + \overline{F} \\
&\leq \varphi(\|x\|) + \overline{F}.
\end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned}|(Vx)(t)| &\leq |q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t)} \frac{|\xi(t,s,x(\gamma(s)))|}{(t-s)^{1-\alpha}}ds \\ &\leq \|q\| + \frac{\Psi(\|x\|)}{\Gamma(\alpha)}\int_0^{\rho(t)} \frac{ds}{(\rho(t)-s)^{1-\alpha}} \\ &\leq \|q\| + \frac{\Psi(\|x\|)}{\Gamma(\alpha+1)}L^{\alpha}. \end{aligned} \tag{7}$$

By linking (6), (7) and assumption (VI) we can conclude that there exists a positive number r_0 such that the operator $W = F.V$ transforms the ball B_{r_0} into itself. Now, from estimates (6) and (7) and from the fact established above we get

$$\|FB_{r_0}\| \leq \varphi(r_0) + \overline{F}, \tag{8}$$

$$\|VB_{r_0}\| \leq \|q\| + \frac{\Psi(r_0)}{\Gamma(\alpha+1)}L^{\alpha}. \tag{9}$$

In addition, consider the set Q including all nonnegative functions $x \in B_{r_0}$. Then, according to our assumptions the operator W transforms the set Q into itself. Now, from (8) and (9) we get

$$\|FQ\| \leq \varphi(r_0) + \overline{F}, \tag{10}$$

$$\|VQ\| \leq \|q\| + \frac{\Psi(r_0)}{\Gamma(\alpha+1)}L^{\alpha}. \tag{11}$$

In what follows we prove that W is continuous on the set Q. To do this, first note that that the continuity of the operator F is an immediate consequence of assumption (II) and a well known result concerning the continuity of the superposition operator [11]. Next, we prove that V is continuous on the set Q. Thus, let us fix arbitrarily $\varepsilon > 0$ and $x_0 \in Q$. For an arbitrary $x \in Q$ such that $\|x - x_0\| \leq \varepsilon$ and arbitrarily fixed $t \in I$ we have

$$\begin{aligned}|(Vx)(t) - (Vx_0)(t)| &\leq \frac{1}{\Gamma(\alpha)}\Big|\int_0^{\rho(t)} \frac{\xi(t,s,x(\gamma(s)))}{(\rho(t)-s)^{1-\alpha}}ds - \int_0^{\rho(t)} \frac{\xi(t,s,x_0(\gamma(s)))}{(\rho(t)-s)^{1-\alpha}}\Big|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t)} \frac{|\xi(t,s,x(\gamma(s))) - \xi(t,s,x_0(\gamma(s)))|}{(\rho(t)-s)^{1-\alpha}}ds \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{\rho(t)} \frac{\overline{\omega}(\xi,\varepsilon)}{(\rho(t)-s)^{1-\alpha}}ds \\ &\leq \frac{\overline{\omega}(\xi,\varepsilon)}{\Gamma(\alpha+1)}L^{\alpha}, \end{aligned}$$

where

$$\overline{\omega}(\xi,\varepsilon) = \sup\{|\xi(t,s,a) - \xi(t,s,b)| : t \in I, s \in [0,L], a; b \in [0,r_0]; |a-b| \leq \varepsilon\}.$$

With regard to assumption (IV) we have that $\overline{\omega}(\xi,\varepsilon) \to 0$ as $\varepsilon \to 0$. This implies the desired continuity of the operator V on the set Q. Finally, we conclude that W is continuous on the set Q. First, let us fix a nonempty subset X of the set Q. Next, choose a number $\varepsilon > 0$ and take $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may presume that $t_1 < t_2$. Then we get

$$\begin{aligned}|(Fx)(t_2) - (Fx)(t_1)| &\leq |f(t_2,x(t_2)) - f(t_2,x(t_1))| + |f(t_2,x(t_1)) - f(t_1,x(t_1))| \\ &\leq \varphi(|x(t_2) - x(t_1)|) + \omega_{r_0}(f,\varepsilon) \\ &\leq \varphi(\omega(x,\varepsilon)) + \omega_{r_0}(f,\varepsilon), \end{aligned}$$

where we denoted

$$\omega_{r_0}(f,\varepsilon)=\sup\{|f(t_2,x)-f(t_1,x)|:t_1,t_2\in I,|t_2-t_1|\le\varepsilon,x\in[-r_0,r_0]\}.$$

As a consequence, we infer $\omega(Fx,\varepsilon)\le\varphi(\omega(x,\varepsilon))+\omega_{r_0}(f,\varepsilon)$ and we have

$$\omega_0(FX)\le\varphi(\omega_0(X)). \tag{12}$$

Moreover, we obtain

$$\begin{aligned}|(Vx)(t_2)-(Vx)(t_1)|&\le\omega(q,\varepsilon)+\frac{\omega_{\psi(r_0)}(\xi,\varepsilon)}{\alpha\Gamma(\alpha)}[\rho(t_2)]^\alpha+\frac{\Psi(\|x\|)}{\alpha\Gamma(\alpha)}(\rho(t_2)-\rho(t_1))^\alpha\\&+\frac{\Psi(\|x\|)}{\alpha\Gamma(\alpha)}[(\rho(t_1)^\alpha-\rho(t_2)^\alpha+(\rho(t_2)-\rho(t_1))^\alpha]\\&\le\omega(q,\varepsilon)+\frac{\omega_{r_0}(\xi,\varepsilon)}{\Gamma(\alpha+1)}L^\alpha+\frac{\Psi(r_0)}{\Gamma(\alpha+1)}[\omega(\rho,\varepsilon)]^\alpha+\frac{\Psi(r_0)}{\Gamma(\alpha+1)}[\omega(\rho,\varepsilon)]^\alpha\\&=\omega(q,\varepsilon)+\frac{1}{\Gamma(\alpha+1)}[\omega_{r_0}(\xi,\varepsilon)L^\alpha+2\Psi(r_0)[\omega(\rho,\varepsilon)]^\alpha].\end{aligned}$$

Hence, $\omega(VX,\varepsilon)\le\omega(q,\varepsilon)+\frac{1}{\Gamma(\alpha+1)}[\omega_{\psi(r_0)}(\xi,\varepsilon)L^\alpha+2\Psi(r_0)[\omega(\rho,\varepsilon)]^\alpha]$ and consequently

$$\omega_0(VX)=0. \tag{13}$$

Now assume that $t_1,t_2\in I$ and $t_1<t_2$. Then, taking an arbitrary function $x\in X$ we have

$$\begin{aligned}[|(Vx)(t_2)-(Vx)(t_1)|-[(Vx)(t_2)-(Vx)(t_1)]&\le|q(t_2)-q(t_1)|-[q(t_2)-q(t_1)]\\&+\frac{1}{\Gamma(\alpha)}\Big|\int_0^{\rho(t_2)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_2)-s)^{1-\alpha}}-\int_0^{\rho(t_1)}\frac{\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}\Big|ds\\&-\frac{1}{\Gamma(\alpha)}\Big[\int_0^{\rho(t_2)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_2)-s)^{1-\alpha}}-\int_0^{\rho(t_1)}\frac{\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}\Big]ds.\end{aligned} \tag{14}$$

By considering our assumptions, we have

$$\begin{aligned}&\int_0^{\rho(t_2)}\frac{\xi(t_2,s,x)(\gamma(s)))}{(\rho(t_2)-s)^{1-\alpha}}ds-\int_0^{\rho(t_1)}\frac{\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds\\&=\int_0^{\rho(t_1)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_2)-s)^{1-\alpha}}ds+\int_{\rho(t_1)}^{\rho(t_2)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_2)-s)^{1-\alpha}}ds\\&-\int_0^{\rho(t_1)}\frac{\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds+\int_0^{\rho(t_1)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds-\int_0^{\rho(t_1)}\frac{\xi(t_2,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds\\&\ge p\Big\{\int_0^{\rho(t_1)}\Big(\frac{1}{(\rho(t_2)-s)^{1-\alpha}}-\frac{1}{(\rho(t_1)-s)^{1-\alpha}}\Big)ds+\int_{\rho(t_1)}^{\rho(t_2)}\frac{1}{(\rho(t_2)-s)^{1-\alpha}}ds\Big\}\\&+\int_0^{\rho(t_1)}\frac{\xi(t_2,s,x(\gamma(s)))-\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds\\&\ge p\frac{[\rho(t_2)]^\alpha-[\rho(t_1)]^\alpha}{\alpha}+\int_0^{\rho(t_1)}\frac{\xi(t_2,s,x(\gamma(s)))-\xi(t_1,s,x(\gamma(s)))}{(\rho(t_1)-s)^{1-\alpha}}ds,\end{aligned} \tag{15}$$

where

$$p=\min\{\xi(t,s,x):t\in I,s\in[0,L],x\in[-r_0,r_0]\}.$$

Since the function $t \to \xi(t, s, x)$ is nondecreasing on I, (15) implies that

$$(Vx)(t_2) - (Vx)(t_1) \geq 0.$$

The above inequality linking with (14) allows us to deduce that $d(Vx) = 0$. As a result

$$d(VX) = 0. \tag{16}$$

Now, from (12), (13), (16), Theorem 3.2, assumption (III) and the definition of μ_d , we get

$$\mu_d(FX) \leq \varphi(\mu_d(X)), \quad \mu_d(VX) = 0. \tag{17}$$

By linking (10), (11), (17) and assumption (VI) and in view of Corollary 2.26 we conclude that the operator W has a fixed point x in the set Q. Notice that with regard to Remark 2.23 and Remark 2.25 the function $x = x(t)$ is a nonnegative and nondecreasing solution of the functional integral equation (5). □

Now, we present an example to illustrate our theory.

Example 3.4. *Consider the following functional integral equation*

$$x(t) = [\frac{t^2}{1+t^4} \ln(1 + \frac{1}{10}|x(t)|)] \times [t^2 e^{-2t} + \frac{1}{\Gamma(\frac{1}{2})} \int_0^{t^2} \frac{s + 3\sqrt{x(\sqrt{s})}}{3(t^2 - s)^{\frac{1}{2}}} ds], \tag{18}$$

where $t \in I = [0, 1]$. Notice that this equation is a particular case of equation (5) with

$$q(t) = t^2 e^{-2t}, \quad f(t, x) = \frac{t^2}{1+t^4} \ln(1 + \frac{1}{10}|x(t)|), \quad \xi(t, s, x) = \frac{s}{3} + \sqrt{x}, \quad \gamma(s) = \sqrt{s}.$$

In addition, $\alpha = \dfrac{1}{2}$ and $\rho(t) = t^2$. It is easy to see that equation (18) satisfies all the hypotheses needed in Theorem 5. Indeed, we have $\varphi(r) = \ln(1 + \dfrac{1}{10} r)$. Moreover, we have

$$|\xi(t, s, x)| \leq \frac{1}{3} + \sqrt{x}$$

and

$$\begin{aligned} |f(t, x) - f(t, y)| &= \frac{t^2}{1+t^4} \Big| \ln\Big(1 + \frac{1}{10}|x|\Big) - \ln\Big(1 + \frac{1}{10}|y|\Big) \\ &\leq \ln \frac{1 + \frac{1}{10}|x|}{1 + \frac{1}{10}|y|} \\ &\leq \ln\Big(1 + \frac{1}{10} . \frac{|x| - |y|}{1 + \frac{1}{10}|y|}\Big) \\ &< \ln\Big(1 + \frac{1}{10}|x - y|\Big) \\ &= \varphi(|x - y|). \end{aligned}$$

Therefore, we observe that the function $\Psi(r)$ appearing in assumption (V) may be written in the form $\Psi(r) = \dfrac{1}{3} + \sqrt{r}$. Furthermore, we have $L = 1$. In addition, we have that $\|q\| = \dfrac{1}{2e}$ and $L = 1$. So the inequality appearing in assumption (VI) takes the form

$$\ln(1 + \frac{r}{10})(\frac{1}{2e} + \frac{1 + 3\sqrt{r}}{3\Gamma(\frac{3}{2})}) \leq r.$$

Note $\Gamma(\frac{3}{2}) = 0.8856...$ *so obviously this inequality has a positive solution* r_0*. For example,* $r_0 = \dfrac{5}{100}$*. Moreover, we have that*

$$\ln(1+\frac{r_0}{10}) + \frac{1}{2e} + \frac{1+3\sqrt{r_0}}{3\Gamma(\frac{3}{2})} < 1.$$

Consequently, all the conditions of Theorem 3.3 are satisfied. Hence, Theorem 3.3 guarantees that equation 18 has a nondecreasing solution in the space $C(I)$*.*

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