



Meshless Approximation Method of One-Dimensional Oscillatory Fredholm Integral Equations

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Abstract. In this findings, a numerical meshless solution algorithm for 1D oscillatory Fredholm integral equation (OFIE) is put forward. The proposed algorithm is based on Levin's quadrature theory (LQT) incorporating multi-quadric radial basis function (MQ-RBF). The procedure involves local approach of MQ-RBF differentiation matrix. The proposed method is specially designed to handle the case when the kernel function (KF) involves stationary point(s) (SP(s)). In addition to that, the model without SP(s) is also considered. The main advantage of the meshless procedure is that it can be easily extended to multi-dimensional geometry. These models have several physical applications in the area of engineering and sciences. The existence of the SP(s) in such models has numerous applications in the field of scattering and acoustics etc. (see [1, 2, 4, 6–8]). The proposed meshless method is accurate and cost-effective and provides a trustworthy platform to solve OFIE(s).

1. Introduction

The Fredholm integral equations (FIE(s)) have increased acceptance in the field of engineering and science. KF of FIE(s) might be oscillatory and non-oscillatory. A boundary value problem having oscillator can be converted to its equivalent FIE having an oscillatory kernel and vice versa. However, the oscillation phenomenon is widely used in the fields of electromagnetic and quantum mechanics [9–12]. Taking this into account, A 1D FIE having oscillatory kernel model of the second kind can be articulated as:

$$s(x) = f(x) + \int_a^b h(x, y)e^{t\Omega O(x,y)}s(y)dy, \quad x \in [a, b], \quad (1)$$

where f, h, O and s are smooth functions ($f, h, O, s \in C^\infty[a, b]$). O is an oscillator function and Ω is frequency of the oscillator. As Ω gets higher and higher value, the model given in (1) become highly OFIE.

The oscillatory integral of the FIE is given by:

$$I = \int_a^b h(x, y)e^{t\Omega O(x,y)}s(y)dy, \quad (2)$$

2010 *Mathematics Subject Classification.* 45B05

Keywords. Levin's Quadrature Theory, Local RBF, MQ-RBF Differentiatin Matrix, Stationary Point.

Received: 26 September 2017; Revised: 24 February 2018; Accepted: 25 September 2018

Communicated by Marko Petković

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where (2) may or may not have SP(s). If $\frac{dO}{dy} = 0$, at any point $\zeta \in [a, b]$, then (2) is said to have a SP at $x = \zeta$, otherwise (2) is called integral without SP(s). Consequently, (1) is called FIE with or without SP(s).

Several methods are reported in the literature for numerical solution of FIE(s) [1, 2, 4, 6, 8, 9, 12–16]. Some numerical techniques have been developed for either oscillatory integrals or non-oscillatory integral equations [7, 8, 14, 18, 20] rather than oscillatory integral equations, where the quadrature methods fails to deliver. The main challenge of (2) is how to evaluate it accurately and efficiently especially if the KF is highly oscillatory while incorporating SP(s). The literature is adequate and very few methods have been reported in the past in this scenario, (see [1, 2, 4, 6, 9, 19, 20, 23]).

The literature details of the evaluation methods for the models in (1) and (2) when the KF is with and free of SP(s) are given in our earlier findings [1, 2, 4] and the references therein. Huybrechs [8] has suggested method of steepest descent for solution of the oscillatory integrals. In his technique, the condition of analyticity for the integrand in the complex plane makes the method more complicated. Ref. [19] discussed the numerical solution of the integral equation. In Ref. [7] a quadrature method is proposed which is based on the contained integration around the SP(s), but the authors used a traditional quadrature procedure but it is less efficient. The author of the paper [6] has proposed modified Levin's quadrature technique for the numerical solution of the oscillatory integral equations given in (1) on Chebyshev-type of nodal points. This approach could be difficult to implement when the nodal points distribution is either non-uniform or uniform while the geometry considered is irregular. Recently, we have suggested accurate numerical methods for numerical solution of oscillatory FIE(s) which incorporate very few nodal points and hence efficient as well [1–5].

In RBFs meshless methods, the structure of the domain may be determined by the application and not the numerical method. RBFs can be applied both globally and locally. The global meshless method is constructed on undertaking the whole domain into account, has some drawbacks, like dense ill-conditioning and sensitivity of shape parameter. In order to avert the bad effects of the global method, a local meshless method has been suggested by the researchers [1, 2, 4, 17, 21, 22]. Local meshless methods are comparatively slightly insensitive to the variation of the shape parameter than the global counterparts. The local method can obtain a very similar result to the global method while only using a small subset of available points, and hence it is cost-effective. In the last few years, hidden potentials of different types of local meshless methods have been explored in a variety of applications.

In the present work, we have extended and improved the work in our earlier findings [3, 4] to the proposed new algorithm to solve OFIE(s). The new algorithm is kernel based meshless approximation. This is part and parcel of local MQ-RBF differential matrix [1, 2, 4] (a meshless differential matrix technique) and LQT [1, 2, 4, 19, 20]. The challenging aspect of this new algorithm is that it can accurately handle the case when the KF involve SP(s). Furthermore, our proposed method works efficiently for both non-uniform and uniform collocation points.

2. Proposed numerical procedure

The proposed method is based on meshless collocation technique for solving oscillatory-integrals and -integral equations. The proposed meshless approach can be easily extended to multidimensional applications. There are two approaches namely global and local RBF's collocation, as discussed in details in [1, 2, 4].

In the local RBF collocation method, the m^{th} derivatives of a general function $p(y)$ are approximated [1, 2] at the centre y_j of the j th stencil of size n_j , enclosed by a set of neighbouring nodes y_j , for $j = 0, 1, \dots, N$ and $\{y_{j_1}, y_{j_2}, \dots, y_{j_{n_j}}\} \subset \{y_0, y_1, \dots, y_N\}$, $n_j < N$ as:

$$p^{(m)}(y_j) \approx \sum_{i=1}^{n_j} \omega_i^{(m)} p(y_{j_i}), \quad j = 0, 1, \dots, N. \quad (3)$$

To get $\omega_i^{(m)}$, $i = 1, \dots, n_j$, we use MQ-RBF which is defined as:

$$\Phi(\|y - y_k\|, \epsilon) = \Phi(\sqrt{1 + \epsilon^2(y - y_k)^2}), k = j_1, j_2, \dots, j_{n_j}, \quad (4)$$

where ϵ is the shape parameter. ϵ affects both the conditioning and accuracy of the MQ-RBF method for a fixed number of centers. Substituting (4) into (3) for p at fixed center y_j as given below:

$$\Phi^{(m)}(\|y_j - y_k\|, \epsilon) = \sum_{i=1}^{n_j} \omega_{j_i}^{(m)} \Phi(\|y_{j_i} - y_k\|, \epsilon), \quad k = j_1, j_2, \dots, j_{n_j}. \tag{5}$$

(5) can be written in its matrix equivalent form as:

$$\begin{bmatrix} \Phi_{j_1}^{(m)}(y_j) \\ \Phi_{j_2}^{(m)}(y_j) \\ \vdots \\ \Phi_{j_{n_j}}^{(m)}(y_j) \end{bmatrix} = \begin{bmatrix} \Phi_{j_1}(y_{j_1}) & \Phi_{j_2}(y_{j_1}) & \cdots & \Phi_{j_{n_j}}(y_{j_1}) \\ \Phi_{j_1}(y_{j_2}) & \Phi_{j_2}(y_{j_2}) & \cdots & \Phi_{j_{n_j}}(y_{j_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{j_1}(y_{j_{n_j}}) & \Phi_{j_2}(y_{j_{n_j}}) & \cdots & \Phi_{j_{n_j}}(y_{j_{n_j}}) \end{bmatrix} \begin{bmatrix} \omega_{j_1}^{(m)} \\ \omega_{j_2}^{(m)} \\ \vdots \\ \omega_{j_{n_j}}^{(m)} \end{bmatrix}, \tag{6}$$

where

$$\Phi_k(y_l) = \Phi(\sqrt{1 + \epsilon^2(y_l - y_k)^2}), \quad k = j_1, j_2, \dots, j_{n_j},$$

for each $l = j_1, j_2, \dots, j_{n_j}$.

Equation (6) can be written in matrix form as:

$$\Phi_{n_j}^{(m)} = A_{n_j} \omega_{n_j}^{(m)}, \tag{7}$$

where

$$\begin{aligned} \Phi_{n_j}^{(m)} &= \left[\Phi_{j_1}^{(m)}(y_j) \quad \Phi_{j_2}^{(m)}(y_j) \quad \cdots \quad \Phi_{j_{n_j}}^{(m)}(y_j) \right]^T, \\ A_{n_j} &= \begin{bmatrix} \Phi_{j_1}(y_{j_1}) & \Phi_{j_2}(y_{j_1}) & \cdots & \Phi_{j_{n_j}}(y_{j_1}) \\ \Phi_{j_1}(y_{j_2}) & \Phi_{j_2}(y_{j_2}) & \cdots & \Phi_{j_{n_j}}(y_{j_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{j_1}(y_{j_{n_j}}) & \Phi_{j_2}(y_{j_{n_j}}) & \cdots & \Phi_{j_{n_j}}(y_{j_{n_j}}) \end{bmatrix}, \end{aligned} \tag{8}$$

and

$$\omega_{n_j}^{(m)} = \left[\omega_{j_1}^{(m)} \quad \omega_{j_2}^{(m)} \quad \cdots \quad \omega_{j_{n_j}}^{(m)} \right]^T.$$

From (7) we get,

$$\omega_{n_j}^{(m)} = (A_{n_j})^{-1} \Phi_{n_j}^{(m)}. \tag{9}$$

Substituting the values of the weight coefficients $\omega_{n_j}^{(m)}$ from (9) into (3), we have

$$p^{(m)}(y_j) = (\omega_{n_j}^{(m)})^T \mathbf{p}_{n_j} = \mathbf{D}_j^{(m)} \mathbf{p}_{n_j}, \tag{10}$$

where

$$\mathbf{p}_{n_j} = \left[p(y_{j_1}) \quad p(y_{j_2}) \quad \cdots \quad p(y_{j_{n_j}}) \right]^T, \quad j = 0, 1, \dots, N,$$

and $\mathbf{D}_j^{(m)}$ are weights w.r.t the local stencil of size n_j and are shown in Fig. 1).

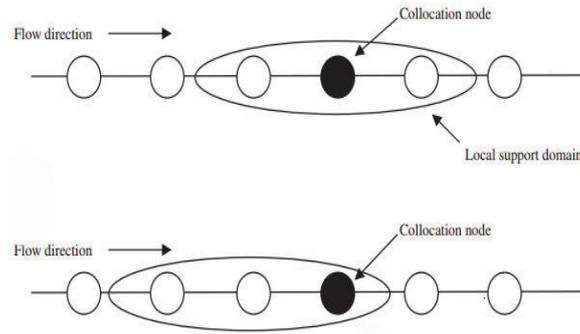


Figure 1: [21] Local support domain's flow diagram (Stencil size 3).

Consequently, we obtained a local RBF m^{th} differential matrix $\mathbf{D}^{(m)}$ [1] of the form:

$$\mathbf{D}_{(N+1) \times (N+1)}^{(m)} = \begin{bmatrix} \mathbf{D}_0^{(m)} \\ \mathbf{D}_1^{(m)} \\ \vdots \\ \mathbf{D}_N^{(m)} \end{bmatrix}. \tag{11}$$

The differential matrix proposed in (11) is having $N + 1 - N_i$ zero entries. This sparse behavior of the local RBF differential matrix procedure makes it efficient. Since in our procedure only first order derivative is involved, therefore replacing m by 1 in above (11) yields:

$$\mathbf{D}^{(1)} = \begin{bmatrix} \mathbf{D}_0^{(1)} \\ \mathbf{D}_1^{(1)} \\ \vdots \\ \mathbf{D}_N^{(1)} \end{bmatrix}. \tag{12}$$

Incorporating the above differential matrix in the proposed method, we get what we call "local meshless differential quadrature (LMDQ)" collocation technique. The LMDQ is based on LQT [20].

Discretizing (1) on $x_k, k = 0, 1, \dots, M$ we have

$$s(x_k) = f(x_k) + \int_a^b h(x_k, y) e^{\iota \Omega O(x_k, y)} s(y) dy, \tag{13}$$

where (13) has oscillatory integrals of the form

$$I_k = \int_a^b h(x_k, y) e^{\iota \Omega O(x_k, y)} s(y) dy. \tag{14}$$

Equation (14) with KF free-of-SP(s) can be evaluated using modified Levin's quadrature and MQ-RBF [1, 2, 4].

2.1. Free-of-SP(s) procedure

We calculate the integrals (14) that are free-of-SP(s). Accordingly, the integral (14) can be transformed to [20]:

$$\frac{dp(y)}{dy} + \iota \Omega \frac{dO(x_k, y)}{dy} p(y) = h(x_k, y) s(y). \tag{15}$$

Putting the value of $h(x_k, y)s(y)$ from (15) into (14), we get:

$$I_j = \int_a^b \left(\frac{dp(y)}{dy} + \iota \Omega \frac{dO(x_k, y)}{dy} p(y) \right) e^{\iota \Omega O(x_k, y)} dy. \tag{16}$$

Equation (16) can be equivalently written as:

$$I_j = \int_a^b \frac{d}{dy} \left(p(y) e^{\iota \Omega O(x_k, y)} \right) dy = p(b) e^{\iota \Omega O(x_k, b)} - p(a) e^{\iota \Omega O(x_k, a)}. \tag{17}$$

For numerical solution of ODE (15); replacing the derivative term by a local RBF differentiation matrix, $\mathbf{D}^{(1)}$ in (12) into (15) yields:

$$\mathbf{D}^{(1)} p(y) + \iota \Omega O'(x_k, y) p(y) = h(x_k, y) s(y).$$

Thus for $k = 0, 1, \dots, M$, we obtain

$$\left(\mathbf{D}^{(1)} + \iota \Omega \sum_k \right) \mathbf{P}_k = \text{diag}(\mathbf{H}_k) \boldsymbol{\psi}, \tag{18}$$

where

$$\begin{aligned} \mathbf{P}_k &= [P(y_0), P(y_1), \dots, P(y_N)]^T; \\ \boldsymbol{\psi} &= [s(y_0), s(y_1), \dots, s(y_N)]^T; \\ \mathbf{H}_k &= [h(x_k, y_0), h(x_k, y_1), \dots, h(x_k, y_N)]^T; \end{aligned}$$

and \sum_k is a diagonal matrix with diagonal entries:

$\{O'(x_k, y_0), O'(x_k, y_1), \dots, O'(x_k, y_N)\}$ and; $\text{diag}(\mathbf{H}_k) \boldsymbol{\psi} = \mathbf{H}_k \otimes \boldsymbol{\psi}$ is a special matrix; \otimes denotes Hadamard product having entries of the form: $h(x_k, y_j) s(y_j)$, $j = 0, 1, 2, \dots, N$. One can obtain the numerical solution \mathbf{P}_k of ODE(s) 15 as

$$\mathbf{P}_k = \left(\mathbf{D}^{(1)} + \iota \Omega \sum_k \right)^{-1} \text{diag}(\mathbf{H}_k) \boldsymbol{\psi}. \tag{19}$$

As from (17), it is clear that we need only the first and the last entry of a vector having zero entries in between, therefore we need a vector of the form:

$$\mathbf{Q}_k = [e^{\iota \Omega O(x_k, b)}, 0, \dots, 0, -e^{\iota \Omega O(x_k, a)}]. \tag{20}$$

Thus one can write (17) using (19) and (20) as

$$\begin{aligned} I_k &= \mathbf{Q}_k \mathbf{P}_k, \\ I_k &= \mathbf{Q}_k \left(\mathbf{D}^{(1)} + \iota \Omega \sum_k \right)^{-1} \text{diag}(\mathbf{H}_k) \boldsymbol{\psi}. \end{aligned} \tag{21}$$

Let

$$\mathbf{V}_k = \mathbf{Q}_k \left(\mathbf{D}^{(1)} + \iota \Omega \sum_k \right)^{-1} \text{diag}(\mathbf{H}_k), \tag{22}$$

where \mathbf{V}_k are vectors of order $1 \times (N + 1)$, then

$$I_k = \mathbf{V}_k \boldsymbol{\psi}, \tag{23}$$

where the order of $\boldsymbol{\psi}$ is $(N + 1) \times 1$. By virtue of (23), one can write (13) as

$$s(x_k) = f(x_k) + I_k, \tag{24}$$

or

$$s(x_k) = f(x_k) + V_k\psi. \tag{25}$$

To get ψ , a square mesh is taken in both x and y direction i.e $x_k = y_j$. Thus $M = N$ will result in making ψ to be as:

$$\psi = [s(x_0), s(x_1), \dots, s(x_N)]^T.$$

One can get distinct linear equations from (25) for different j ; which leads to the following linear system:

$$\psi = F + V\psi; \tag{26}$$

where the order of F is $(N + 1) \times 1$ and the order of V is $(N + 1) \times (N + 1)$, having the form:

$$V = \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_N \end{bmatrix}, \quad F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}.$$

We have used SVD to avoid menace of ill-conditioning of (26). Finally, the unknown ψ is achieved from (26) as

$$\psi = (I - V)^{-1}F, \tag{27}$$

which is the required discrete solution of the oscillatory FIE.

2.2. Fixed SP(s) procedure

We consider here the case when (14) is having a KF with SP(s). Let $\zeta \in [a, b]$ be a fixed SP. Choose a suitable constant $\eta > 0$ such that (14) is subdivided into the following three integrals (see [1, 2, 4] for details) as:

$$I_k = \int_a^{\zeta-\eta} h(x_k, y)s(y)e^{i\Omega O(x_k, y)}s(y)dy + \int_{\zeta-\eta}^{\zeta+\eta} h(x_k, y)s(y)e^{i\Omega O(x_k, y)}s(y)dy + \int_{\zeta+\eta}^b h(x_k, y)s(y)e^{i\Omega O(x_k, y)}s(y)dy = I_k^1 + I_k^2 + I_k^3, \quad k = 0, 1, \dots, M. \tag{28}$$

To evaluate (28), one has to establish connectivity between local and global nodes through barycentric interpolation [1, 2, 4, 24]. The integrals I_k^1, I_k^2 and I_k^3 can now be evaluated through modified Levin's quadrature incorporating MQ-RBF differential matrix [1, 2, 4], such that (28) becomes:

$$I_k = \left[\sum_{q=1}^3 \mathbf{Q}_k^q (\mathbf{D}_q + i\Omega\sigma_k^q)^{-1} \text{diag}(\mathbf{H}_k^q)\mathbf{M}_q^t \right] \psi, \tag{29}$$

where for each $I_k^q, q = 1, 2, 3$ in their respective sub-domain $[a^q, b^q]$ we have;

\mathbf{D}_q is the MQ-RBF differential matrix;

\mathbf{Q}_k^q is a vector of the form $\mathbf{Q}_k^q = [e^{i\Omega O(x_k, b^q)}, 0, \dots, 0, -e^{i\Omega O(x_k, a^q)}]$;

σ_j^q is a diagonal matrix with diagonal entries $\{o'(x_k, y_0), o'(x_k, y_1), \dots, o'(x_k, y_N)\}$;

\mathbf{M}_q^l is the local interpolation matrix [1];

\mathbf{H}_j^q, ψ are vectors of the form $\mathbf{H}_k^q = [f(x_k, y_0), f(x_k, y_1), \dots, y_N]^T$;

and $\psi = [s(y_0), s(y_1), \dots, s(y_N)]^T$. Equation (29) in simplified form can be written as:

$$I_k = \mathbf{U}_k \psi, \tag{30}$$

where

$$\mathbf{U}_k = \sum_{q=1}^3 \mathbf{Q}_k^q (\mathbf{D}_q + i\Omega \sigma_k^q)^{-1} \text{diag}(\mathbf{H}_k^q) \mathbf{M}_q^l,$$

Upon substitution of (29) into (13) we get:

$$s(x_k) = f(x_k) + I_k. \tag{31}$$

The rest of the procedure is similar to the case when KF is without SP. Following the steps from (24) and onward solution to the oscillatory integral equation, having KF with a SP is obtained.

3. Numerical Experiments

In this section, we test the method on benchmark problems given in [1, 6]. The models given here are examples of the OFIE having phase function with and free of SP(s). To check the accuracy, relative error norm has been checked both against varying nodal points and frequencies. The relative error norm is defined as follows:

$$\epsilon_r = \frac{|\psi(t_k) - \tilde{\psi}(t_k)|}{|\psi(t_k)|}, \quad k = 0, 1, \dots, N,$$

where ψ and $\tilde{\psi}$ represents the exact and approximate solution respectively. In the proposed method LMDQ, the stencil size is taken 3 throughout the calculation. All the computations in this section are performed using MATLAB R2015a on Haier Core i3 having 256 GB SSD drive with 8.0 GB RAM.

Problem 3.1. Consider the following OFIE [1, 6]:

$$\psi(t) = h(t) + \int_{-1}^1 f(t, z) \psi(z) e^{i\Omega O(t, z)} dz,$$

with

$$O(t, z) = \frac{t^2}{20} + \left(z + \frac{6}{5}\right)^2,$$

$$h(t) = \cos(10t) - \left(-e^{\frac{1}{20}i\Omega t^2 + \frac{1}{25}i\Omega} + e^{\frac{1}{20}i\Omega t^2 + \frac{121}{25}i\Omega}\right) e^{-t^2},$$

and

$$f(t, z) = \frac{i\Omega \left(2z + \frac{12}{5}\right)}{\cos(10z)} e^{-t^2}.$$

The exact solution of the problem is $\psi(t) = \cos(10t)$. This test problem is having kernel with phase function free-of-SP(s). Figures 2 and 3 show accuracy in terms of relative error norm ϵ_r . In Fig. 2, $N = 11$ is fixed while Ω is increasing mildly. Figure 2 shows that the error ϵ_r is around 10^{-6} for mild oscillation of Ω i.e. between 100 – 600. The error ϵ_r is decreasing for increasing nodal points N while keeping $\Omega = 10^6$ fixed as witnessed by 3 (left). In Fig. 3 (right), we see that accuracy is increasing for fixed number of nodes $N = 20$ and higher value of Ω . From this example, it is clear that accuracy of the proposed procedure LMDQ is good while incorporating very few nodes.

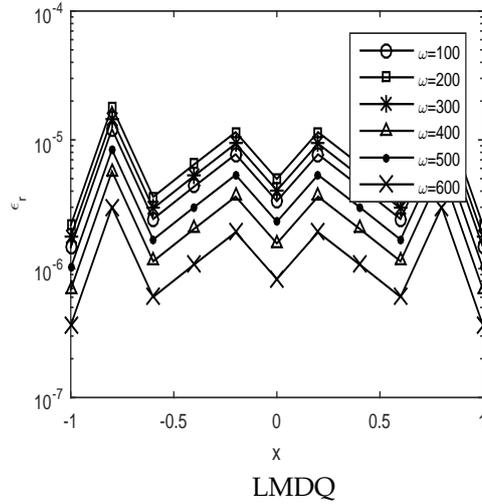


Figure 2: Problem 3.1, Error ε_r for mild oscillation of Ω between 100 – 600 and $N = 11$ fixed.

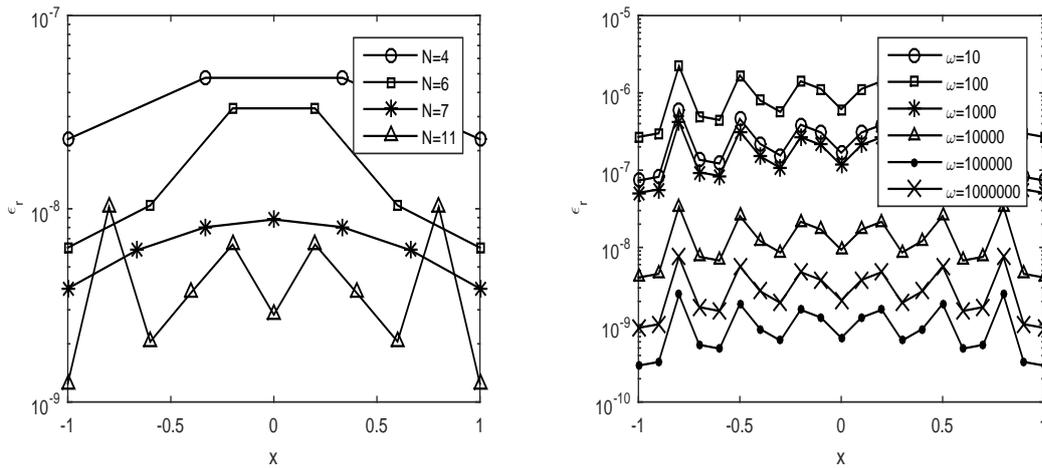


Figure 3: Problem 3.1, Error ε_r , (left) for N varies between 4 – 11 and $\Omega = 10^6$ fixed and (right) for Ω varies between 10 – 10^6 and $N = 21$ fixed.

Problem 3.2. Consider [1, 6]:

$$s(t) = f(t) + \int_{-1}^1 h(t, u) e^{i\Omega\left(\frac{t^2}{20} + (u - \frac{3}{10})\right)} s(u) du,$$

where

$$f(t) = 1 + \frac{\sin(5t)}{t^2 + \frac{1}{5}} - e^{-2t^3 - 6} \left\{ e^{i\Omega\left(\frac{t^2}{20} + \frac{49}{100}\right)} - e^{i\Omega\left(\frac{t^2}{20} + \frac{169}{100}\right)} \right\},$$

and

$$h(t, u) = \frac{-12u + i\Omega\left(2u - \frac{3}{5}\right)}{1 + \frac{\sin(5u)}{u^2 + \frac{1}{5}}} e^{-2t^3 - 6u^2}.$$

The exact solution of the problem is $s(t) = 1 + \frac{\sin(5t)}{t^2 + \frac{1}{5}}$. In this second example, the KF of the model is having a SP at $\zeta = 0.3$. Results are shown in Figs. 4, 5 and 6. A new domain partition methodology is proposed to solve such type of models. We have observed from Fig. 4 (right) that the LMDQ is more accurate compared to Ref.[6] in Fig. 4 (left) as accuracy of LMDQ is in the range of 10^{-3} to 10^{-5} for mild oscillation of Ω . In Fig. 5 Ω is fixed to be 10^6 ; while in Fig. 6 nodal points are fixed to be 21. We have witnessed that the proposed numerical method is accurate both against increasing nodal points and frequencies as observed from Figs. 5 and 6. The proposed method LMDQ is also efficient as it incorporate very few nodal points to get accurate results as compared to [6] in presence of SP. Hence the proposed algorithm provide a reliable source for solution of the model in hand.

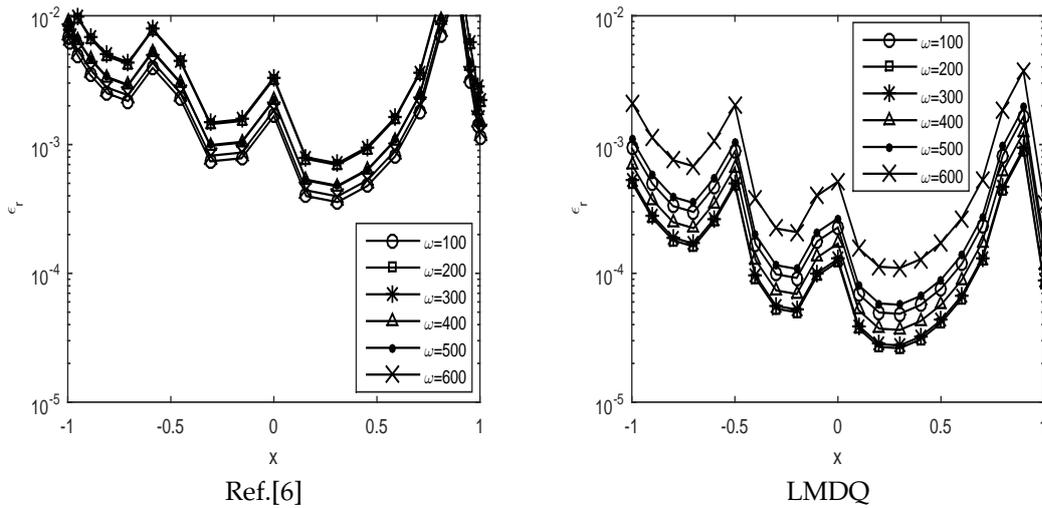


Figure 4: Problem 3.2, ϵ_r (left to right) for mild oscillation of Ω with LMDQ having $\epsilon = 1e - 2$ while $N = 21$ fixed.

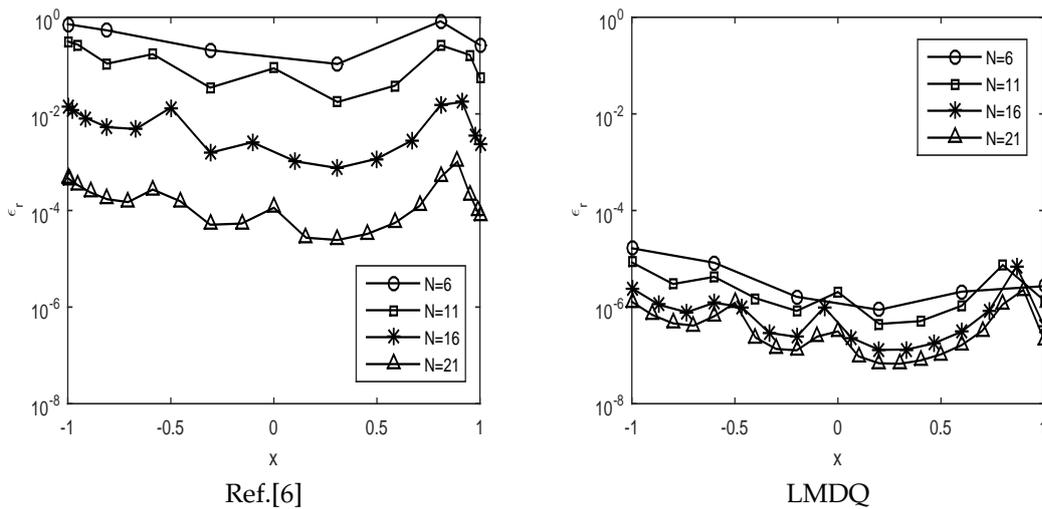


Figure 5: Problem 3.2, error ϵ_r (left to right) for N varies between 6 – 21 and $\Omega = 10^6$ fixed.

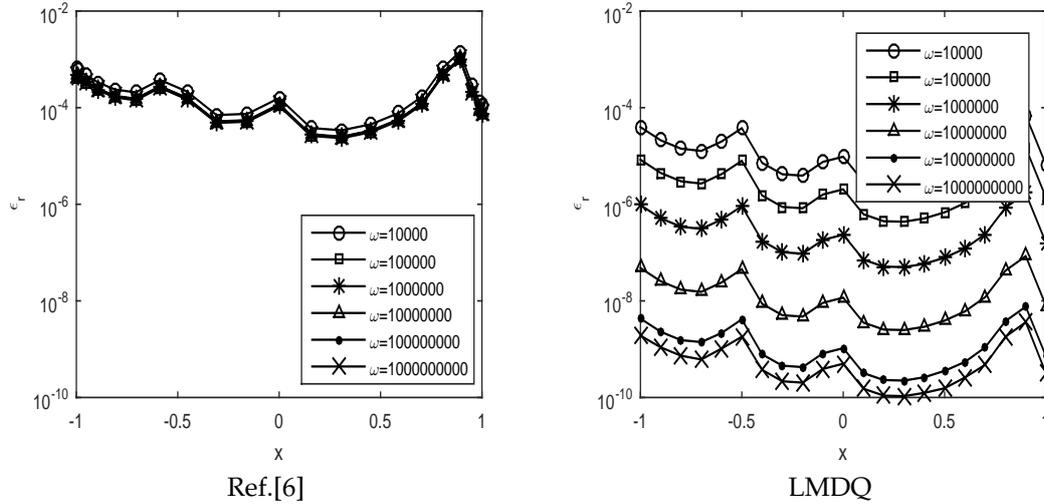


Figure 6: Problem 3.2, error ϵ_r (left to right) for Ω varies between $10^4 - 10^9$ and $N = 21$ fixed.

Problem 3.3. Consider the following OFIE:

$$\psi(t) = h(t) + \int_{-1}^1 f(t, z)\psi(z)e^{i\Omega g(t,z)} dz,$$

with

$$g(t, z) = \frac{t^4}{15} + \left(z + \frac{6}{5}\right)^2,$$

$$h(t) = \tan^2(\cos(5\sqrt{t+1})) + \frac{1}{\Omega} e^{\frac{1}{15}i\Omega t^4} \{e^{\frac{121}{25}i\Omega} - e^{\frac{1}{25}i\Omega}\},$$

and

$$f(t, z) = \frac{\left(2z + \frac{12}{5}\right)}{\tan^2(\cos(5\sqrt{t+1}))}.$$

The exact solution of the problem is $\psi(t) = \tan^2(\cos(5\sqrt{t+1}))$. This problem is for the oscillatory FIE having a KF without SP. In this example, we have also observed that the performance of the proposed method LMDQ with $\epsilon = 1e - 2$ is good for a small range of oscillations $\Omega = 5, 10, 100, 1000$ while incorporating uniform nodes as shown in Fig. 7. Figure 8 shows accuracy in terms of relative error norm ϵ_r for fixed $N = 10$ while Ω mildly vary between $100 - 600$ on non-uniform nodal points. In this figure, we have scattered and randomized nodal points within the domain and found that the accuracy is more or less the same for mild oscillations of frequencies. In Fig. 9, we have checked the performance of the proposed procedure LMDQ for a fixed number of small nodes $N = 10$ both for mild and high oscillations of frequencies on three types of nodal points discretization: Chebyshev, non-uniform, and uniform. It is clear from this Fig. that the proposed method is having converging behavior with the increase in the value of Ω , which varies mildly to left of Fig. 9 and high to right of Fig. 9. We have also observed the effect of ϵ on the proposed method LMDQ both for uniform, non-uniform, and Chebyshev type of nodal points while $N = 10$ and $\Omega = 1e4$ are kept fixed as shown in Figs. 10, 11, and 12. From these Figs., we have seen the fluctuation of accuracy in terms of relative error norm for LMDQ against a different range of ϵ and found its sensitivity. Selection of ϵ is still an open problem besides some techniques are developed which are problems oriented and hence can't be generalized. Using

LMDQ procedure, we have seen that for all the three types: i.e. uniform, non-uniform, and Chebyshev type of nodes, the optimum value for the problem in hand could be anyone in the range $1e - 7$ to 0.1 . In the current model we have used the value of the ϵ to be $1e - 2$. Thus choosing a good ϵ while incorporating very few nodes, the proposed procedure LMDQ is observed to perform well in terms of accuracy and efficiency. Thus the proposed methods are producing accurate and efficient results in free-of-SP approach for different type of uniform and non-uniform nodes.

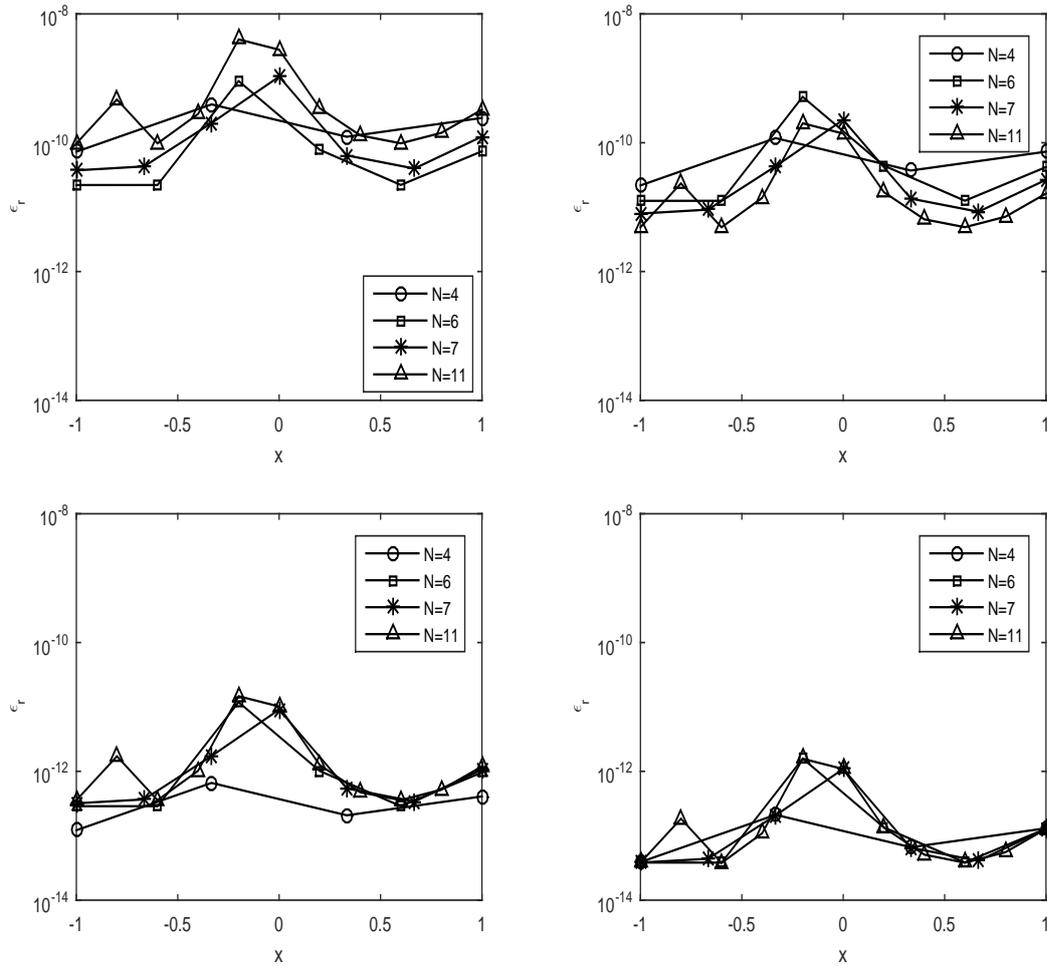


Figure 7: Test 3.3, LMDQ accuracy in terms of ϵ_r (left to right and top to bottom) while keeping $\epsilon = 1e - 2$ against $\Omega = 5, 10, 100, 1000$ on uniform nodes.

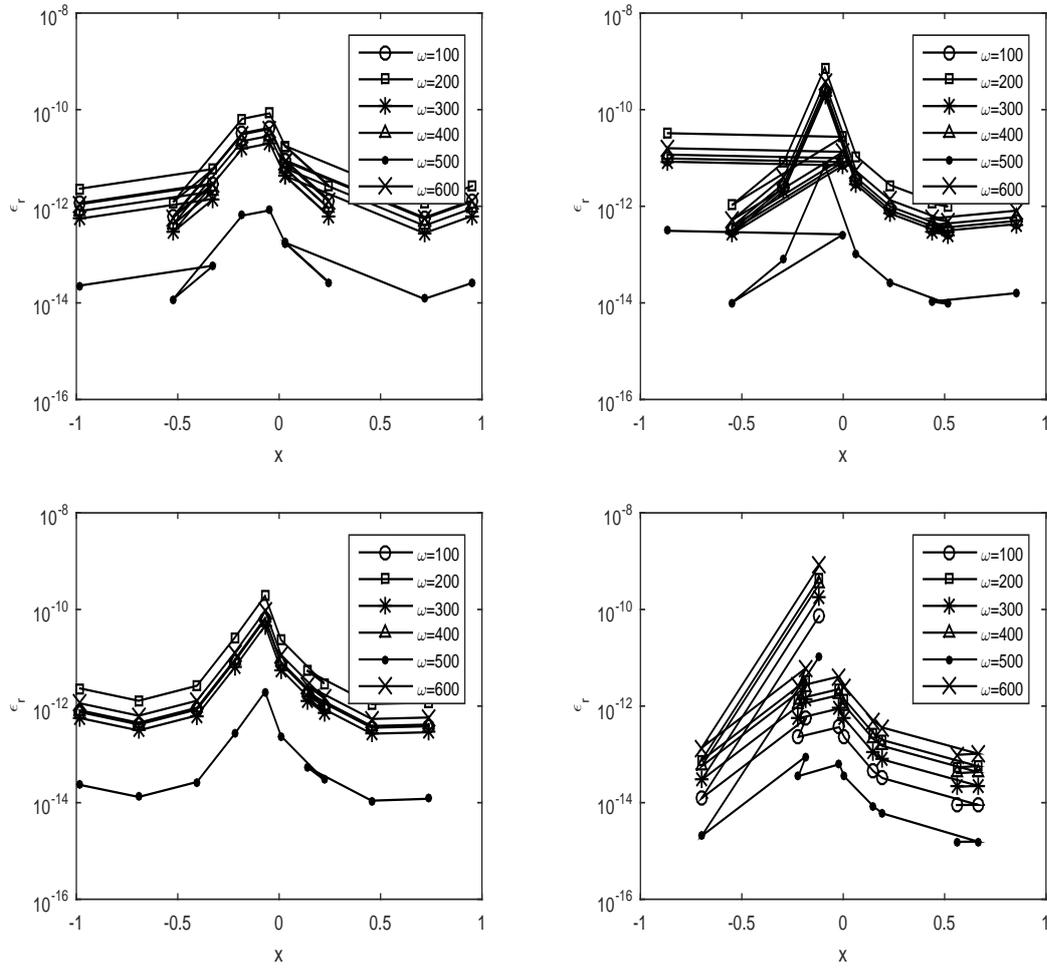


Figure 8: Problem 3.3, LMDQ accuracy in terms of ϵ_r (left to right and top to bottom) for mild oscillations against non-uniform nodes.

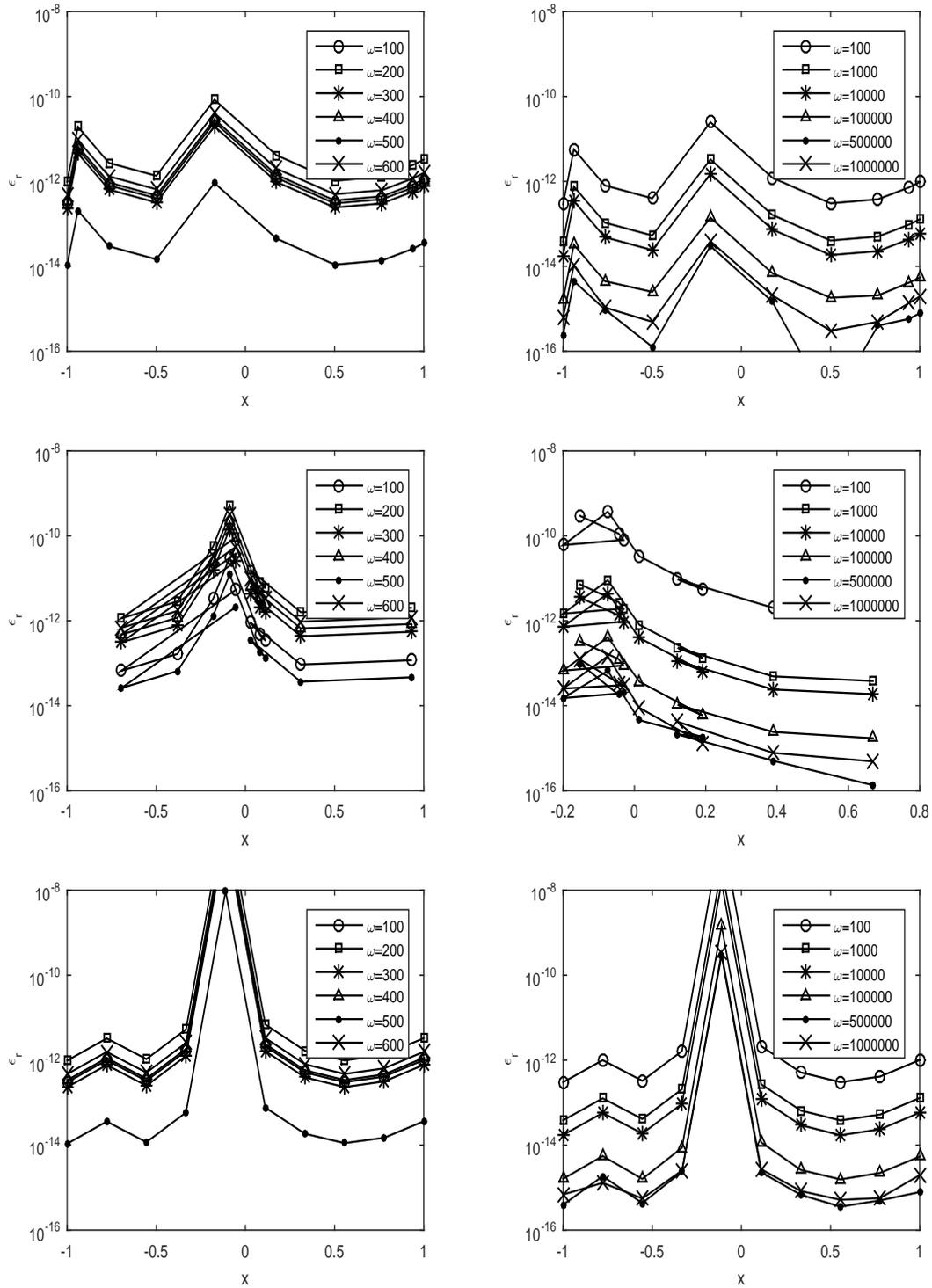


Figure 9: Problem 3.3, LMDQ accuracy in terms of ϵ_r (left: mild and right: high oscillations) against Chebyshev (top), non-uniform (mid), and uniform (bottom) nodes respectively.

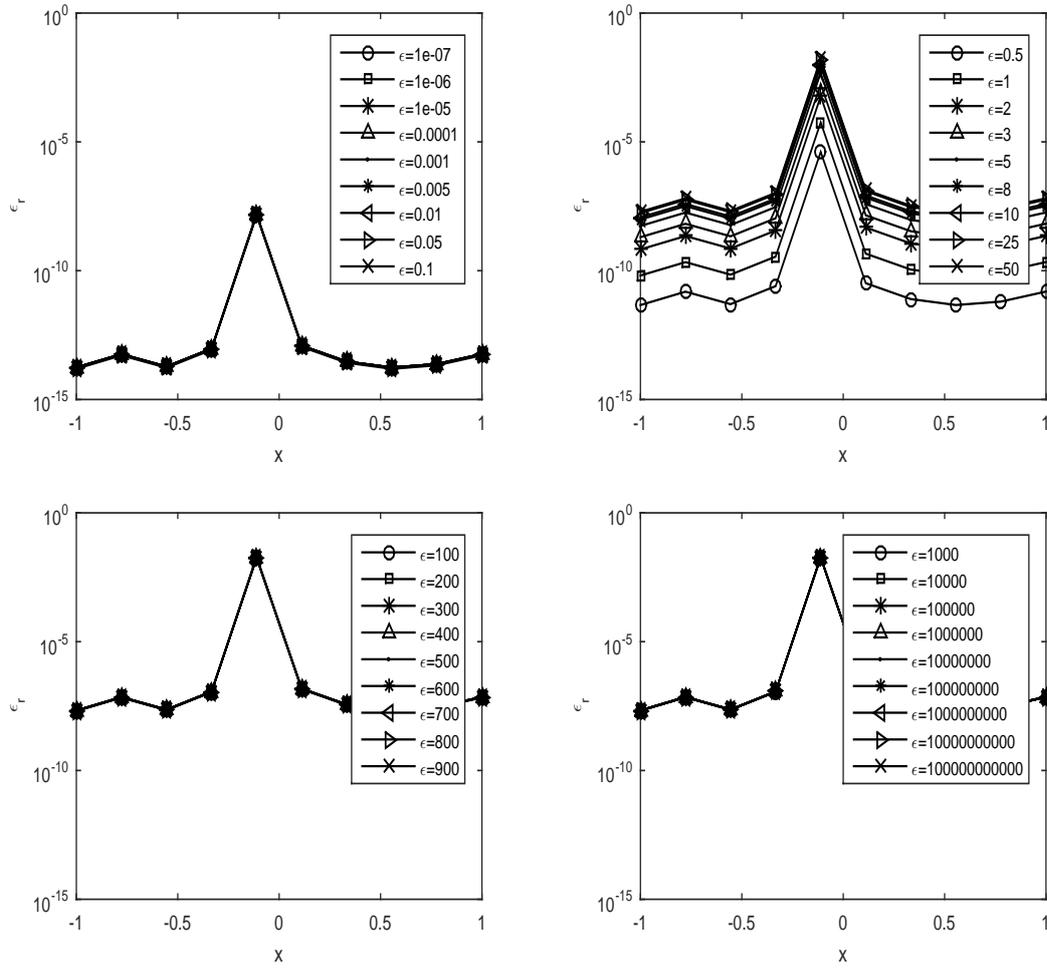


Figure 10: Problem 3.3, Effect of ϵ on LMDQ accuracy in terms of ϵ_r while keeping uniformly discretized $N = 10$ and, $\Omega = 1e4$ fixed.

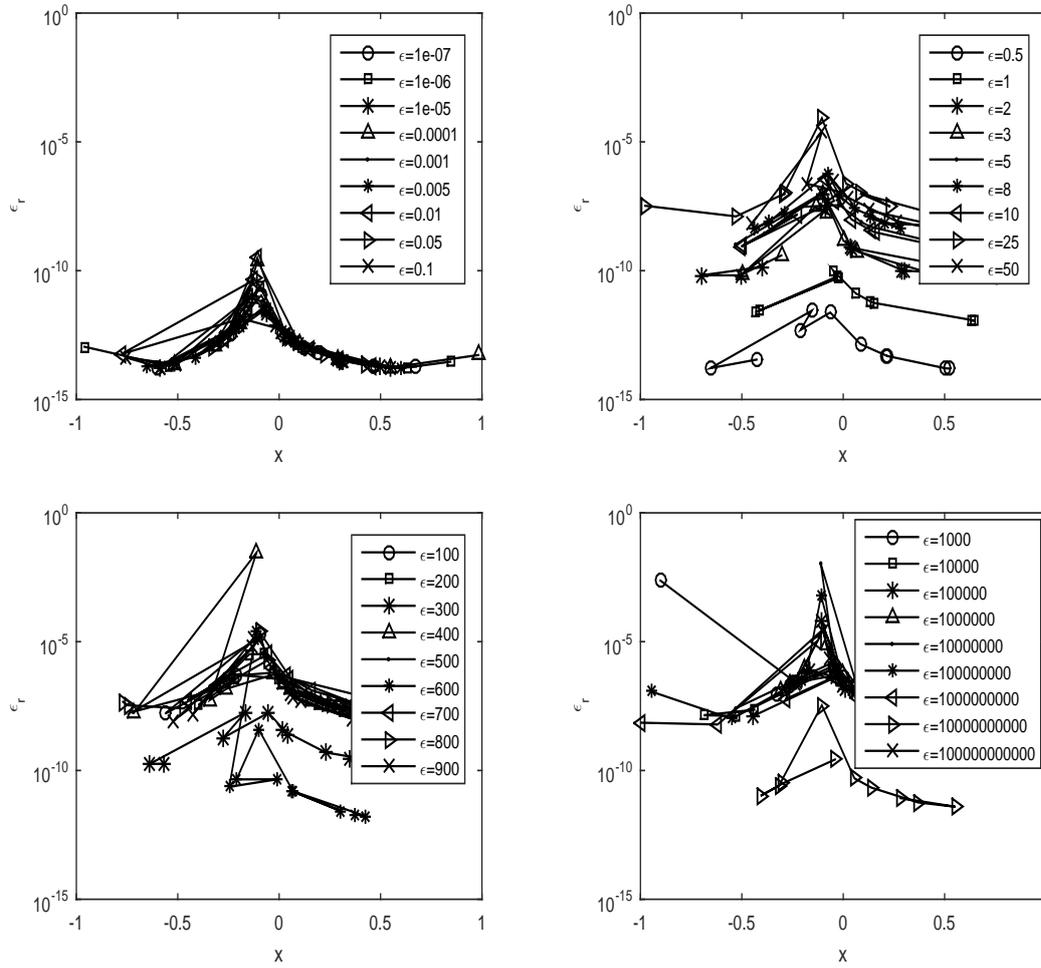


Figure 11: Problem 3.3, Effect of ϵ on LMDQ accuracy in terms of ϵ_r while keeping non-uniformly discretized $N = 10$ and, $\Omega = 1e4$ fixed.

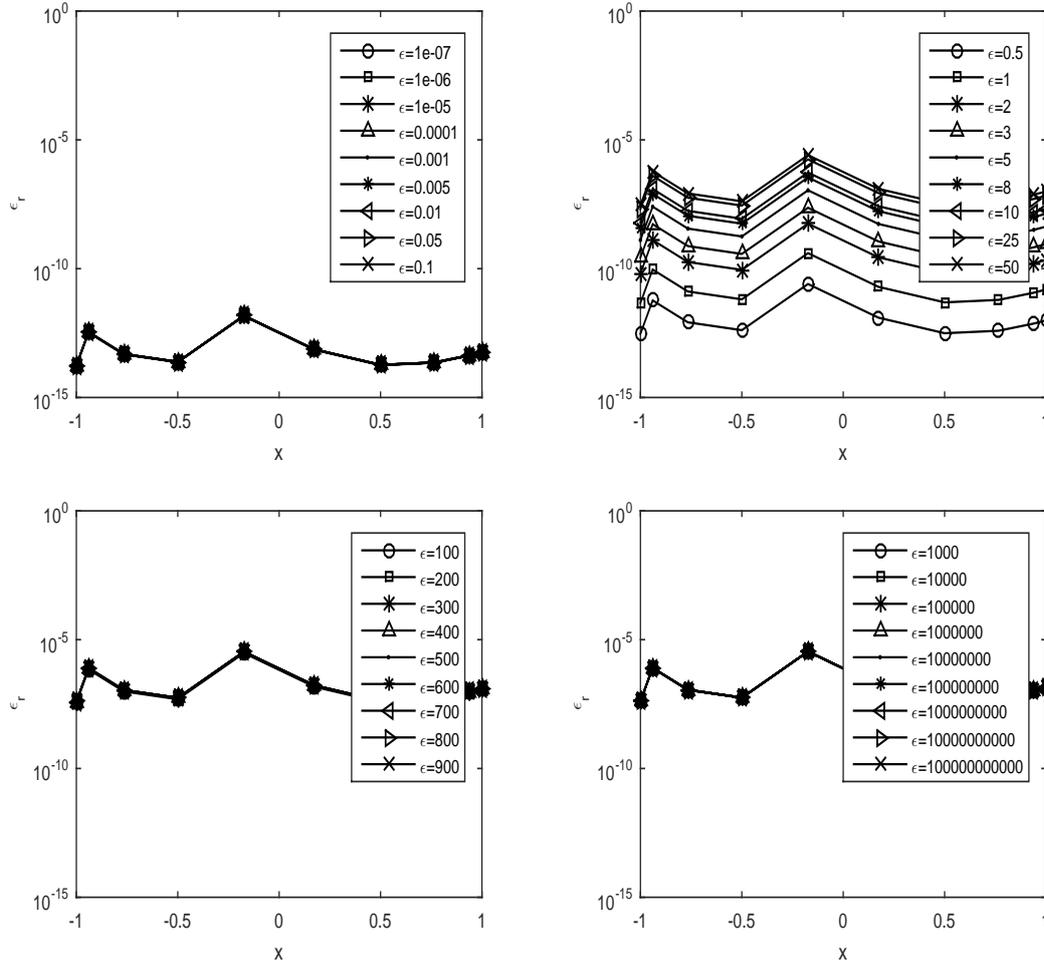


Figure 12: Problem 3.3, Effect of ϵ on LMDQ accuracy in terms of ϵ_r while keeping Chebyshev discretized nodes $N = 10$ and, $\Omega = 1e4$ fixed.

4. Conclusions

The proposed numerical method LMDQ is a local RBF meshless procedure. It is based on the LQT. We have seen that LMDQ is a good alternate numerical scheme for the solution of one-dimensional OFIE. The challenging aspect of the proposed method is that it also evaluate OFIE having KF with SP(s) along with phase function free of SP(s) through a simple algorithm. Performance of the method is validated computationally producing accurate results.

Acknowledgment: The authors of the current paper are thankful to the anonymous referee for the valuable comments which improved its quality. Moreover, the first author appreciates the financial assistance provided by CECOS University of IT and Emerging Sciences Peshawar, Pakistan.

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