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Fixed Point Theorems for Countably Asymptotically Φ−Nonexpansive Maps in Locally Convex Spaces and Application

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Abstract. In this paper, we introduce the concept of a countably asymptotically Φ−nonexpansive operator. In addition, we establish new fixed point results for some countably asymptotically Φ−nonexpansive and sequentially continuous maps, fixed-point results of Krasnosel'skii type in locally convex spaces. Moreover, we present Leray-Schauder-type fixed point theorems for countably asymptotically Φ−nonexpansive maps in locally convex spaces. Apart from that we show the applicability of our results to the theory of Volterra integral equations in locally convex spaces. The main condition in our results is formulated in terms of the axiomatic measure of noncompactness. Our results improve and extend in a broad sense recent ones obtained in literature.

1. Introduction

Many nonlinear problems involve the study of nonlinear equations of the form

$$T(x) + S(x) = x, \; x \in K,$$

where K is a closed convex subset of a Banach space X (see [12]).
A mapping T defined on a nonempty convex closed subset K of a Banach space X is said to be asymptotically Φ−nonexpansive if there exists a sequence $(k_n)_n \subseteq [1, \infty[$ with $\lim k_n = 1$ as $n \longrightarrow \infty$ such that for all bounded subsets D of K,

$$\Phi(T^n(D)) \leq k_n \Phi(D). \tag{1.1}$$

In 1997, Vijayaraju [19] proved some fixed point theorems for asymptotically Φ−nonexpansive mapping in Banach spaces where Φ be a Kuratoski measure of noncompactness.

In 2016, Ben Amar, O'Regan and Touati [4] established some Krasnoselskii type fixed point theorems for the sum of two operators T and S, where T is asymptotically Φ−nonexpansive in Banach spaces with Φ is a measure of weak noncompactness.

In this paper, we introduce the concept of a countably asymptotically Φ−nonexpansive (i.e., By assuming the condition (1.1) holds only for countable bounded sets D in K) in locally convex spaces with we propose of cleaner axiomatic definition of measure of noncompactness.

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The present paper is organized as follows. After some preliminaries, in Section 3 we shall extend the results of Ben Amar, Derbel, O'Reagan and Xiang [[2], Theorem 3.1 and 3.2] (see Theorem 3.2 and 3.4), the result of Ben Amar and Mnif [[3], Theorem 3.3] (see Theorem 3.4) and results of Vijayaraju [[19], Theorem 2.1, 2.2 and 2.3] (see Corollary 3.9, Coraollary 3.12 and Theorem 3.14) in locally convex spaces. Moreover, we can proved the results of [19] are also true for countably asymptotically $\Phi-$nonexpansive mapping not necessarily asymptotically $\Phi-$nonexpansive mapping.

Recently, Khchine, Maniar and Taoudi [11] established a collection of new fixed point theorems for operators of the form $T+S$ on an bounded convex K subset of a locally convex space $(X,(p_\alpha)_{\alpha\in I})$ where T is assumed to be $p_\alpha-$contraction (or $p_\alpha-$nonexpansive or $p_\alpha-$expansive) operator while S is assumed to be continuous and S is $T-$convex-power condensing about x_0 and $n_0\in\mathbb{N}^*$ w.r.t Φ (i.e., for any bounded set N of K with $\Phi(N)>0$, we have

$$\Phi(\mathcal{F}^{(n_0,x_0)}(T,S,N))<\Phi(N)$$

where Φ be a measure of noncompactness on X,
$\mathcal{F}^{(n_0,x_0)}(T,S,N)=\mathcal{F}^{(1,x_0)}(T,S,\overline{conv}(\mathcal{F}^{(n_0-1,x_0)}(T,S,N)\cup\{x_0\}))$ and
$\mathcal{F}^{(1,x_0)}(T,S,N)=\mathcal{F}(T,S,N)=\{x\in K: x=T(x)+S(y)\ for\ some\ y\in N\}$).
In Section 4, we obtain some new forms of Krasnosel-skii's fixed point theorems for operators of the form $T+S$ with for each $n\in\mathbb{N}$, T^n is $p_\alpha-$contraction with a constant $k_n\in]0,1[$ such that $k_n\longrightarrow 1$ as $n\longrightarrow\infty$, and for each countable bounded subset D of K, we have

$$\Phi(\mathcal{F}(T^n,S,D))\leq\frac{1}{k_n}\Phi(D),$$

(or T is asymptotically $p_\alpha-$nonexpansive with a sequence $(k_n)_n\subseteq]1,\infty[$ and for each countable bounded subset D of K and $\lambda\in]0,1[$, we have

$$\Phi(\mathcal{F}(\lambda T^n,\lambda S,D))\leq\lambda k_n\Phi(D),$$

or for each $n\in\mathbb{N}$, T^n is $p_\alpha-$expansive with a constant $k_n\in]1,\infty[$ such that $k_n\longrightarrow 1$ as $n\longrightarrow\infty$, and for each countable bounded subset D of K, we have

$$\Phi(\mathcal{F}(T^n,S,D))\leq k_n\Phi(D).)$$

Moreover, we establish a Krasnoselskii type fixed point theorem for the sum of two sequentially continuous operators T and S with T is countably asymptotically $\Phi-$nonexpansive (see Theorem 4.7). Note our result (Theorem 4.7) improves and generalizes Theorem 2.5 in [18] and Theorem 3.3 in [4]. In addition, we present a Leray-Schauder alternative type of Krasnosel'skii fixed point theorem for countably asymptotically $\Phi-$nonexpansive (see Theorem 4.11). We note that this result (Theorem 4.11) improves Theorem 3.4 in [4].
In the last section of this paper we show the applicability of our result (Theorem 4.3) to the theory of the nonlinear integral equation

$$x(t)=g(x(t))+h(t)+\int_0^t f(s,x(s))ds$$

in a locally convex space.

2. Preliminaries

Let (X,Γ) denote a locally convex Hausdorff space with $\mathcal{P}=(p_\alpha)_{\alpha\in I}$ a family of seminorms that generated the topology of X with the zero element θ, where I is any index set.
We write $x_n\longrightarrow x$ the convergence in (X,Γ) (i.e., for each $\alpha\in I$, $p_\alpha(x_n-x)\longrightarrow 0\ as\ n\longrightarrow\infty$) and $x_n\rightharpoonup x$ to denote the weak convergence.

Definition 2.1. *Let K be a nonempty subset of X. A mapping $T:K\longrightarrow K$ is called* *(a)* $p_\alpha-$*contraction mapping, if for each* $\alpha \in I$, *there is a real number* $0 \leq k_\alpha < 1$ *such that*

$$p_\alpha(T(x) - T(y)) \leq k_\alpha p_\alpha(x - y) \text{ for all } x,\ y \in K.$$

(b) $p_\alpha-$*nonexpansive mapping, if for each* $\alpha \in I$ *we have*

$$p_\alpha(T(x) - T(y)) \leq p_\alpha(x - y) \text{ for all } x,\ y \in K.$$

(c) $p_\alpha-$*expansive mapping, if for each* $\alpha \in I$, *there is a real number* $k_\alpha > 1$ *such that*

$$p_\alpha(T(x) - T(y)) \geq k_\alpha p_\alpha(x - y) \text{ for all } x,\ y \in K.$$

(d) asymptotically $p_\alpha-$*nonexpansive, if for each* $\alpha \in I$

$$p_\alpha(T^n(x) - T^n(y)) \leq k_n p_\alpha(x - y)$$

for all $x,\ y \in K$ *and for all* $n \in N$, *where* $(k_n)_n \subseteq [1, \infty[$ *such that* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$.

(e) asymptotically regular, if for each $\alpha \in I$ *and for each* $x \in K$

$$p_\alpha(T^n(x) - T^{n-1}(x)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

(f) asymptotically regular with respect to S *with* $S : K \longrightarrow X$ *be a mapping, if for each* $\alpha \in I$ *and for each* $x \in K$

$$p_\alpha(T^n(x) - T^{n-1}(x) + S(x)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We state the following Banach's contraction principle that we be repeatedly used in the sequel.

Theorem 2.2. *([6], Theorem 2.2) Let* K *be a nonempty sequentially complete subset of* X. *If* T *is* $p_\alpha-$*contraction mapping of* K *into itself, then* T *has a unique fixed point* u *in* K *and* $T^n(x) \longrightarrow u$ *as* $n \longrightarrow \infty$ *for each* $x \in K$.

Lemma 2.3. *[11] Let* $(X, (p_\alpha)_{\alpha\in I})$ *be a sequentially complete Hausdorff locally convex space and* K *be a closed subset of* X. *Assume* $T : K \longrightarrow X$ *is* $p_\alpha-$*expansive mapping and* $K \subset T(K)$. *Then, there exists a unique point* $x \in K$ *such that* $T(x) = x$.

Theorem 2.4. *[16] Let* K *be a nonempty compact convex subset of* X. *If* T *is continuous mapping of* K *into itself, then* T *has a fixed point in* K.

Definition 2.5. *Let* K *be a nonempty subset of* X. *An operator* $T : K \longrightarrow X$ *is said to be sequentially continuous, if for every sequence* $(x_n) \subset K$ *with* $p_\alpha(x_n - x) \longrightarrow 0$ *as* $n \longrightarrow \infty$, $\alpha \in I$ *and* $x \in K$, *we have, for each* $\alpha \in I$, $p_\alpha(T(x_n) - T(x)) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Now, we give an axiomatic definition of measures of noncompactness in locally convex spaces.

Definition 2.6. *[1]*

Let (X, Γ) *be a Hausdorff topological vector space with zero element* θ. *Let* C *be a lattice with a least element denoting by* 0_C. *A function* Φ *defined on* $\mathcal{P}_{bd}(X)$ *(i.e.,* $\mathcal{P}_{bd}(X) := \{D \subset X : D \text{ is nonempty and bounded}\}$*) with values in* C *will be called a measure of noncompactness (MNC, for each) on* X *if it satisfies the following conditions:*

(i) $\Phi(\overline{conv}(\Omega)) \leq \Phi(\Omega)$ *for each* $\Omega \in \mathcal{P}_{bd}(X)$, *where the symbol* $\overline{conv}(\Omega)$ *denotes the closed convex hull of* Ω *in* X.

(ii) Monotonicity: For any bounded subsets $\Omega_1,\ \Omega_2$ *of* X *we have,* $\Omega_1 \subset \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2)$.

(iii) Nonsingularity: $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ *for any* $a \in X$ *and* $\Omega \in \mathcal{P}_{bd}(X)$.

(iv) $\Phi(\Omega) = 0$ *if and only if* Ω *is relatively compact in* X.

In the case when C has additionally the structure of a cone in a linear space over the field of real numbers, we will say that an measure of noncompactness Φ is positively homogeneous provided $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and for $\Omega \in \mathcal{P}_{bd}(X)$. Moreover, Φ is referred to as subadditive if $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in \mathcal{P}_{bd}(X)$.

Remark 2.7. *If Γ is the weak topology on X, the measure of noncompactness Φ is called the measure of weak noncompactness (MWNC, for each) on X.*

A handy and useful example of an measure of noncompactness in a complete locally convex space is defined as follows:

$$\mu_\alpha(K) = \inf\{d > 0 : K \subset \cup_{i=1}^n K_i, \text{ with } diam_{p_\alpha}(K_i) \leq d, \ i = 1, ..., n\},$$

for each bounded subset K of X. This measure of noncompactness is called the Kuratowskii measure of noncompactness of K with respect to the family of seminorms $(p_\alpha)_{\alpha\in I}$.

Definition 2.8. *Let K be a nonempty subset of X and Φ be a measure of noncompactness in X. A mapping $T : K \longrightarrow K$ is called*

(a) $\Phi-$condensing, if $\Phi(T(D)) < \Phi(D)$ for any bounded sets $D \subseteq K$ with $\Phi(D) > 0$.

(b) countably $\Phi-$condensing, if $\Phi(T(D)) < \Phi(D)$ for any countable bounded sets $D \subseteq K$ with $\Phi(D) > 0$.

(c) asymptotically $\Phi-$nonexpansive, if there exists a sequence $(k_n)_n \subseteq [1, \infty[$ with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ such that $\Phi(T^n(D)) \leq k_n\Phi(D)$ for all $n \geq 1$ and D is a bounded subset of K.

(c) countably asymptotically $\Phi-$nonexpansive, if there exists a sequence $(k_n)_n \subseteq [1, \infty[$ with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ such that $\Phi(T^n(D)) \leq k_n\Phi(D)$ for all $n \geq 1$ and D is a countable bounded subset of K.

Definition 2.9. *Let C be a subset of a topological (Hausdorff) space X.*

(1) C is countably compact, if every sequence in C has a cluster-point in C (i.e., A point $x \in X$ is a cluster point of a sequence $(x_n)_n$ if for every neighbourhood V of x, there are infinitely many natural numbers n such that $x_n \in V$).

(2) C is sequentially compact, if every sequence in C has a convergent subsequence with limit in C.

(3) C is relatively countably compact, if every sequence in C has a cluster-point in X.

(4) C is relatively sequentially compact, if every sequence in C has a convergent subsequence with limit in X.

Some facts on the relation of these notions: It is easy to see that

(1) Every (relatively) compact set is (relatively) countably compact.

(2) Every (relatively) sequentially compact set is (relatively) countably compact.

Now, we recall the following definition from the literature [9].

Definition 2.10. *A Hausdorff topological space X is said to be angelic if for every relatively countably compact set $C \subseteq X$, the following hold:*

(i) C is relatively compact,

(ii) for each $x \in \overline{C}$, there exists a sequence $(x_n)_n \subseteq C$ such that $x_n \longrightarrow x$.

All metrizable locally convex spaces equipped with the weak topology are angelic (see the Eberlein-Šmulian theorem [8]).

Remark 2.11. *If X is angelic, then any sequentially continuous map on a compact set is continuous.*

Lemma 2.12. *[11] Let $(X,(p_\alpha)_{\alpha\in I})$ be a Hausdorff locally convex space and $S : K \longrightarrow X$ be a p_α-contraction with constant k_α. Then for each $\alpha \in I$, and for all bounded subset D of K we have*

$$\mu_\alpha(S(D)) \leq k_\alpha \mu_\alpha(D).$$

Let $\rho > 0$, $J = [0,\rho] \subset \mathbb{R}$ be an interval, and $\tilde{P} = (\tilde{p}_\alpha)_{\alpha\in I}$ be a family of seminorms defined by $\tilde{p}_\alpha(u) = \max_{t\in J} p_\alpha(u(t))$ for each $u \in C(J,X)$. It is easy to check that the space of continuous functions from J to X, $E = C(J,X)$ endowed with the topology generated by the family $\tilde{P}$ is a complete Hausdorff locally convex space. Let μ_α and $\tilde{\mu}_\alpha$ the Kuratowskii's measure of noncompactness of $(X,\mathcal{P})$ and $(E,\tilde{\mathcal{P}})$, respectively. For later use, we recall the following auxiliary result.

Lemma 2.13. *([7], p.412) Let $(X,(p_\alpha)_{\alpha\in I})$ be a complete Hausdorff locally convex space and let $J = [0,\rho] \subset \mathbb{R}$ be an interval*

(i) Let H be a bounded set of $C(J,X)$, then $\sup_{t\in J} \mu_\alpha(H(t)) \leq \mu_\alpha(H(J))$ for each $\alpha \in I$. Here, $H(t) = \{x(t) : x \in H\}$ and $H(J) = \cup_{t\in J} H(t)$.

(ii) Let H be a bounded equicontinuous set of $C(J,X)$, then

(a) $\tilde{\mu}_\alpha(H) = \sup_{t\in J} \mu_\alpha(H(t)) = \mu_\alpha(H(J))$ for each $\alpha \in I$,

(b) for each $u_0 \in C(J,X)$, $\overline{conv}(\{H,x_0\})$ is a bounded equicontinuous subset in $C(J,X)$,

(c) for all $\alpha \in I$, $t \longmapsto \mu_\alpha(H(t)) \in C(J,\mathbb{R}^+)$ and for each $0 \leq t_0 \leq t \leq \rho$ we have $\mu_\alpha\{\int_{t_0}^t u(s)ds : u \in H\} \leq \int_{t_0}^t \mu_\alpha(\{u(s) : s \in H\})ds$.

3. Fixed points of countably asymptotically Φ−nonexpansive mappings

Our first result was motivated by ideas in [[2], Lemma 3.1].

Lemma 3.1. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic and $T : K \longrightarrow K$ be a countably $\Phi-$condensing mapping with bounded range. Suppose T maps compact sets into relatively compact sets, then there is a convex relatively compact subset H of X such that $T(H) \subseteq H$.*

Proof. Let $x_0 \in X$ and $\mathcal{F} := \{D \subseteq K : D \text{ is convex}, x_0 \in D \text{ and } T(D) \subseteq D\}$. Obviously $\mathcal{F}$ is non-empty, since $conv(T(K) \cup \{x_0\}) \in \mathcal{F}$. Let $H = \bigcap_{D\in\mathcal{F}} D$. Note H is convex and $x_0 \in H$. If $x \in H$, then $T(x) \in D$ for all $D \in \mathcal{F}$ and hence $T(H) \subseteq H$. Therefore, $H \in \mathcal{F}$. We now show H is relatively compact. Let $H_* = conv(T(H) \cup \{x_0\})$, and we have $H_* \subseteq H$, which implies that $T(H_*) \subseteq T(H) \subseteq H_*$. Therefore, $H_* \in \mathcal{F}$ and $H \subseteq H_*$. Hence, $H = H_* = conv(T(H) \cup \{x_0\})$. Let $a = sup\{\Phi(C) : C \text{ is a countable subset of } K\}$. Now let C_n be a sequence of countable subsets of H with $\Phi(C_n) \longrightarrow a$ as $n \longrightarrow \infty$. Let $C = \cup_{k\geq 1} C_k$, and since C is a countable subset of H, we obtain $a \geq \Phi(C) \geq \Phi(C_k) \longrightarrow a$. Then $\Phi(C) = a$.
Let $x \in C$ there exist $p_x \in \mathbb{N}^*$ and $y_1,...,y_{p_x} \in \{x_0\} \cup T(H)$ such that $x = \Sigma_{i=1}^{p_x} \lambda_i y_i$ with $\lambda_i \geq 0$, $\forall\, i \in \langle 1,p_x\rangle$ and $\Sigma_{i=1}^{p_x} \lambda_i = 1$. Let $J_x = \{j \in \langle 1,p_x\rangle : y_j = x_0\}$. For every $i \in \langle 1,p_x\rangle \backslash J_x$, $y_i = T(a_i)$ with $a_i \in H$. Let $\mathcal{M}_x = \{a_i \in K : i \in \langle 1,p_x\rangle\backslash J_x\}$, and $\mathcal{M} = \cup_{x\in D} \mathcal{M}_x$. Since C is a countable subset of H, we have $\mathcal{M}$ is a countable subset of H. Note $x = \Sigma_{i\in J_x} \lambda_i x_0 + \Sigma_{i\in\langle 1,p_x\rangle\backslash J_x} \lambda_i T(a_i) \in conv(\{x_0\} \cup T(\mathcal{M}))$. Then, $C \subseteq conv(\{x_0\} \cup T(\mathcal{M}))$. Note since $T(K)$ is bounded, then so also are the sets K, $\mathcal{M}$ and C. We have $\Phi(C) \leq \Phi(T(\mathcal{M}))$. If $\Phi(\mathcal{M}) > 0$, then $\Phi(C) < \Phi(\mathcal{M}) \leq a$, and we obtain $\Phi(C) < a$, a contradiction. Hence $\Phi(\mathcal{M}) = 0$. So $\overline{\mathcal{M}}$ is compact. Then $T(\overline{\mathcal{M}})$ is relatively compact. Therefore $\Phi(D) \leq \Phi(T(\mathcal{M})) \leq \Phi(T(\overline{\mathcal{M}})) = 0$. Thus $\Phi(C) = 0$, i.e., $a = 0$. Let $(x_n)_n \subseteq H$. Since $\{x_n : n \in \mathbb{N}\}$ is a countable subset of H we have $\Phi(\{x_n : n \in \mathbb{N}\}) \leq a = 0$. Then $\overline{H}$ is sequentially compact. Thus $\overline{H}$ is countably compact. By the angelicity of $\overline{H}$, we have $\overline{H}$ is compact. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic and $T : K \longrightarrow K$ is a sequentially continuous and countably $\Phi-$condensing mapping with bounded range. Then, T has a fixed point.*

Proof. From Lemma 3.1, there is a convex subset and relatively compact H of K with $T(H) \subseteq H$. Note $T : \overline{H} \longrightarrow K$ is sequentially continuous, $\overline{H}$ is compact so $T : \overline{H} \longrightarrow K$ is continuous. Thus $T(\overline{H}) \subseteq \overline{T(H)} \subseteq \overline{H}$. In inclusion $T_{|\overline{H}} : \overline{H} \longrightarrow \overline{H}$ is continuous and $\overline{H}$ is compact. From Theorem 2.4, T has a fixed point in K. □

In the following result, we consider the case of a Banach space X endowed with its weak topology. This topology is locally convex and it is induced by the family of seminorms $p_f(x) =\mid f(x) \mid$ for all $f \in X^*$.

Corollary 3.3. *Let K be a closed and convex subset of a Banach space X and let ω be a MWNC on X. Then for every countably ω-condensing and weakly sequentially continuous map $T : K \longrightarrow K$ with bounded range has a fixed point.*

Theorem 3.4. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$, $U \subseteq K$ be an open subset of K with $\theta \in U$ and Φ is a measure of noncompactness on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. Let $T : \overline{U} \longrightarrow K$ be a sequentially continuous countably $\Phi-$condensing mapping with bounded range. Then, either*

$$T \text{ has a fixed point in } \overline{U}, \text{or} \tag{3.1}$$

$$\text{there are a } u \in \partial_K U \text{ and } \lambda \in]0,1[\text{ with } u = \lambda T; \tag{3.2}$$

here $\partial_K U$ denotes the boundary of U in K.

Proof. Suppose (3.2) is false and T has no fixed point on $\partial_K U$. Let

$$D := \{x \in \overline{U} : \ x = \lambda T(x) \text{ for some } \lambda \in [0,1]\}.$$

Then, D is nonempty bounded since $\theta \in D$ and $T(\overline{U})$ is bounded. Note also that $C \subseteq conv(T(C) \cup \{\theta\})$ for any countable subset D of K, and so

$$\Phi(C) \leq \Phi(conv(T(C) \cup \{\theta\})) \leq \Phi(T(C)),$$

which implies (since T is countably $\Phi-$condensing) that $\Phi(D) = 0$. Thus, D is relatively sequentially compact. We next show D is closed. Let $x \in \overline{D}$. By the angelicity of X, there exists $(x_n)_n \subseteq D$ with $x_n \longrightarrow x$. Thus for each $n \geq 0$, $x_n = \lambda_n T(x_n)$ with $(\lambda_n)_n \subseteq [0,1]$. Without loss of generality assume $\lambda_n \longrightarrow \lambda \in [0,1]$ (since [0,1] is compact). Therefore $\lambda_n T(x_n) \longrightarrow \lambda T(x)$. Hence $x = \lambda T(x)$, so $x \in D$. Thus D is sequentially compact. By the angelicity of X, we obtain D is compact. Notice that $\partial_K U \cap D = \emptyset$. Since $(X,(p_\alpha)_{\alpha\in I})$ is a Tychonoff space, there exists a continuous mapping $\mu : \overline{U} \longrightarrow [0,1]$ separating D and $\partial_K U$, i.e., $\mu(D) = 1$ and $\mu(\partial_K U) = 0$. Define $N : K \longrightarrow K$ by

$$N(x) = \begin{cases} \mu(x)T(x), & \text{if } x \in \overline{U}; \\ \theta, & \text{if } x \in K\backslash\overline{U}. \end{cases}$$

Note N is sequentially continuous. Also note for any countable bounded subset C of K with $\Phi(C) > 0$, we have from $N(C) \subset conv(T(C \cap U) \cup \{\theta\})$ that

$$\Phi(N(C)) \leq \Phi(T(C \cap U)),$$

so if $\Phi(C \cap U) = 0$ then $C \cap U$ is relatively compact, so $T(C \cap U)$ is relatively compact and $\Phi(T(C \cap U)) = 0 < \Phi(C)$, whereas if $\Phi(C \cap U) \neq 0$, then $\Phi(T(C \cap U)) < \Phi(C \cap U) \leq \Phi(C)$; in both cases $\Phi(N(C)) < \Phi(C)$. Thus, N is countably $\Phi-$condensing. Theorem 3.2 guarantees that there is an $x \in K$ with $N(x) = x$. Since $\theta \in U$ then $x \in U$. Hence, $x = \mu(x)T(x)$ and since $\mu(x) \in [0,1]$, we have $x \in D$, so $\mu(x) = 1$. Thus $x = T(x)$, i.e., $x \in U$ is a fixed point of T. □

Corollary 3.5. *Let K be a nonempty closed convex subset of a Banach space X, $U \subseteq K$ be a weakly open subset of K with $\theta \in U$ and Φ is a MWNC on X. Assume $T : \overline{U} \longrightarrow K$ is a weakly sequentially continuous, countably $\Phi-$condednsing mapping with bounded range. Then, either*
($A1$) T has a fixed point in $\overline{U}$, or
($A2$) there are a points $x \in \partial_K^w U$ (the weak boundary of U in K) and $k > 1$ with $T(x) = kx$.

Remark 3.6. *Note Corollary 3.5 strictly contains a result of Ben Amar and Mnif (see [3], Theorem 3.3). Indeed the maps considered in [3] satisfy all the hypotheses of our Corollary 3.5 since every Φ-condensing maps is countably Φ-condensing maps but the converse is not always true.*

Definition 3.7. *Let D be a nonempty closed set of a Hausdorff locally convex space $(X, (p_\alpha)_{\alpha\in I})$ and $T : D \longrightarrow E$ be a mapping. T is said to be sequentially semi-closed operator at θ if the conditions $(x_n)_n \subseteq D$, for each $\alpha \in I$ $p_\alpha(x_n - T(x_n)) \longrightarrow 0$ as $n \longrightarrow \infty$ imply that there exists $x \in D$ such that $T(x) = x$.*

Theorem 3.8. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X, (p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous MNC on X. Assume $(X, (p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K$ be a mapping with bounded range satisfying the following conditions*

(*i*) *T is sequentially continuous,*

(*ii*) *T is countably asymptotically $\Phi-$nonexpansive with a sequence $(k_n)_n \subseteq [1, \infty[$,*

(*iii*) *$p_\alpha(T(x) - T^n(x)) \longrightarrow 0$ as $n \longrightarrow \infty$ for each $x \in K$,*

(*iv*) *T is semi-closed operator at θ.*

Then, T has a fixed point in K.

Proof. For fixed $y \in K$, let T_n be a mapping defined by

$$T_n(x) = a_nT^n(x) + (1 - a_n)y, \text{ for all } x \in K \text{ and } n \in \mathbb{N}^*,$$

where $a_n = (1-1/n)/k_n$. Since K is convex, it follows that T_n maps K into itself. Now, using the homogeneity of the MNC and the fact that T is countably asymptotically $\Phi-$nonexpansive, it follows that for any countable bounded D of K with $\Phi(D) > 0$,

$$\begin{aligned} \Phi(T_n(D)) &= \Phi(a_nT^n(D) + (1 - a_n)y) \\ &\leq a_nk_n\Phi(D) \\ &\leq (1 - 1/n)\Phi(D) \\ &< \Phi(D). \end{aligned}$$

Thus T_n is countably $\Phi-$condensing. Next note that since T is sequentially continuous, then T_n is sequentially continuous. Theorem 3.2 guarantees that there is an $x_n \in K$ with

$$x_n = T_n(x_n) = a_nT^n(x_n) + (1 - a_n)y$$

for each $n \in \mathbb{N}^*$. Note that

$$x_n - T^n(x_n) = (1 - a_n)(y - T^n(x_n)) \longrightarrow \theta \text{ as } n \longrightarrow \infty,$$

since $a_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $y - T^n(K)$ is bounded. From assumption (*iii*), we obtain

$$x_n - Tx_n \longrightarrow \theta \text{ as } n \longrightarrow \infty.$$

Finally, since T is semi-closed operator at θ, we obtain

$$\theta \in (I - T)(K).$$

Hence there is a point $x \in K$ such that $x = T(x)$. □

From Theorem 3.8 we can deduce the following result, which extends in a broad sense [[19], Theorem 2.1].

Corollary 3.9. *Let K be a nonempty closed convex subset of a Banach space $(X, \| . \|)$ and Φ is a positive homogeneous MNC on X. Let $T : K \longrightarrow K$ be a mapping with bounded range satisfying the following condition*

(i) *T is sequentially continuous,*

(ii) *T is countably asymptotically $\Phi-$nonexpansive with a sequence $(k_n)_n \subseteq [1, \infty[$,*

(iii) *$\| T(x) - T^n(x) \| \longrightarrow 0$ as $n \longrightarrow \infty$ for each $x \in K$,*

(iv) *T is semi-closed operator at θ.*

Then, T has a fixed point in K.

Theorem 3.10. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X, (p_\alpha)_{\alpha\in I})$, $U \subseteq K$ be an open subset of K with $\theta \in U$ and Φ is a positive homogeneous MNC on X. Assume $(X, (p_\alpha)_{\alpha\in I})$ is angelic. Let $T : \overline{U} \longrightarrow \overline{U}$ be a mapping with bounded range satisfying the following conditions*

(i) *T is sequentially continuous,*

(ii) *T is countably asymptotically $\Phi-$nonexpansive with a sequence $(k_n)_n \subset [1, \infty[$,*

(iii) *$p_\alpha(T(x) - T^n(x)) \longrightarrow_{n\longrightarrow\infty} 0$ for each $x \in \overline{U}$,*

(iv) *T is semi-closed operator at θ.*

Then, either

$$T \text{ has a fixed point in } \overline{U}, \text{ or} \tag{3.3}$$

for some $n \in \mathbb{N}$,

$$\text{there are a } u \in \partial_K U \text{ and } \lambda \in]0, 1[\text{ with } u = \lambda T^n(u); \tag{3.4}$$

Proof. Suppose (3.4) is false (i.e., for all $n \in \mathbb{N}$, there are no $u \in \partial_K U$ and $\lambda \in]0, 1[$ with $u = \lambda T^n(u)$). Define $T_n = a_n T^n$, $n \in \mathbb{N}^*$ where $a_n = (1 - 1/n)/k_n$. Since $\theta \in \overline{U}$ and K is convex, it follows that T_n maps $\overline{U}$ into K. Clearly $T_n(\overline{U})$ is bounded. Consider any countable subset D of $\overline{U}$. Using the homogeneity of the MNC, we have

$$\Phi(T_n(D)) = a_n \Phi(T^n(D)).$$

Now since T is countably asymptotically $\Phi-$nonexpansive, so we deduce T_n is countably $\Phi-$condensing. Since T is sequentially continuous, T_n is sequentially continuous. If there exist a $u \in \partial_K U$ and $L > 1$ with $T_n(u) = Lu$, then

$$u = \frac{1}{L} a_n T^n(u).$$

This is impossible since $(1/L)a_n \in]0, 1[$. From Theorem 3.4, there exists $x_n \in U$ with

$$x_n = T_n(x_n) = a_n T^n(x_n).$$

Note that

$$x_n - T^n(x_n) = (a_n - 1)T^n(x_n) \longrightarrow \theta \text{ as } n \longrightarrow \infty$$

since $a_n \longrightarrow 1$ and $T^n(\overline{U}) \subseteq T(\overline{U})$ is bounded. The argument in Theorem 3.8 guarantees that T has a fixed point. □

Theorem 3.11. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X, (p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous MNC on X. Assume $(X, (p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K$ be a mapping with bounded range satisfying the following conditions*

(i) *T is sequentially continuous,*

(*ii*) *T is countably asymptotically* $\Phi-$*nonexpansive with a sequence* $(k_n)_n \subset [1,\infty[$*,*

(*iii*) *if for each* $\alpha \in I$, $p_\alpha(x_n - y_n) \longrightarrow 0$*, then* $p_\alpha(T(x_n) - T(y_n)) \longrightarrow 0$*,*

(*iv*) *T is an asymptotically regular self mapping of K,*

(*v*) *T is semi-closed operator at* θ*.*

Then T has a fixed point in K.

Proof. Define a map T_n from K to K as in the proof of Theorem 3.8. Proceeding as in Theorem 3.8, there is a point $x_n \in K$ such that

$$x_n - T^n(x_n) \longrightarrow \theta \text{ as } n \longrightarrow \infty. \tag{3.5}$$

From assumption (*iv*), we have

$$T^n(x_n) - T^{n-1}(x_n) \longrightarrow \theta \text{ as } n \longrightarrow \infty. \tag{3.6}$$

Next note that, for all $n \in N^*$

$$x_n - T^{n-1}(x_n) = (x_n - T^n(x_n)) + (T^n(x_n) - T^{n-1}(x_n)). \tag{3.7}$$

Using (3.5) and (3.6) in (3.7), we get

$$x_n - T^{n-1}(x_n) \longrightarrow \theta \text{ as } n \longrightarrow \infty.$$

From assumption (*iii*), we have

$$T(x_n) - T^n(x_n) \longrightarrow \theta. \text{ as } n \longrightarrow \infty.$$

Next note that, for all $\alpha \in I$

$$\begin{aligned} p_\alpha(x_n - T(x_n)) &= p_\alpha(x_n - T^n(x_n) - T(x_n) + T^n(x_n)) \\ &\leq p_\alpha(x_n - T^n(x_n)) + p_\alpha(T(x_n) - T^n(x_n)) \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Finally, since T is semi-closed operator at θ, we obtain

$$\theta \in (I - T)(K).$$

Thus, there is a point $x \in K$ such that $x = T(x)$. □

Corollary 3.12. *Let Let K be a nonempty convex closed subset of a Banach space* $(X, \| \; . \; \|)$ *and* Φ *is a positive homogeneous MNC on X. Let* $T : K \longrightarrow K$ *be a mapping with bounded range satisfying the following conditions*

(*i*) *T is sequentially continuous,*

(*ii*) *T is countably asymptotically* $\Phi-$*nonexpansive with a sequence* $(k_n)_n \subseteq [1,\infty[$*,*

(*iii*) *if* $\| x_n - y_n \| \longrightarrow 0$ *as* $n \longrightarrow \infty$*, then* $\| T(x_n) - T(y_n) \| \longrightarrow 0$ *as* $n \longrightarrow \infty$*,*

(*vi*) *T is an asymptotically regular self mapping of K,*

(*v*) *T is semi-closed operator at* θ*.*

Then T has a fixed point in K.

Remark 3.13. *Note Corollary 3.12 strictly contains a result of Vijayaraju (see [19], Theorem 2.2)). Since every asymptotically* $\Phi-$*nonexpansive is countably asymptotically* $\Phi-$*nonexpansive maps. Moreover, if T is lipschtitz then T satisfy condition* (*iii*) *of our Corollary 3.12.*

Theorem 3.14. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$, $U \subseteq K$ be an open subset of K with $\theta \in U$ and Φ is a positive homogeneous MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. Let $T : \overline{U} \longrightarrow \overline{U}$ be a mapping with bounded range satisfying the following conditions*

(*i*) *T is sequentially continuous,*

(*ii*) *T is countably asymptotically $\Phi-$nonexpansive with a sequence $(k_n)_n \subseteq [1,\infty[$,*

(*iii*) *if for each $\alpha \in I$, $p_\alpha(x_n - y_n) \longrightarrow 0$, then $p_\alpha(T(x_n) - T(y_n)) \longrightarrow 0$,*

(*vi*) *T is an asymptotically regular self mapping of $\overline{U}$,*

(*v*) *T is semi-closed operator at θ.*

Then, either

$$T \text{ has a fixed point in } \overline{U}, \tag{3.8}$$

or, for some $n \in \mathbb{N}$,

$$\text{there are a } u \in \partial_K U \text{ and } \lambda \in]0,1[\text{ with } u = \lambda T^n(u); \tag{3.9}$$

Proof. Suppose (3.9) is false. Define $T_n = a_n T^n$, $n \in N^*$ where $a_n = (1 - 1/n)/k_n$. The argument in Theorem 3.10 guarantees that there is a $x_n \in \overline{U}$ such that

$$x_n - T^n(x_n) \longrightarrow \theta \text{ as } n \longrightarrow \infty.$$

The argument in Theorem 3.11 guarantees that there is a $x \in \overline{U}$ such that $x = T(x)$. □

Remark 3.15. *Theorem 3.14 improves and generalizes Theorem 2.3 in [19] in the context of a Banach space with Φ is a Kuratowski measure of noncompactness.*

4. Fixed points for a sum of two mappings

Let K be a nonempty subset of a Hausdorff locally convex space X and $T : X \longrightarrow X$ and $S : K \longrightarrow X$ be two maps. For any $N \subseteq K$, we set

$$\mathcal{F}(T,S,N) := \{x = T(x) + S(y) : \ y \in N\}.$$

Theorem 4.1. *Let K be a nonempty sequentially complete convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K$, $S : K \longrightarrow X$ be two mappings. Suppose S and T satisfy the following conditions*

(*i*) *S is sequentially continuous,*

(*ii*) *T is an asymptotically $p_\alpha-$nonexpansive with a sequence $(k_n)_n \subseteq [1,\infty[$,*

(*iii*) *T is an asymptotically regular mapping with respect to S,*

(*iv*) *$\Phi(\mathcal{F}(\lambda T^n, \lambda S, D)) \leq \lambda k_n \Phi(D)$ if D is a countable bounded subset of K and $\lambda \in]0,1[$,*

(*v*) *$\lambda T^n(x) + \lambda S(y) \in K$, for any $x, \ y \in K$ and $\lambda \in]0,1[$ and $T(K) + S(K)$ is bounded.*

Then, $T + S$ has a fixed point in K.

Proof. Let y be a fixed element of K. Define $F_n = a_n T^n + a_n S(y)$, $n \in \mathbb{N}^*$ where $a_n := (1 - 1/n)/k_n$. From assumption (v), we have $F_n(K) \subseteq K$. Since T is asymptotically $p_\alpha-$nonexpansive, it follows that

$$\begin{aligned} p_\alpha(F_n(x) - F_n(z)) &= a_n p_\alpha(T^n(x) - T^n(z)) \\ &\leq a_n k_n p_\alpha(x - z) \\ &\leq (1 - 1/n) p_\alpha(x - z) \quad \textit{for all } x, z \in K \textit{ and } \alpha \in I. \end{aligned}$$

Hence F_n is $p_\alpha-$contraction from K into itself. From Theorem 2.2, F_n has a unique fixed point, say $\tau_n(y)$ in K. Therefore,

$$\tau_n(y) = F_n(\tau_n(y)) = a_n(T^n(\tau_n(y)) + S(y)), \ \textit{for each } n \in \mathbb{N}^*.$$

We now show $\tau_n : K \longrightarrow K$ is sequentially continuous. Let $x,\ y \in K$ be arbitrary. Then we have, for each $n \in \mathbb{N}^*$

$$\begin{aligned} p_\alpha(\tau_n(x) - \tau_n(y)) &\leq a_n p_\alpha(T^n(\tau_n(x)) - T^n(\tau_n(y))) + a_n p_\alpha(S(x) - S(y)) \\ &\leq (1 - 1/n) p_\alpha(\tau_n(x) - \tau_n(y)) + a_n p_\alpha(S(x) - S(y)) \ \textit{ for each } \alpha \in I. \end{aligned}$$

Therefore,

$$p_\alpha(\tau_n(x) - \tau_n(y)) \leq \frac{n-1}{k_n} p_\alpha(S(x) - S(y)), \ \textit{for each } n \in \mathbb{N}^*.$$

Since S is sequentially continuous, so is τ_n. It remains to show that τ_n is countably $\Phi-$condensing. Consider any countable subset D of K. Then, we have for each $n \in \mathbb{N}^*$

$$\begin{aligned} \Phi(\tau_n(D)) &= \Phi(\mathcal{F}(a_n T^n, a_n S, D)) \\ &\leq a_n k_n \Phi(D) \\ &\leq (1 - 1/n)\Phi(D). \end{aligned}$$

In particular τ_n is countably $\Phi-$condensing. Now, Theorem 3.2 gaurantees the existence of $x_n \in K$ satisfying

$$x_n = \tau_n(x_n) = a_n T^n(x_n) + a_n S(x_n). \tag{4.1}$$

We can use this argument for all $n \in \mathbb{N}^*$. Note that

$$x_n - T^n(x_n) - S(x_n) = (a_n - 1)(T^n(x_n) + S(x_n)) \longrightarrow \theta \textit{ as } n \longrightarrow \infty, \tag{4.2}$$

since $a_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $T^n(K) + S(K) \subset T(K) + S(K)$ is bounded.
Since T is an asymptotically regular with respect to S, it follows that

$$T^n(x_n) - T^{n-1}(x_n) + S(x_n) \longrightarrow \theta \textit{ as } n \longrightarrow \infty. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$x_n - T^{n-1}(x_n) \longrightarrow \theta \textit{ as } n \longrightarrow \infty. \tag{4.4}$$

Now,

$$\begin{aligned} p_\alpha(x_n - (T + S)(x_n)) &\leq p_\alpha(x_n - T^n(x_n) - S(x_n)) + p_\alpha(T^n(x_n) - T(x_n)) \\ &\leq p_\alpha(x_n - T^n(x_n) - S(x_n)) + k_1 p_\alpha(T^{n-1}(x_n) - x_n) \quad . \end{aligned}$$

Thus

$$x_n - T(x_n) - S(x_n) \longrightarrow \theta \textit{ as } n \longrightarrow \infty. \tag{4.5}$$

Now let

$$M := \{x_n : \ n \in \mathbb{N}^*\}$$

Note that M is bounded since $x_n = a_n T^n(x_n) + S(x_n)$ for any $n \in \mathbb{N}^*$ and note also that $T^n(K) + S(K) \subseteq T(K) + S(K)$ is bounded. We claim that M is relatively compact. If not then by (4.1), we have

$$M \subseteq \mathcal{F}(a_n T^n, a_n S, M).$$

From assumption (iv), we obtain

$$\begin{aligned}\Phi(M) &\le a_n k_n \Phi(M)\\ &\le (1-1/n)\Phi(M)\\ &< \Phi(M),\end{aligned}$$

a contradiction. Thus M is relatively compact. By the angelicity of X there is a subsequence $(x_{\varphi(n)})_n$ of the sequence $(x_n)_n$ such that

$$x_{\varphi(n)} \longrightarrow x \quad for\ some\ x \in K.$$

From assumption (ii), we have

$$p_\alpha(T(x_{\varphi(n)}) - T(x)) \le k_1\, p_\alpha(x_{\varphi(n)} - x),\ for\ each\ \alpha \in I.$$

Then

$$T(x_{\varphi(n)}) \longrightarrow T(x).$$

Since S is sequentially continuous, it follows that

$$(I-T-S)(x_{\varphi(n)}) \longrightarrow (I-T-S)(x).$$

From (4.5), we get

$$(I-T-S)(x_{\varphi(n)}) \longrightarrow \theta.$$

By the uniqueness of limit (since X is Hausdorff), we obtain

$$(I-T-S)(x) = \theta.$$

Then, $T+S$ has a fixed point. □

Theorem 4.2. *Let K be a nonempty sequentially complete convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T: K \longrightarrow K$, $S: K \longrightarrow X$ be two mappings with S is sequentially continuous. Suppose S and T satisfy the following conditions*

(i) *for all $n \in \mathbb{N}$, T^n is $p_\alpha-$contraction with a constant $k_n \in]0,1[$ such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$,*

(ii) *$\Phi(\mathcal{F}(T^n,S,D)) \le \frac{1}{k_n}\Phi(D)$ if $n \in \mathbb{N}$ and D is a countable bounded subset of K,*

(iii) *T is asymptotically regular with respect to S,*

(iv) *$\lambda T^n(x) + \lambda S(y) \in K$ for all $x,\ y \in K,\ n \in \mathbb{N}$ and $\lambda \in]0,1]$ with $T(K)$ and $S(K)$ are bounded.*

Then, $T+S$ has a fixed point in K.

Proof. Let y be a fixed element of K. Define $F_n^y = T^n + S(y),\ n \in N^*$. From assumptions (i) and (iv), we obtain F_n^y is a contraction mapping from K into itself. From Theorem 2.2 F_n^y has a unique fixed point point, say $L_n(y)$ in K. Then for all $y \in K$,

$$L_n(y) = T^n(L_n(y)) + S(y).$$

Note that for any subset N of K, we have $L_n(N) = \mathcal{F}(T^n,S,N)$.
Now let

$$\tau_n(y) = b_n L_n(y)$$

where $b_n := k_n(1-1/n) < 1$.
From assumption (iv), we obtain $\tau_n(K) \subset K$. We next show τ_n is countably $\Phi-$condensing. Consider any countable bounded subset D of K.Then, from assumption (ii), we obtain

$$\begin{aligned}\Phi(\tau_n(D)) &= b_n\Phi(L_n(D))\\ &\le (b_n/k_n)\Phi(D)\\ &\le (1-1/n)\Phi(D).\end{aligned}$$

In particular τ_n is countably $\Phi-$condensing. We now show τ_n is sequentially continuous. Let $x,\ y \in K$, for each $\alpha \in I$ we obtain

$$\begin{aligned} p_\alpha(L_n(x) - L_n(y)) &= p_\alpha(T^n(L_n(x)) - T^n(L_n(y)) + S(x) - S(y)) \\ &\leq k_n p_\alpha(L_n(x) - L_n(y)) + p_\alpha(S(x) - S(y)). \end{aligned}$$

Then

$$p_\alpha(L_n(x) - L_n(y)) \leq \frac{1}{1-k_n} p_\alpha(S(x) - S(y)).$$

Since S is sequentially continuous, so is L_n. Hence, τ_n is sequentially continuous. Theorem 3.2 guarantees the existence of $x_n \in K$ such that

$$x_n = \tau_n(x_n) = b_n L_n(x_n) = b_n(T^n(L_n(x_n)) + S(x_n)) = b_n T^n((1/b_n)x_n) + b_n S(x_n).$$

Therefore, for each $\alpha \in I$

$$\begin{aligned} p_\alpha(x_n - b_n T^n(x_n) - S(x_n)) &= p_\alpha(b_n(T^n(x_n/b_n) - T^n(x_n)) + (b_n - 1)S(x_n)) \\ &\leq b_n k_n p_\alpha(x_n/b_n - x_n) + (b_n - 1)p_\alpha(S(x_n)) \\ &\leq b_n k_n((1/b_n) - 1)p_\alpha(x_n) - (1 - b_n)p_\alpha(S(x_n)) \\ &\leq (1 - b_n)(k_n p_\alpha(x_n) - p_\alpha(S(x_n))) \\ &\longrightarrow 0 \ as\ n \longrightarrow \infty. \end{aligned}$$

since $b_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $S(K)$ and $(x_n)_n$ are bounded (since $(x_n)_n \subseteq conv(T(K)+S(K)\cup\{\theta\})$ and $T(K)+S(K)$ is bounded). Hence

$$x_n - b_n T^n(x_n) - S(x_n) \longrightarrow \theta \ as\ n \longrightarrow \infty.$$

Next note that

$$x_n - T^n(x_n) - S(x_n) = x_n - b_n T^n(x_n) - S(x_n) - (1 - b_n)T^n(x_n) \longrightarrow \theta \ as\ n \longrightarrow \infty$$

since $T^n(K) \subseteq T(K)$ is bounded and $b_n \longrightarrow 1\ as\ n \longrightarrow \infty$. The same argument as in Theorem 4.1 guarantees that

$$x_n - T(x_n) - S(x_n) \longrightarrow \theta \ as\ n \longrightarrow \infty$$

Now let

$$M := \{x_n :\ n \in \mathbb{N}^*\}$$

Note that M is bounded. We claim that M is relatively compact. If not then by assumptions (i) and (ii), we have

$$\begin{aligned} \Phi(M) &= \Phi(\{x_n :\ n \in \mathbb{N}\}) \\ &= b_n \Phi(L_n(M)) \\ &\leq (1 - 1/n)\Phi(M) \\ &< \Phi(M), \end{aligned}$$

a contradiction. Thus M is relatively compact. The argument in Theorem 4.1 guarantees the existence of $x \in K$ such that

$$x = T(x) + S(x).$$

□

In our next result, we examine Theorems 3.1 of [11] for the case when $(X, (p_\alpha)_{\alpha\in I})$ is angelic and $n_0 = 1$. Also, we show that the condition "S *is* $T-$convex-power condensing about x_0 w.r.t. Φ", i.e.,

$$\Phi(\mathcal{F}(T, S, D)) < \Phi(D) \qquad (a)$$

for every bounded subset D of K with $\Phi(D) > 0$ where Φ is a measure of noncompactness on X, can be relaxed by assuming (a) holds only for countable bounded sets D in K such that $\Phi(D) > 0$.

Theorem 4.3. *Let K be a nonempty sequentially complete bounded convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : X \longrightarrow X,\ S : K \longrightarrow X$ be two mappings. Suppose S and T satisfy the following conditions*

- (i) *S is sequentially continuous,*
- (ii) *T is $p_\alpha-$contraction with a constant k_α,*
- (iii) $\Phi(\mathcal{F}(T,S,D)) < \Phi(D)$ *if D is a countable bounded subset of K with* $\Phi(D) > 0$,
- (iv) $T(x) + S(y) \in K$ *for all $x,\ y \in K$.*

Then, $T + S$ has a fixed point in K.

Proof. Let y be a fixed element of K. Define $F^y = T + S(y)$. From assumptions (ii) and (iv), we get $F^y(x)$ is a $p_\alpha-$contraction mapping from K into itself. Theorem 2.2 guarantees that there is unique fixed point of F^y, say $\tau(y)$. Since for any $y \in K$ $\tau(y) = T(\tau(y)) + S(y)$, we have $\tau(N) = \mathcal{F}(T,S,N)$, for any bounded subset N of K. Then, τ is countably $\Phi-$condensing. We now show τ is sequentially continuous. Let $x,\ y \in K$, for each $\alpha \in I$ we obtain

$$\begin{aligned} p_\alpha(\tau(x) - \tau(y)) &= p_\alpha(T(\tau(x)) - T(\tau(y)) + S(x) - S(y)) \\ &\leq k_\alpha p_\alpha(\tau(x) - \tau(y)) + p_\alpha(S(x) - S(y)). \end{aligned}$$

Then

$$p_\alpha(\tau(x) - \tau(y)) \leq \frac{1}{1-k_\alpha} p_\alpha(S(x) - S(y)).$$

Since S is sequentially continuous, so is τ. Theorem 3.2 guarantees the existence of $x \in K$ such that $x = \tau(x) = T(x) + S(x)$. This completes the proof. □

Remark 4.4. *Note that if X is complete then the assumption (iv) of our Theorem 4.3 can be replaced with "if $x = T(x) + S(y),\ y \in K$ imply $x \in K$".*

Theorem 4.5. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K,\ S : K \longrightarrow X$ be two mappings with S is sequentially continuous. Suppose S and T satisfy the following conditions*

- (i) *for all $n \in \mathbb{N}$, T^n is $p_\alpha-$expansive with a constant $k_n \in]1,\infty[$ such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$,*
- (ii) $\Phi(\mathcal{F}(T^n,S,D)) \leq k_n\Phi(D)$ *if D is a countable bounded subset of K,*
- (iii) *$z \in S(K)$ implies $K \subset z + T^n(K)$ for any $n \in \mathbb{N}$ and $T(K) + S(K)$ is bounded,*
- (iv) $p_\alpha(T(x) - T^n(x)) \longrightarrow 0$ *as $n \longrightarrow \infty$, for each $x \in K$.*

Then, $T + S$ has a fixed point in K.

Proof. Let y be a fixed element of K. Define $F_n^y = T^n + S(y),\ n \in \mathbb{N}$. From assumption ($iii$), we have $K \subset F_n(K)$. Note that since T^n is $p_\alpha-$expansive, F_n^y is $p_\alpha -$ *expansive*. Lemma 2.3 guarantees that there is a unique fixed point of F_n^y, say $L_n(y) \in K$. Note that

$$L_n(y) = T^n(L_n(y)) + S(y),\ for\ each\ y \in K,\ n \in N.$$

and $L_n(K) \subset K$. Let $z \in K$ and $\tau_n(y) = a_nL_n(y) + (1 - a_n)z$ where $a_n := (1 - 1/n)/k_n$. Note that $\tau_n(K) \subseteq conv(L_n(K) \cup \{z\}) \subseteq K$. Now, using the homogeneity of the MNC and the fact that $L_n(D) = \mathcal{F}(T^n,S,D)$, it follows that for any countable bounded D of K with $\Phi(D) > 0$,

$$\begin{aligned}\Phi(\tau_n(D)) &= a_n\Phi(L_n(D) + (1 - a_n)z)\\ &= a_n\Phi(\mathcal{F}(T^n, S, D))\\ &\leq (a_n/k_n)\Phi(D)\\ &\leq (1 - 1/n)\Phi(D)\\ &< \Phi(D).\end{aligned}$$

Thus, τ_n is countably $\Phi-$condensing. We next show τ_n is sequentially continuous. To see this, let $x,\ y \in K$. Notice that

$$L_n(x) - L_n(y) = S(x) - S(y) + T^n(L_n(x)) - T^n(L_n(y)).$$

Then, for each $\alpha \in I$

$$p_\alpha(L_n(x) - L_n(y) + (S(y) - S(x))) \geq k_n p_\alpha(L_n(x) - L_n(y)).$$

Thus

$$p_\alpha(L_n(x) - L_n(y)) \leq \frac{1}{k_n - 1} p_\alpha(S(y) - S(x)).$$

Since S is sequentially continuous we obtain L_n is sequentially continuous, so is τ_n. Theorem 3.2 guarantees that there is an $x_n \in K$ with

$$x_n = \tau_n(x_n) = a_n L_n(x_n) + (1 - a_n)z$$

for each $n \in \mathbb{N}^*$. Note that

$$x_n - L_n(x_n) = (1 - a_n)(z - L_n(x_n)) \longrightarrow \theta,\ as\ n \longrightarrow \infty$$

since $a_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $L_n(K) \subset T(K) + S(K)$ *is bounded*. Next note that

$$L_n(x_n) = x_n/a_n - (1 - a_n)/a_n z \in K.$$

Therefore

$$x_n - T^n(x_n/a_n - (1 - a_n)/a_n z) - S(x_n) \longrightarrow \theta,\ as\ n \longrightarrow \infty. \tag{4.6}$$

From assumption (*iv*), it follows that

$$T(x_n/a_n - (1 - a_n)/a_n z) - T^n(x_n/a_n - (1 - a_n)/a_n z) \longrightarrow \theta,\ as\ n \longrightarrow \infty. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$x_n - T(x_n/a_n - (1 - a_n)/a_n z) - S(x_n) \longrightarrow \theta,\ as\ n \longrightarrow \infty. \tag{4.8}$$

Now let

$$M := \{x_n :\ n \in \mathbb{N}\}.$$

Note that M is bounded since $M \subset conv(T(K) + S(K) \cup \{z\})$ and $T(K) + S(K)$ is bounded. We claim that M is relatively compact. If not then by assumptions (*i*) and (*ii*), we have

$$\begin{aligned}\Phi(M) &= \Phi(\{x_n :\ n \in \mathbb{N}\})\\ &= a_n\Phi(L_n(M))\\ &\leq (1 - 1/n)\Phi(M)\\ &< \Phi(M),\end{aligned}$$

a contradiction. Thus M is relatively compact. By the angelicity of X, there is a subsequence $(x_{\varphi(n)})_n$ of the sequence $(x_n)_n$ such that

$$x_{\varphi(n)} \longrightarrow x \quad for\ some\ x \in K.$$

Since S is sequentially continuous, we have $S(x_{\varphi(n)}) \longrightarrow S(x)$. Then $x_{\varphi(n)} - S(x_{\varphi(n)}) \longrightarrow (I-S)(x)$ and from (4.7) we obtain

$$x_{\varphi(n)} - T(x_{\varphi(n)}/a_{\varphi(n)} - (1-a_{\varphi(n)})/a_{\varphi(n)}z) - S(x_{\varphi(n)}) \longrightarrow \theta,$$

where $(a_{\varphi(n)})_n$ be a subsequence of the sequence $(a_n)_n$. Hence

$$T(x_{\varphi(n)}/a_{\varphi(n)} - (1-a_{\varphi(n)})/a_{\varphi(n)}z) \longrightarrow (I-S)(x). \tag{4.9}$$

We now show $(I-S)(K) \subset T(K)$. Let $y = (I-S)(a)$, with $a \in K$. From assumption (iii), we obtain $a \in S(a)+T(K)$. Then there is a $u \in K$ such that $a = S(a) + T(u)$. Thus $y = T(u) \in T(K)$. Therefore $(I-S)(K) \subset T(K)$. Since T is $p_\alpha-$expansive, we have $T^{-1} : T(K) \longrightarrow K$ is $p_\alpha-$contraction.
From (4.9), we have

$$x_{\varphi(n)}/a_{\varphi(n)} - (1-a_{\varphi(n)})/a_{\varphi(n)}z \longrightarrow T^{-1}((I-S)(x)).$$

Since $a_{\varphi(n)} \longrightarrow 1$, we have

$$x_{\varphi(n)}/a_{\varphi(n)} - (1-a_{\varphi(n)})/a_{\varphi(n)}z \longrightarrow x.$$

Since X is Hausdorff, it follows that $T^{-1}((I-S)(x)) = x$. Then $(I-S)(x) = T(x)$. Hence, $T+S$ has a fixed point. □

In our next result, we examine Theorems 3.15 of [11] for the case when $(X,(p_\alpha)_{\alpha\in I})$ is angelic, $n_0 = 1$ and we show that the condition "*S is T*−convex-power condensing about x_0 w.r.t. Φ" can be relaxed by assuming (a) holds only for countable bounded sets D in K such that $\Phi(D) > 0$.

Theorem 4.6. *Let K be a nonempty closed convex bounded subset of a sequentially complete Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K$, $S : K \longrightarrow X$ be two mappings. Suppose S and T satisfy the following conditions*

- (i) *S is sequentially continuous,*
- (ii) *T is $p_\alpha-$expansive with a constant $k_\alpha \in]1,\infty[$*
- (iii) *$\Phi(\mathcal{F}(T,S,D)) < \Phi(D)$ if D is a countable bounded subset of K with $\Phi(D) > 0$,*
- (iv) *$z \in S(K)$ implies $K \subset z + T(K)$.*

Then, $T+S$ has a fixed point in K.

Proof. Define a map F^y as in the proof of Theorem 4.3. By assumption (iv), we have $K \subset F^y(K)$ and by assumption (ii), we obtain F^y is $p_\alpha-$expansive. From Lemma 2.3, there is a unique fixed point $\tau(y) \in K$ of F^y. Note that $\tau(y) = T(\tau(y)) + S(y)$ and $\tau(K) \subset K$. Let N be a subset of K, we have $\tau(N) = \mathcal{F}(T,S,N)$. Then, τ is countably Φ−condensing from K into itself. Now, we claim that τ is sequentially continuous. To see this, let $x,\ y \in K$. Notice that

$$\tau(x) - \tau(y) = S(x) - S(y) + T(\tau(x) - T(\tau(y)).$$

Then, for each $\alpha \in I$

$$p_\alpha(\tau(x) - \tau(y) + (S(y) - S(x))) \geq k_\alpha p_\alpha(\tau(x) - \tau(y)).$$

Thus

$$p_\alpha(\tau(x) - \tau(y)) \leq \frac{1}{k_\alpha - 1} p_\alpha(S(y) - S(x)).$$

Since S is sequentially continuous, so is τ. Theorem 3.2 guarantees the existence of $x \in K$ such that $x = \tau(x) = T(x) + S(x)$.

□

Theorem 4.7. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ and Φ is a positive homogeneous and subadditive MNC on X. Assume $(X,(p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : K \longrightarrow K$, $S : K \longrightarrow X$ be two sequentially continuous mappings. Suppose S and T satisfy the following conditions*

(i) *$S(D)$ is relatively compact if D is a countable bounded subset of K,*

(ii) *T is a countably asymptotically $\Phi-$nonexpansive mapping with a sequence $(k_n)_n \subset [1, \infty[$,*

(iii) *T is an asymptotically regular with respect to S,*

(iv) *if for each $\alpha \in I$ and for each sequences $(x_n)_n$ and $(y_n)_n$ such that $p_\alpha(x_n - y_n) \longrightarrow 0$, then $p_\alpha(T(x_n) - T(y_n)) \longrightarrow 0$,*

(v) *for all $n \in \mathbb{N}$, $\lambda T^n(x) + \lambda S(y) \in K$, for all $x,\ y \in K$, $\lambda \in]0, 1[$ and $T(K) + S(K)$ is bounded.*

Then, $T + S$ has a fixed point in K.

Proof. We define a map F_n from K to K by

$$F_n(x) = a_n(T^n(x) + S(x)) \ for\ all\ n \in N\ and\ x \in K.$$

where $a_n := (1 - 1/n)/k_n$.
Now, using the homogeneity and the subadditivity of the measure of noncompactness Φ and the fact that T is an countably asymptotically $\Phi-$nonexpansive, it follows that for any countable bounded D of K with $\Phi(D) > 0$,

$$\begin{array}{rcl} \Phi(F_n(D)) & = & \Phi(a_n(T^n + S)(D)) \\ & \leq & \Phi(a_n T^n(D) + a_n S(D)) \\ & \leq & a_n k_n \Phi(D) + a_n \Phi(S(D)) \\ & \leq & (1 - 1/n)\Phi(D) \\ & < & \Phi(D). \end{array}$$

Hence F_n is countably $\Phi-$condensing. Next note that since T and S are sequentially continuous, it follows that F_n is sequentially continuous. Theorem 3.2 guarantees that there is $x_n \in K$ with

$$F_n(x_n) = a_n T^n(x_n) + a_n S(x_n) = x_n.$$

Hence

$$x_n - T^n(x_n) - S(x_n) = (a_n - 1)(T^n(x_n) + S(x_n)) \longrightarrow \theta \ as\ n \longrightarrow \infty \tag{4.10}$$

since $a_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $T^n(K) + S(K) \subset T(K) + S(K)$ is bounded.
From assumption (iii) and (4.10), we obtain

$$x_n - T^{n-1}(x_n) \longrightarrow \theta \ as\ n \longrightarrow \infty.$$

From assumption (iv), we have

$$p_\alpha(T(x_n) - T^n(x_n)) \longrightarrow 0,\ for\ any\ \alpha \in I.$$

Next note that for each $\alpha \in I$

$$\begin{array}{rcl} p_\alpha(x_n - (T + S)(x_n)) & \leq & p_\alpha(x_n - T^n(x_n) - S(x_n)) + p_\alpha(T^n(x_n) - T(x_n)) \\ & \longrightarrow & 0\ as\ n \longrightarrow \infty. \end{array}$$

Now let

$$M := \{x_n :\ n \in \mathbb{N}^*\}$$

Note that M is bounded since $M \subset conv(T(K) + S(K) \cup \{\theta\})$ and $T(K) + S(K)$ is bounded. We claim that M is relatively compact. If not then by assumptions (i) and (ii), we have

$$\begin{array}{rcl} \Phi(M) & = & \Phi(\{x_n :\ n \in \mathbb{N}\}) \\ & \leq & a_n \Phi(T^n(M)) + a_n \Phi(S(M)) \\ & \leq & a_n k_n \Phi(M) \\ & < & \Phi(M), \end{array}$$

a contradiction. Thus M is relatively compact and the argument in Theorem 4.1 guarantees that there is an $x \in K$ such that

$$x = T(x) + S(x).$$

□

Remark 4.8. *Theorem 4.7 extends and generalizes Theorem 2.5 in [18] in the case when X is angelic. Indeed, since T is asymptotically p_α−nonexpansive, we obtain T satisfying condition (iv) of our Theorem 4.7 and T is asymptotically μ_α−nonexpansive. Then T is countably asymptotically μ_α−nonexpansive.*

As a consequence of Theorem 4.7, we may state the following result.

Corollary 4.9. *Let K be a nonempty convex subset of a Banach space X and Φ is a positive homogeneous and subadditive MWNC on X. Let $T : K \longrightarrow K$, $S : K \longrightarrow X$ be two weakly sequentially continuous mappings that satisfy the following assumptions:*

(*i*) *$S(D)$ is weakly relatively compact if D is a countable bounded subset of K,*

(*ii*) *T is an countably asymptotically Φ−nonexpansive mapping with a sequence $(k_n)_n \subset [1, \infty[$,*

(*iii*) *T is weakly asymptotically regular with respect to S,*

(*iv*) *if for each sequences $(x_n)_n$ and $(y_n)_n$ such that $x_n - y_n \rightharpoonup \theta$, we have $T(x_n) - T(y_n) \rightharpoonup \theta$,*

(*v*) *for all $n \in \mathbb{N}$, $\lambda T^n(x) + \lambda S(y) \in K$, for all $x,\ y \in K$, $\lambda \in]0, 1[$ and $T(K) + S(K)$ is bounded.*

Then, $T + S$ has a fixed point in K.

Remark 4.10. *Note Corollary 4.9 strictly contains a result of Ben Amar, O'Regan and Touati (see [[4], Theorem 3.3]). Indeed every asymptotically ω-nonexpansive maps is countably asymptotically ω-nonexpansive but the converse is not always true.*

Theorem 4.11. *Let K be a nonempty closed convex subset of a Hausdorff locally convex space $(X, (p_\alpha)_{\alpha\in I})$, $U \subseteq K$ be an open subset of K with $\theta \in U$ and Φ is a positive homogeneous and subadditive MNC on X. Assume $(X, (p_\alpha)_{\alpha\in I})$ is angelic. In addition, let $T : \overline{U} \longrightarrow \overline{U}$ and $S : \overline{U} \longrightarrow X$ be two mappings that satisfy the following conditions*

(*i*) *$S(D)$ is relatively compact if D is a countable bounded subset of $\overline{U}$,*

(*ii*) *T is a countably asymptotically Φ-nonexpansive mapping with a sequence $(k_n)_n \subseteq [1, \infty[$,*

(*iii*) *T is asymptotically regular with respect to S,*

(*iv*) *if for each $\alpha \in I$ and for each sequences $(x_n)_n$ and $(y_n)_n$ such that $p_\alpha(x_n - y_n) \longrightarrow 0$, then $p_\alpha(T(x_n) - T(y_n)) \longrightarrow 0$,*

(*v*) *for all $n \in \mathbb{N}$, $\lambda T^n(x) + \lambda S(y) \in K$, for all $x,\ y \in \overline{U}$, $\lambda \in]0, 1[$ and $T(\overline{U}) + S(\overline{U})$ is bounded.*

Then, either

$$T + S \text{ has a fixed point in } \overline{U}, \text{ or} \tag{4.11}$$

for some $n \in \mathbb{N}$,

$$\text{there are an } u \in \partial_K U \text{ and } \lambda \in]0, 1[\text{ with } u = \lambda(T^n + S)(u). \tag{4.12}$$

Proof. Suppose that (4.12) does not hold. Let

$$a_n := \frac{1 - 1/n}{k_n}\ \forall\, n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. We first show the mapping $F_n = a_nT^n + a_nS$ is countably $\Phi-$condensing. To see that, let D be a countable bounded subset of $\overline{U}$. Using the homogeneity and the subadditivity of the MNC Φ, we obtain

$$\Phi(F_n(D)) \leq \Phi(a_nT^n(D) + a_nS(D)) \leq a_n\Phi(T^n(D)) + a_n\Phi(S(D)).$$

Now $S(D)$ is relatively compact is relatively compact and T is countably asymptotically Φ-nonexpansive, so we deduce that F_n is countably $\Phi-$condensing. Also $\theta \in U$ and $T(\overline{U}) + S(\overline{U})$ is bounded guarantee that F_n maps $\overline{U}$ into K. If there exist an $u \in \partial_K U$ and $k > 1$ with $F_n(u) = u$, then

$$u = \frac{1}{k}a_nT^n(u) + \frac{1}{k}a_nS(u).$$

This is impossible since $(1/k)a_n \in]0, 1[$. From Theorem 3.4, there exists $x_n \in \overline{U}$ with

$$x_n = F_n(x_n) = a_nS(x_n) + a_nT^n(x_n).$$

The argument in Theorem 4.7 guarantees that there exists $x \in U$ such that $x = T(x) + S(x)$. □

Remark 4.12. *Note Theorem 4.11 improves and generalizes Theorem 3.4 in [4] in the context of a Banach space equipped with its weak topology and a measure of weak noncompactness.*

Let $(X, (p_\alpha)_{\alpha\in I})$ be a Hausdorff locally convex space satisfying the condition

$$\sup_{\alpha\in I} p_\alpha(x) < \infty \; for \; each \; x \in X, \tag{C}$$

where $\mathcal{P} = (p_\alpha)_{\alpha\in I}$ is the family of seminorms that generates the topology Γ of X. We define the following function as follows:

$$q(x) = \sup_{\alpha\in I} p_\alpha(x), \; x \in I.$$

Remark 4.13. *1. Let $(X, (p_\alpha)_{\alpha\in I})$ be a sequentially complete Hausdorff locally convex space, E. V. Teixeira [15] has considered the space*

$$X_b = \{x \in X : \sup_{\alpha\in I} p_\alpha(x) < \infty\},$$

called the set of bounded elements of X. He showed that (X_b, q) is a Banach space. For more details, see [[15], Proposition 2.5].

2. *Let $(X, \| \, . \, \|)$ be a Banach space and let $\mathcal{P}$ be the family of seminorms $\{p_f : x \longmapsto | f(x) | : \; f \in X^*, \; \| f \|_{X^*} \leq 1\}$. The topology Γ generated by $\mathcal{P}$ is called the weak topology. The space $(X, \mathcal{P})$, is a Hausdorff locally convex space satisfying the condition (C). Furthermore $X_b = X$ and $q(x) = \| x \|$.*

3. *Let $(X^*, \| \, . \, \|_{X^*})$ be a dual space, endowed with its weak* topology, i.e. generated by the family of seminorms $\mathcal{F} = \{p_x : f \longmapsto | f(x) | : \; x \in X \text{ and } \; \| x \| \leq 1\}$. In this case, $X_b = X^*$ and $q(f) = \| f \|_{X^*}$ for each $f \in X^*$.*

Definition 4.14. *Let K be a nonempty subset of X. A mapping $T : K \longrightarrow K$ is called $q-$asymptotically regular with respect to S with $S : K \longrightarrow X$ be a mapping, if for each $x \in K$,*

$$q(T^n(x) - T^{n-1}(x) + S(x)) \longrightarrow 0 \; as \; n \longrightarrow \infty.$$

Let $(X, (p_\alpha)_{\alpha\in I})$ be a Hausdorff locally convex space satisfying the condition (C) such that (X, q) is a Banach space. Let $(x_n)_n$ be a sequence in X, we write $x_n \longrightarrow x$ the convergence in $(X, (p_\alpha)_{\alpha\in I})$ (i.e., for each $\alpha \in I$, $p_\alpha(x_n - x) \longrightarrow 0 \; as \; n \longrightarrow \infty$) and $x_n \overset{q}{\longrightarrow} x$ the convergence in (X, q) (i.e., $q(x_n - x) \longrightarrow 0 \; as \; n \longrightarrow \infty$).

Theorem 4.15. *Let $(X, (p_\alpha)_{\alpha\in I})$ be a Hausdorff locally convex space satisfying the condition (C) such that (X, q) is a Banach space. Assume that $(X, \{p_\alpha\}_{\alpha\in I})$ is angelic. In addition let Φ be a measure of noncompatness on X. Let K be a nonempty closed convex subset of X, and $T : X \longrightarrow X$, $S : K \longrightarrow X$ be two sequentially continuous mappings satisfying*

(i) *T is asymptotically $q-$nonexpansive (i.e., $q(T^n(x)-T^n(y)) \leq k_n q(x-y)$ for all $x,\ y \in K$ with $(k_n)_n$ be a sequence of $[1,\infty[$ and $k_n \longrightarrow 1$),*

(ii) *T is $q-$asymptotically regular with respect to S,*

(iii) *$\Phi(\mathcal{F}(\lambda T^n, \lambda S, D)) \leq \lambda k_n \Phi(D)$ if D is a countable bounded subset of K, $\lambda \in]0,1[$,*

(iv) *if $x = \lambda(T^n(x) + S(y))$, $y \in K$, $n \in N$ and $\lambda \in (0,1)$, then $x \in K$ and $T^n(K) + S(K)$ is bounded.*

Then, $T + S$ has a fixed point in K.

Proof. For each fixed $y \in K$, we define a map F_n^y by

$$F_n^y(x) = a_n T^n(x) + a_n S(y) \text{ for all } x \in K,$$

where $a_n := (1 - 1/n)/k_n$.
From assumption (i), we have F_n^y is $q-$contraction from X into itself and so it has a unique fixed point in X by the Banach contraction principle. Let us denote by $\tau_n : K \longrightarrow X$ the map which assigns to each $y \in K$ the unique $\tau_n(y)$ in X such that $\tau_n(y) = a_n(T^n(\tau_n(y)) + S(y))$. From assumption ($iv$), we have $\tau_n(K) \subset K$. Note that $\tau_n(N) = \mathcal{F}(a_n T^n, a_n S, N)$ for any subset N of K, then τ_n is countably $\Phi-$condensing. It remains to show that $\tau_n : K \longrightarrow K$ is sequentially continuous in $(X, (p_\alpha)_{\alpha\in I})$. Let $(x_m)_m$ be a sequence of K converging to some x. Since for each $n \in \mathbb{N}$,

$$\{\tau_n(x_m) : \ m \in \mathbb{N}\} \subseteq \mathcal{F}(a_n T^n, a_n S, \{x_m : \ m \in \mathbb{N}\}),$$

we have

$$\Phi(\{\tau_n(x_m) : \ m \in \mathbb{N}\}) \leq a_n k_n \Phi(\{x_m : \ m \in \mathbb{N}\}).$$

Since $(x_m)_m$ is a sequence of K converging, we have $\Phi(\{x_m : \ m \in \mathbb{N}\}) = 0$.
Therefore $\Phi(\{\tau_n(x_m) : \ m \in \mathbb{N}\}) = 0$. So there is a subsequence $(x_{\varphi(m)})_m$ of $(x_m)_m$ such that $\tau_n(x_{\varphi(m)}) \longrightarrow z_n$ *as* $m \longrightarrow \infty$. Since T is sequentially continuous, we have $T^n(\tau_n(x_{\varphi(m)})) \longrightarrow T^n(z_n)$ as $m \longrightarrow \infty$ and since S is sequentially continuous, we obtain $S(x_{\varphi(m)}) \longrightarrow S(x)$. Taking into account that $\tau_n(x_{\varphi(m)}) = a_n T^n(\tau_n(x_{\varphi(m)})) + a_n S(x_{\varphi(m)})$, for each $m,\ n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, we have $z_n = a_n T^n(z_n) + a_n S(x)$. Hence $z_n = F_n^x(z_n)$ and by uniqueness of τ_n, we conclude that $z_n = \tau_n(x)$. Therefore $\tau_n(x_{\varphi(m)}) \longrightarrow \tau_n(x)$ as $m \longrightarrow \infty$. Now, we show that $\tau_n(x_m) \longrightarrow \tau_n(x)$ as $m \longrightarrow \infty$. Suppose the contrary. Then there exists a neighborhood V_n of $\tau_n(x)$ and a subsequence $(x_{\varphi(m)})_m$ of $(x_m)_m$ such that $\tau_n(x_{\varphi(m)}) \notin V_n$, for all $n \in \mathbb{N}$. Then, arguing as before, we may extract a subsequence $(x_{\varphi(\psi(m))})$ of $(x_{\varphi(m)})$ such that $\tau_n(x_{\varphi(\psi(m))}) \longrightarrow \tau_n(x)$, which is absurd since $\tau_n(x_{\varphi(m)}) \notin V_n$ for all $\beta \in L$. Hence τ_n is sequentially continuous. Now, Theorem 3.2 guarantees the existence of $x_n \in K$ such that $x_n = \tau_n(x_n) = a_n T^n(x_n) + a_n S(x_n)$, for all $n \in \mathbb{N}$. We can use this argument for all $n \in \mathbb{N}^*$. Note that

$$q(x_n - T^n(x_n) - S(x_n)) = (a_n - 1)q(T^n(x_n) + S(x_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{4.13}$$

since $a_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $T^n(K) + S(K) \subset T(K) + S(K)$ is bounded.
Since T is an $q-$asymptotically regular with respect to S, it follows that

$$q(T^n(x_n) - T^{n-1}(x_n) + S(x_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.14}$$

From (4.13) and (4.14), we obtain

$$q(x_n - T^{n-1}(x_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.15}$$

Now,

$$\begin{aligned} q(x_n - (T+S)(x_n)) &\leq q(x_n - T^n(x_n) - S(x_n)) + q(T^n(x_n) - T(x_n)) \\ &\leq q(x_n - T^n(x_n) - S(x_n)) + k_1 q(T^{n-1}(x_n) - x_n) \end{aligned} .$$

Thus

$$q(x_n - T(x_n) - S(x_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.16}$$

Now let

$$M := \{x_n : n \in \mathbb{N}^*\}$$

The same reasoning in the proof of Theorem 4.1, we can conclude M is relatively compact. By the angelicity of X there is a subsequence $(x_{\varphi(n)})_n$ of the sequence $(x_n)_n$ such that

$$x_{\varphi(n)} \longrightarrow x \quad for\ some\ x \in K.$$

Since T and S are sequentially continuous, it follows that

$$(I - T - S)(x_{\varphi(n)}) \longrightarrow (I - T - S)(x).$$

From (4.16), we get

$$(I - T - S)(x_{\varphi(n)}) \longrightarrow \theta.$$

By the uniqueness of limit (since X is Hausdorff), we obtain $(I - T - S)(x) = \theta$. Then, $T + S$ has a fixed point. □

Now, we apply Theorem 4.15 to the special case when X is a Banach space endowed with its weak topology.

Corollary 4.16. *Let K be a nonempty closed convex bounded subset of a Banach space $(X, \|\ .\ \|)$, ω is a measure of weak noncompactness on X. Let $T : K \longrightarrow K$, $S : X \longrightarrow X$ be two weakly sequentially continuous mappings. Assume T and S satisfy the following conditions*

- (*i*) $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$, *for each* $x,\ y \in K$ *and* $n \in \mathbb{N}$ *with* $(k_n)_n$ *is a sequence in* $[1, \infty[$ *and* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$,
- (*ii*) *for each* $x \in K$, $\| T^n(x) - T^{n-1}(x) + S(x) \| \longrightarrow 0$ *as* $n \longrightarrow \infty$,
- (*iii*) $\omega(\mathcal{F}(\lambda T^n, \lambda S, D)) \leq \lambda k_n \omega(D)$ *if D is a countable bounded subset of K,* $\lambda \in]0, 1[$,
- (*iv*) *if* $x = \lambda(T^n(x) + S(y))$, $y \in K$ *and* $\lambda \in (0, 1)$ *where* $\lambda \in]0, 1[$, *then* $x \in K$ *and* $T^n(K) + S(K)$ *is bounded.*

Then, $T + S$ has a fixed point in K.

Theorem 4.17. *Let $(X, (p_\alpha)_{\alpha \in I})$ be a Hausdorff locally convex space satisfying the condition (C) such that (X, q) is a Banach space. Assume that $(X, \{p_\alpha\}_{\alpha \in I})$ is angelic. In addition let Φ be a positive homogeneous MNC on X. Let K be a nonempty closed convex subset of X, and $T : X \longrightarrow X$, $S : K \longrightarrow X$ be two sequentially continuous mappings satisfying the following conditions*

- (*i*) *for each* $n \in \mathbb{N}$, T^n *is* $q-$*contraction with a constant* $k_n \in]0, 1[$ *such that* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$ *(i.e.,* $q(T^n(x) - T^n(y)) \leq k_n q(x - y)$ *for all* $x,\ y \in K$*),*
- (*ii*) $\Phi(\mathcal{F}(T^n, S, D)) \leq \frac{1}{k_n}\Phi(D)$ *if D is a countable bounded subset of K,*
- (*iii*) *T is $q-$asymptotically regular with respect to S,*
- (*iv*) $\lambda T^n(x) + \lambda S(y) \in K$, *for all* $x,\ y \in K$ *and* $\lambda \in]0, 1]$. *Moreover $T(K)$ and $S(K)$ are bounded.*

Then, $T + S$ has a fixed point in K.

Proof. The reasoning of Theorem 4.15 yields the desired results. □

In the case of Banach spaces endowed with their weak topologies, Theorem 4.17 states as follows.

Corollary 4.18. *Let K be a nonempty closed convex subset of a Banach space $(X, \|\ .\ \|)$ and ω be a measure of weak noncompactness on X. Assume that $T : K \longrightarrow K$ and $S : K \longrightarrow X$ be to two weakly sequentially continuous mappings satisfying:*

(*i*) $\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|$ *with* $k_n \subset]0, 1[$ *and* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$,

(*ii*) $\omega(\mathcal{F}(T^n, S, D)) \leq \frac{1}{k_n}\omega(D)$ *if* D *is a countable bounded subset of* K,

(*iii*) $\| T^n(x) - T^{n-1}(x) + S(x) \| \longrightarrow 0$ *as* $n \longrightarrow \infty$,

(*iv*) $\lambda(T^n(x) + S(y)) \in K$, *for all* $x, \ y \in K$ *and* $\lambda \in]0, 1]$. *Moreover* $T(K)$ *and* $S(K)$ *are bounded,*

Then, $T + S$ *has a fixed point in* K.

Lemma 4.19. *[11] Let* $(X, (p_\alpha)_{\alpha \in I})$ *be a Hausdorff locally convex space satisfying the condition* $(\mathcal{C})$ *such that* (X, q) *is a Banach space and* K *be a sequentially closed subset of* X. *Let* $S : K \longrightarrow X$ *be a* $q-$*expansive and continuous mapping such that* $K \subset S(K)$. *Then, there exists a unique point* x^* *in* K *such that* $S(x^*) = x^*$.

Theorem 4.20. *Let* $(X, (p_\alpha)_{\alpha \in I})$ *be a Hausdorff locally convex space satisfying the condition* $(\mathcal{C})$ *such that* (X, q) *is a Banach space. Assume that* $(X, (p_\alpha)_{\alpha \in I})$ *is angelic. Let* Φ *be a measure of noncompatness in* X. *Let* K *be a nonempty closed convex subset of* X *and* $T : K \longrightarrow K$, $S : K \longrightarrow X$ *be two mappings with* T *is continuous and* S *is sequentially continuous. Suppose that*

(*i*) *for each* $n \in \mathbb{N}$ T^n *is* $q-$*expansive with a contant* $k_n \in]1, \infty[$ *such that* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$ (*i.e.,* $q(T^n(x) - T^n(y)) \geq k_n q(x - y), \ \text{for all } x, \ y \in K$),

(*ii*) $\Phi(\mathcal{F}(T^n, S, D)) \leq k_n \Phi(D)$ *if* D *is a countable bounded subset of* K,

(*iii*) $z \in S(K)$ *implies* $K \subset z + T^n(K)$ *for any* $n \in \mathbb{N}$ *and* $T(K) + S(K)$ *is bounded,*

(*iv*) *for each* $x \in K$ *and* $\alpha \in I$, $p_\alpha(T(x) - T^n(x)) \longrightarrow 0$ *as* $n \longrightarrow \infty$,

Then $T + S$ *has a fixed point in* K.

Proof. The reasoning of Theorem 4.15 yields the desired results. □

In the case of Banach spaces endowed with their weak topologies, Theorem 4.20 states as follows.

Corollary 4.21. *Let* K *be a nonempty closed convex subset of a Banach space* $(X, \| \, . \, \|)$ *and* ω *be a measure of weak noncompactness on* X. *Assume that* $T : K \longrightarrow K$ *and* $S : K \longrightarrow X$ *be two mappings with* S *is weakly sequentially continuous and* T *is weakly continuous. Suppose that*

(*i*) $\| T^n(x) - T^n(y) \| \geq k_n \| x - y \|$ *with* $k_n \subset]1, \infty[$ *and* $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$,

(*ii*) $\omega(\mathcal{F}(T^n, S, D)) \leq k_n \omega(D)$ *if* D *is a countable bounded subset of* K,

(*iii*) $z \in S(K)$ *implies* $K \subset z + T^n(K)$ *for any* $n \in \mathbb{N}$ *and* $T(K) + S(K)$ *is bounded,*

(*iv*) *for each* $x \in K$, $T(x) - T^n(x) \rightharpoonup \theta$ *as* $n \longrightarrow \infty$.

Then $T + S$ *has a fixed point in* K.

5. Application

5.1. Volterra integral equations

Let $(X, (p_\alpha)_{\alpha \in I})$ be a complete Hausdorff locally convex and angelic space and $J = [0, \rho] \subset \mathbb{R}$ be an interval ($\rho > 0$). In this section, we investigate the existence of solutions to the following Volterra integral equation

$$x(t) = g(x(t)) + h(t) + \int_0^t f(s, x(s))ds, \ t \in J, \tag{5.1}$$

where $f \in \mathcal{C}(J \times X, X)$, $h \in \mathcal{C}(J, X)$ *and* $g \in \mathcal{C}(X, X)$ are given mappings. The integral in (5.1) is understood to be the Riemann integral and solution to (5.1) will be sought in $E = \mathcal{C}(J, X)$. Our application is motivated by earlier works; we quote for instance Chao-dong [7], Hussain and Taoudi [10], Khchine, Maniar and Taoudi [11] and Yuasa [20]. Equation (5.1) will be studied under the following conditions:

($\mathcal{H}_1$) g is $p_\alpha-$contraction with a constant $k_\alpha \in]0,1[$,

($\mathcal{H}_2$) there exist $a : J \longrightarrow [0,\infty[$ a continuous function and a real number $b > 0$ such that $p_\alpha(f(s,u)) \leq a(s)p_\alpha(u) + b$ for a.e. $s \in J$, $\alpha \in I$, and all $u \in X$, with $\int_0^\rho a(s)ds < 1$; and $1 - k_\alpha - \rho A > 0$. Here $A = \max_{s\in J} a(s)$,

($\mathcal{H}_3$) there is a constant $L \geq 0$ such that for any bounded subset D of $C(J,X)$ and for each $0 \leq c \leq d \leq \rho$, we have

$$\mu_{p_\alpha}(f([c,d] \times D)) \leq L\mu_{p_\alpha}(D([c,d])). \tag{b}$$

Khchine, Maniar and Taoudi showed in [11] the integral equation (5.1) has a solution in a complete Hausdorff locally convex space $(X,(p_\alpha)_{\alpha\in I})$ whenever assumptions ($\mathcal{H}_1$)-($\mathcal{H}_3$) are satisfied.
In our next result, we add the condition of angelicity and we show that in the case where $\rho L < 1 - k_\alpha$ for any $\alpha \in I$, the condition ($\mathcal{H}_3$) can be relaxed by assuming that (b) holds only for countable bounded subsets D of $C(J,X)$.

Theorem 5.1. *Assume that* ($\mathcal{H}_1$), ($\mathcal{H}_2$) *and*

($\mathcal{H}_4$) *there is a constant $L \geq 0$ with $\rho L < 1 - k_\alpha$ for any $\alpha \in I$ such that for any countable bounded subset D of $C(J,X)$ and for each $0 \leq c \leq d \leq \rho$, we have*

$$\mu_{p_\alpha}(f([c,d] \times D)) \leq L\mu_{p_\alpha}(D([c,d])).$$

hold. Then, the integral equation (5.1) has a least one continuous solution.

Proof. From assumption ($\mathcal{H}_1$), there exists $x_0 \in X$ such that $x_0 = g(x_0)$. We consider

$$K = \cap_{\alpha\in I}\{x \in K_\alpha :\ p_\alpha(x(t) - x(s)) \leq \mid t - s \mid R_\alpha + \frac{p_\alpha(h(t) - h(s))}{1 - k_\alpha},\ t,\ s \in [0,\rho]\},$$

where $K_\alpha = \{x \in E :\ \tilde{p_\alpha}(x - x_0) \leq r_\alpha\}$, $r_\alpha \geq \frac{\rho b + A\rho p_\alpha(x_0) + \tilde{p_\alpha}(h)}{1-k_\alpha-\rho A}$ *and* $R_\alpha = \frac{Ar_\alpha + Ap_\alpha(x_0)+b}{1-k_\alpha}$, for each $\alpha \in I$. Notice that K is bounded equicontinuous convex closed subset of E containing x_0. We define the following operators $S,\ T : E \longrightarrow E$ by

$$T(x)(t) = g(x(t)) + h(t) - x_0$$

and

$$S(x)(t) = x_0 + \int_0^t f(s,x(s))ds.$$

We shall use same ideas from [11] to show that Step1-Step3.
Step 1: We show that T is a $\tilde{p_\alpha}-$contraction from E into itself.
Let $u,\ v \in E$ and $t \in [0,\rho]$, we have

$$\begin{aligned} p_\alpha(T(u)(t) - T(v)(t)) &= p_\alpha(g(u(t)) - g(v(t))) \\ &\leq k_\alpha p_\alpha(u(t) - v(t)). \end{aligned}$$

This implies

$$\tilde{p_\alpha}(T(u) - T(v)) \leq k_\alpha \tilde{p_\alpha}(u - v).$$

Step 2: We prove that S is sequentially continuous.
Let $(x_n)_n$ be a sequence in K which converges to some $z \in E$. For $\alpha \in I$ and $t \in [0,\rho]$, we get

$$\begin{aligned} p_\alpha((S(x_n)S(z))(t)) &= p_\alpha(\int_0^t (f(s,x_n(s)) - f(s,z(s))ds) \\ &\leq \int_0^\rho f(s,x_n(s)) - f(s,z(s))ds. \end{aligned}$$

Thus,

$$\tilde{p_\alpha}(S(x_n) - S(z)) \leq \int_0^\rho f(s, x_n(s)) - f(s, z(s))ds.$$

The dominated convergence theorem yields $\tilde{p_\alpha}(S(x_n) - S(z)) \longrightarrow 0$ and therefore $S(x_n) \longrightarrow S(z)$.
Step 3: Next, we show that if $x = T(x) + S(y),\ y \in K$, then $x \in K$.
Note that for each $x \in K$, $\alpha \in I$ and $0 \leq t \leq t^{'} \leq \rho$, we have

$$\begin{aligned} p_\alpha(S(x)(t^{'}) - S(x)(t)) &\leq \int_t^{t^{'}} p_\alpha(f(s, x(s)))ds \\ &\leq \int_t^{t^{'}} (a(s)p_\alpha(x(s)) + b)ds \\ &\leq (t^{'} - t)(Ar_\alpha + Ap_\alpha(x_0) + b). \end{aligned}$$

Furthermore, for each $\alpha \in I$

$$\begin{aligned} p_\alpha(S(x)(t) - x_0) &\leq \int_0^t p_\alpha(f(s, x(s)))ds \\ &\leq \int_0^t (a(s) + p_\alpha(x(s)) + b)ds \\ &\leq \rho(Ar_\alpha + Ap_\alpha(x_0) + b) \\ &\leq r_\alpha. \end{aligned}$$

Now, let $y \in K$ and $x \in E$ such that $x = T(x) + S(y)$. For $\alpha \in I$ and $t \in [0, \rho]$, we have

$$\begin{aligned} p_\alpha(x(t) - x_0) &= p_\alpha(T(x)(t) + S(y)(t) - x_0) \\ &\leq p_\alpha(g(x(t)) + h(t) - x_0) + p_\alpha(S(y)(t) - x_0) \\ &\leq p_\alpha(g(x(t)) - x_0) + p_\alpha(h(t)) + p_\alpha(S(y)(t) - x_0) \\ &\leq k_\alpha p_\alpha(x(t) - x_0) + p_\alpha(h(t)) + p_\alpha(S(y)(t) - x_0) \\ &\leq k_\alpha p_\alpha(x(t) - x_0) + \tilde{p_\alpha}(h) + \rho(Ar_\alpha + Ap_\alpha(x_0) + b). \end{aligned}$$

Then,

$$p_\alpha(x(t) - x_0) \leq \frac{1}{(1 - k_\alpha)}[\tilde{p_\alpha}(h) + \rho(Ar_\alpha + Ap_\alpha(x_0) + b)] \leq r_\alpha.$$

Hence, $\tilde{p_\alpha}(x - x_0) \leq r_\alpha$. Furthermore, for each $0 \leq t \leq s \leq \rho$, we have

$$\begin{aligned} p_\alpha(x(t) - x(s)) &\leq p_\alpha(T(x)(t) - T(x)(s)) + p_\alpha(S(y)(t) - S(y)(s)) \\ &\leq (s - t)(Ar_\alpha + Ap_\alpha(x_0) + b) + k_\alpha p_\alpha(x(t) - x(s)) + p_\alpha(h(t) - h(s)). \end{aligned}$$

This implies

$$p_\alpha(x(t) - x(s)) \leq R_\alpha \mid t - s \mid + \frac{p_\alpha(h(t) - h(s))}{1 - k_\alpha}.$$

Therefore, $x \in K$.
Step 4: Now, we claim that $\tilde{\mu}_\alpha(\mathcal{F}(T, S, D)) < \tilde{\mu}_\alpha(D)$ if D is a countable bounded subset of K with $\tilde{\mu}_\alpha(D) > 0$.
For all $x \in \mathcal{F}(T, S, D)$, there is a $y \in D$ such that $x = T(x) + S(y)$. Thus, for $t \in [0, \rho]$, we have

$$\mathcal{F}(T, S, D)(t) \subset T(\mathcal{F}(T, S, D))(t) + S(D)(t).$$

Using Lemma 2.12 and the properties of μ_α, we get

$$\begin{aligned} \mu_\alpha(\mathcal{F}(T, S, D)(t)) &\leq \mu_\alpha(T(\mathcal{F}(T, S, D))(t) + S(D)(t)) \\ &\leq k_\alpha \mu_\alpha(\mathcal{F}(T, S, D)(t)) + \mu_\alpha(S(D)(t)). \end{aligned}$$

Then,

$$\mu_\alpha(\mathcal{F}(T,S,D)(t)) \leq \frac{1}{1-k_\alpha}\mu_\alpha(S(D)(t)). \tag{5.2}$$

Using the properties of μ_α and the mean value theorem for the Reimann integral, we have

$$\begin{aligned}
\mu_\alpha(S(D)(t)) &= \mu_\alpha(\{x_0 + \int_0^t f(s,x(s))ds,\ x \in D\}) \\
&\leq \mu_\alpha(\{\int_0^t f(s,x(s))ds,\ x \in D\}) \\
&\leq \mu_\alpha(t\overline{conv}(\{f(s,x(s)) :\ x \in D\})) \\
&\leq t\mu_\alpha(\overline{conv}(\{f([0,t]\times D)\}) \\
&= t\mu_\alpha(\{f([0,t]\times D)\}) \\
&\leq tL\mu_\alpha(D([0,t]))
\end{aligned}$$

Since K is a bounded and equicontinuous set, so is D. Applying Lemma 2.13, we obtain

$$\mu_\alpha(S(D)(t)) \leq tL\tilde{\mu}_\alpha(D). \tag{5.3}$$

Then, for each $t \in [0,\rho]$ and for all $\alpha \in I$, we get

$$\mu_\alpha(\mathcal{F}(T,S,D)(t)) \leq \frac{tL}{1-k_\alpha}\tilde{\mu}_\alpha(D).$$

Using again Lemma 2.13, we obtain

$$\mu_\alpha(\mathcal{F}(T,S,D)) \leq \frac{\rho L}{1-k_\alpha}\tilde{\mu}_\alpha(D).$$

Then,

$$\tilde{\mu}_\alpha(\mathcal{F}(T,S,D)) \leq \frac{\rho L}{1-k_\alpha}\tilde{\mu}_\alpha(D).$$

Hence $\tilde{\mu}_\alpha(\mathcal{F}(T,S,D)) < \tilde{\mu}_\alpha(D)$ if D is a countable bounded subset of K with $\tilde{\mu}_\alpha(D) > 0$.
Applying Theorem 4.3, we get a fixed point for $T+S$ and hence a solution to (5.1) in E. □

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