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New Results for a Class of Boundary Value Problems Involving Impulsive Fractional Differential Equations

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Abstract. In this paper, a class of boundary value problems involving impulsive fractional differential equations is studied. By constructing Green's function, a natural formula of solutions is derived. By applying fixed point theorems for mixed monotone operator with perturbation and sum operator, some new results on the existence and uniqueness of positive solutions are obtained, and an iterative sequence is constructed to approximate the positive solutions.

1. Introduction

Recently, the nonlinear boundary value problems (BVPs for short) of impulsive differential equations with integer order have been investigated extensively, see [1, 9–13, 27] and the references therein. BVPs of impulsive fractional differential equations play a very important role in theory and applications, see [2, 3, 16, 17, 19, 23–26] for some references along this line. However, as pointed out in [2, 3], the theory of BVPs for impulsive fractional differential equations is in the initial stages and many aspects are yet to be explored.

In [2], by means of contraction mapping principle and Krasnoselskii's fixed point theorem, Ahmad et al. considered the existence of the solutions for the following boundary value problem:

$$\begin{cases} {}^{C}D^{q}_{0+}u(t) = f(t,u(t)), t \in J' = J\backslash\{t_1,t_2,\cdots,t_m\}, J=[0,1], \\ \Delta u(t_k) = I_k(u(t_k)), \Delta u'(t_k) = J_k(u(t_k)), k = 1,2,\cdots,m, \\ u(0)+u'(0) = 0, u(1)+u'(1) = 0, \end{cases}$$

where ${}^{C}D^{q}_{0+}$ is the Caputo fractional derivative of order $q \in (1,2)$. $f : J\times\mathbb{R} \to \mathbb{R}$ is continuous, $I_k, J_k : \mathbb{R} \to \mathbb{R}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k (k = 1,2,\cdots,m)$ respectively for $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$.

Recently, Wang et al. [17] establish a general framework to find a natural formula of solutions for impulsive fractional boundary value problems, which will provide an effective way to deal with such problems. Naturally, one wishes to know whether or not the solution of fractional impulsive differential

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equations can be expressed by using Green's function and if this result holds, can this result be used for the special case with non-impulse. Thus, it is an interesting problem and it is worthwhile to study. On the other hand, we observe that in [2, 3, 14, 16, 17, 19, 23–26], the authors demand that the nonlinear term and the impulse functions are bounded or satisfy Lipschitz conditions and demand the operators to be completely continuous or compactness conditions, clearly, these conditions are very strong. We observe that recent advances in the context of analytical and numerical studies of fractional differential equations can be found in [6, 7].

Inspired by the above literatures, by constructing Green's function and by applying some new fixed point theorems for mixed monotone operator with perturbation and sum operator, we mainly study the existence and uniqueness of positive solutions for the following boundary value problem:

$$\begin{cases} {}^{C}D^{q}_{0+}u(t) = f(t,u(t),u(t)) + g(t,u(t)), \\ \qquad\qquad t \in J' = J\backslash\{t_1,t_2,\cdots,t_m\}, J=[0,1], \\ \Delta u(t_k) = I_k(u(t_k),u(t_k)), \Delta u'(t_k) = J_k(u(t_k),u(t_k)), k = 1,2,\cdots,m, \\ \alpha u(0) + \beta u'(0) = \eta_1, \alpha u(1) + \beta u'(1) = \eta_2, \end{cases} \tag{1}$$

and its special case with non-impulse

$$\begin{cases} {}^{C}D^{q}_{0+}u(t) = f(t,u(t),u(t)) + g(t,u(t)), t \in J = [0,1], \\ \alpha u(0) + \beta u'(0) = \eta_1, \alpha u(1) + \beta u'(1) = \eta_2, \end{cases} \tag{2}$$

where $\alpha, \beta, \eta_1, \eta_2$ are real constants with $\beta \geq \alpha > 0$, $\eta_1 \geq \eta_2 \geq 0$. ${}^{C}D^{q}_{0+}$ is the Caputo fractional derivative of order $q \in (1,2)$. $f : J\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$, $g : J\times\mathbb{R}^+ \to \mathbb{R}^+$ are continuous. $I_k, J_k \in C(\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$. The impulsive point set $\{t_k\}_{k=1}^m$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim\limits_{h\to 0^+} u(t_k + h)$, $u(t_k^-) = \lim\limits_{h\to 0^-} u(t_k + h), k = 1,2,\cdots,m$.

Our work presented has the following new features. Firstly, we present a natural formula of solutions for a system associated with the problem (1) by using Green's function and obtain the properties of Green's function. We find that the expression of the solution by using Green's function is simpler than those of [16, 17, 23](see Remark 3.2). Secondly, our new results of the problem (1) can be used for the problem (2). The problem (2) is a special case of the problem (1) with $I_k \equiv 0$ and $J_k \equiv 0$ (see Remark 3.4). Thirdly, our new results do not demand that the nonlinear term and the impulse functions are bounded or satisfy Lipschitz conditions, clearly, these conditions are very strong. Moreover, our new results can guarantee the existence of a unique positive solution without assuming operators to be completely continuous or compactness conditions and an iterative sequence is constructed to approximate it. The results of the above-mentioned works are generalized and significantly improved (see Remark 3.14). Hence we improve the results of [2, 3, 16, 17, 19, 23–26] to some degree. So it is worthwhile to investigate the problems (1) and (2).

2. Preliminaries and previous results

Let $E = \{u(t) : u(t) \in C(J)\}$ denote a real Banach space with the supremum norm. Let

$$\begin{aligned} &PC(J) = \{u \in E : u : J \to \mathbb{R}^+, u \in C(J'), u(t_k^+), u(t_k^-) \\ &\text{exist with} \quad u(t_k^-) = u(t_k), k = 1,2,\cdots,m\}, \\ &PC^1(J) := \{u \in PC(J) : u' \in PC(J)\} \\ &P = \{u \in PC(J) : u(t) \geq 0, t \in J\}, \\ &P_w = \{u \in P : u \sim w, w > \theta\}. \end{aligned}$$

Obviously, $PC(J) \subset E$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t\in J} |u(t)|,$$

$PC^1(J)$ is also a Banach space with the norm

$$\|u\|_{PC^1} = \|u\|_{PC} + \|u'\|_{PC}.$$

$P \subset PC(J)$ is a normal cone.

A function $u \in PC^1(J)$ is said to be a positive solution of the problem (1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], \mathbb{R}^+)$ satisfies ${}^C D^q_{0+} u_k(t) = f(t, u_k(t), u_k(t)) + g(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_k(t)$ on $[0, t_k)$ is just $u_{k-1}(t)$ and the conditions $\Delta u(t_k) = I_k(u(t_k), u(t_k)), \Delta u'(t_k) = J_k(u(t_k), u(t_k)), k = 1, 2, \cdots, m$ with $\alpha u(0) + \beta u'(0) = \eta_1, \alpha u(1) + \beta u'(1) = \eta_2$.

Definition 2.1. [8, 18] The fractional integral of order $q > 0$ for a function $f : [0, +\infty) \to \mathbb{R}$ is defined as

$$I^q_{0+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) \mathrm{d}s, t > 0.$$

provided the right side is pointwise defined on $[0, +\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. (Generalization of classical Caputo derivative) [8, 18] The Caputo fractional derivative of order $q > 0$ for a function $f : [0, +\infty) \to \mathbb{R}$ is defined as

$${}^C D^q_{0+} f(t) = \frac{1}{\Gamma(n - q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(s) - \sum_{k=0}^{n-1} \frac{s^k}{k!} f^{(k)}(0)}{(t - s)^{q-n+1}} \mathrm{d}s, t > 0, n - 1 < q < n.$$

In the case $f(t) \in C^n[0, +\infty)$, then we have ${}^C D^q_{0+} f(t) = I^{n-q}_{0+} f^{(n)}(t)$, $t > 0, n - 1 < q < n$. That is to say that Definition 2.2 is just the classical Caputo fractional derivative. In Definition 2.2, the integrable function f can be discontinuous. This fact can support us to consider impulsive problems.

Definition 2.3. [4, 5] $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i = 1, 2) \in P, u_1 \le u_2, v_1 \ge v_2$ imply $A(u_1, v_1) \le A(u_2, v_2)$. The quantity $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.4. [22] An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \ge tA(x), \forall t \in (0, 1), x \in P.$$

Definition 2.5. [22] Let β be a real number with $0 < \beta < 1$. An operator $A : P \to P$ is said to be β-concave if it satisfies

$$A(tx) \ge t^\beta A(x), \forall t \in (0, 1), x \in P.$$

Lemma 2.6. [17] For $q > 0$, the general solution of the fractional differential equation ${}^C D^q_{0^+} u(t) = 0$ is given by

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n - 1, n = [q] + 1$ and $[q]$ denotes the integer part of the real number q.

In view of Lemma 2.6, it follows that

$$I^q_{0^+}({}^C D^q_{0^+} u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \cdots, n - 1, n = [q] + 1$.

Lemma 2.7. [21] Let $w > \theta$, $\beta \in (0, 1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies

$$A(tx, t^{-1} y) \ge t^\beta A(x, y), \quad \forall t \in (0, 1), x, y \in P.$$

$B : P \to P$ is an increasing sub-homogeneous operator. Assume that
(i) there is $w_0 \in P_w$ such that $A(w_0, w_0) \in P_w$ and $Bw_0 \in P_w$;

(ii) there exists a constant $\delta_0 > 0$ such that $A(x, y) \geq \delta_0 Bx, \forall x, y \in P$.
Then:
(1) $A : P_w \times P_w \to P_w$ and $B : P_w \to P_w$;
(2) There exist $u_0, v_0 \in P_w$ and $\gamma \in (0, 1)$ such that

$$rv_0 \leq u_0 < v_0, u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0.$$

(3) The operator equation $A(x, x) + Bx = x$ has a unique solution x^* in P_w;
(4) For any initial values $x_0, y_0 \in P_w$, constructing successively sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1},$$
$$y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, n = 1, 2, \cdots$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Remark 2.8. *If we take $B = \theta$ in Lemma 2.7, then the corresponding conclusion is still true (see Corollary 2.2 in [21]).*

Lemma 2.9. *[20] Let P be a normal cone in a real Banach space E, $A : P \to P$ be an increasing γ-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*
(i) there is $w > \theta$ such that $Aw \in P_w$ and $Bw \in P_w$;
(ii) there exists a constant $\delta_0 > 0$ such that $Ax \geq \delta_0 Bx, \forall x \in P$.
Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_w. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \cdots$ for any initial value $y_0 \in P_w$, we have $y_n \to x^*$ as $n \to \infty$.

Remark 2.10. *If we take $B = \theta$ in Lemma 2.9, then the corresponding conclusion is still true (see [15]).*

3. Main results

Firstly, we present a natural formula of solutions for a system associated with the problem (1) by adopting the view of Wang et al. [17] and by using Green's function and obtain the properties of Green's function.

Lemma 3.1. *Given $h(t) \in C(J, \mathbb{R}^+), 1 < q < 2$, the unique solution of*

$$\begin{cases} {}^C D^q_{0+} u(t) = h(t), t \in J' = J \backslash \{t_1, t_2, \cdots, t_m\}, J = [0, 1], \\ \Delta u(t_k) = I_k(u(t_k), u(t_k)), \Delta u'(t_k) = J_k(u(t_k), u(t_k)), k = 1, 2, \cdots, m, \\ \alpha u(0) + \beta u'(0) = \eta_1, \alpha u(1) + \beta u'(1) = \eta_2, \beta \geq \alpha > 0, \eta_1 \geq \eta_2 \geq 0, \end{cases} \tag{3}$$

is formulated by

$$u(t) = \int_0^1 G_1(t, s)h(s)ds + \sum_{i=1}^m G_2(t, t_i)J_i(u(t_i), u(t_i)) + \sum_{i=1}^m G_3(t, t_i)I_i(u(t_i), u(t_i)) + G_4(t), t \in J,$$

where

$$G_1(t, s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} + \dfrac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} \\ + \dfrac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, 0 \leq s \leq t \leq 1, \\ \dfrac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \dfrac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, 0 \leq t \leq s \leq 1, \end{cases} \tag{4}$$

$$G_2(t,t_i)=\begin{cases}\dfrac{\beta-\alpha t_i}{\alpha}+\dfrac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}, 0\le t_i<t\le 1, i=1,2,\cdots,m,\\ \dfrac{\beta-\alpha t}{\alpha}+\dfrac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}, 0\le t<t_i\le 1, i=1,2,\cdots,m,\end{cases} \tag{5}$$

$$G_3(t,t_i)=\begin{cases}\dfrac{\alpha+\beta-\alpha t}{\alpha}, 0\le t_i<t\le 1, i=1,2,\cdots,m,\\ \dfrac{\beta-\alpha t}{\alpha}, 0\le t<t_i\le 1, i=1,2,\cdots,m,\end{cases} \tag{6}$$

$$G_4(t)=\frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2}, 0\le t\le 1. \tag{7}$$

Proof. Applying Lemma 2.6, Eq. (3.1) is reduced to an equivalent integral equation (3.6):

$$u(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\mathrm{d}s-c_k-d_kt, \forall t\in(t_k,t_{k+1}], \tag{8}$$

where $t_0=0, t_{m+1}=1$. Consequently,

$$u'(t)=\frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}h(s)\mathrm{d}s-d_k, t\in(t_k,t_{k+1}].$$

In the light of $\alpha u(0)+\beta u'(0)=\eta_1$, and $\alpha u(1)+\beta u'(1)=\eta_2$, we have

$$-\alpha c_0-\beta d_0=\eta_1, \tag{9}$$

$$\alpha\int_0^1\frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s+\beta\int_0^1\frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s-\alpha c_m-(\alpha+\beta)d_m=\eta_2. \tag{10}$$

In view of $\Delta u(t_k)=I_k(u(t_k),u(t_k))$, and $\Delta u'(t_k)=J_k(u(t_k),u(t_k))$, we have

$$c_m=c_0-\sum_{k=1}^m (I_k(u(t_k),u(t_k))-J_k(u(t_k),u(t_k))t_k), \tag{11}$$

$$d_m=d_0-\sum_{k=1}^m J_k(u(t_k),u(t_k)). \tag{12}$$

By (3.7), (3.9) and (3.10), we get

$$d_0=-\frac{\alpha}{\beta}c_m-\frac{\alpha}{\beta}\sum_{k=1}^m (I_k(u(t_k),u(t_k))-J_k(u(t_k),u(t_k))t_k)-\frac{\eta_1}{\beta}, \tag{13}$$

$$\begin{aligned}d_m=&-\frac{\alpha}{\beta}c_m-\frac{\alpha}{\beta}\sum_{k=1}^m (I_k(u(t_k),u(t_k))-J_k(u(t_k),u(t_k))t_k)\\&-\sum_{k=1}^m J_k(u(t_k,u(t_k)))-\frac{\eta_1}{\beta}.\end{aligned} \tag{14}$$

By (3.8)-(3.9) and (3.11)-(3.12), we derive

$$\begin{aligned}d_0=&\int_0^1\frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s+\frac{\beta}{\alpha}\int_0^1\frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s\\&+\sum_{i=1}^m (I_i(u(t_i),u(t_i))-J_i(u(t_i),u(t_i))t_i)\\&+\frac{\alpha+\beta}{\alpha}\sum_{i=1}^m J_i(u(t_i),u(t_i))+\frac{\eta_1-\eta_2}{\alpha}.\end{aligned} \tag{15}$$

$$\begin{aligned} c_0 = &-\frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s - \frac{\beta^2}{\alpha^2}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\ &-\frac{\beta}{\alpha}\sum_{i=1}^{m}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &-\frac{\beta(\alpha+\beta)}{\alpha^2}\sum_{i=1}^{m} J_i(u(t_i),u(t_i)) - \frac{\beta\eta_2 - (\alpha+\beta)\eta_1}{\alpha^2}. \end{aligned} \tag{16}$$

From (3.9)-(3.10) and (3.13)-(3.14), we get

$$\begin{aligned} d_k &= d_0 - \sum_{i=1}^{k} J_i(u(t_i),u(t_i)) \\ &= \int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s + \frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\ &\quad + \sum_{i=1}^{m}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &\quad + \frac{\alpha+\beta}{\alpha}\sum_{i=1}^{m} J_i(u(t_i),u(t_i)) - \sum_{i=1}^{k} J_i(u(t_i),u(t_i)) + \frac{\eta_1-\eta_2}{\alpha}. \end{aligned} \tag{17}$$

$$\begin{aligned} c_k &= c_0 - \sum_{i=1}^{k}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &= -\frac{\beta}{\alpha}\int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s - \frac{\beta^2}{\alpha^2}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\ &\quad -\frac{\beta}{\alpha}\sum_{i=1}^{m}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &\quad -\frac{\beta(\alpha+\beta)}{\alpha^2}\sum_{i=1}^{m} J_i(u(t_i),u(t_i)) - \sum_{i=1}^{k}(I_i(u(t_i),u(t_i)) \\ &\quad - J_i(u(t_i),u(t_i))t_i) - \frac{\beta\eta_2 - (\alpha+\beta)\eta_1}{\alpha^2}. \end{aligned} \tag{18}$$

Hence, for $k = 1, 2, \cdots, m$, (3.15) and (3.16) imply

$$\begin{aligned} &c_k + d_k t \\ &= -\frac{\beta-\alpha t}{\alpha}\int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s - \frac{\beta^2-\alpha\beta t}{\alpha^2}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\ &\quad -\frac{\beta-\alpha t}{\alpha}\sum_{i=1}^{m}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &\quad -\frac{(\alpha+\beta)(\beta-\alpha t)}{\alpha^2}\sum_{i=1}^{m} J_i(u(t_i),u(t_i)) \\ &\quad -\sum_{i=1}^{k}(I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\ &\quad -\sum_{i=1}^{k} J_i(u(t_i),u(t_i))t - \frac{\alpha\eta_1 + (\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2}, \end{aligned} \tag{19}$$

Now substituting (3.13) and (3.14) into (3.6), for $t \in [0, t_1]$, we obtain

$$
\begin{aligned}
& u(t) \\
&= \int_0^t \frac{(t-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s + \frac{\beta-\alpha t}{\alpha}\int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s \\
&\quad + \frac{\beta^2-\alpha\beta t}{\alpha^2}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\
&\quad + \frac{\beta-\alpha t}{\alpha}\sum_{i=1}^m (I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\
&\quad + \frac{(\alpha+\beta)(\beta-\alpha t)}{\alpha^2}\sum_{i=1}^m J_i(u(t_i),u(t_i)) + \frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2} \\
&= \int_0^t \frac{(t-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s + \frac{\beta-\alpha t}{\alpha}\left(\int_0^{\mathrm{t}} + \int_{\mathrm{t}}^1\right)\frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s \\
&\quad + \frac{\beta^2-\alpha\beta t}{\alpha^2}\left(\int_0^{\mathrm{t}} + \int_{\mathrm{t}}^1\right)\frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\
&\quad + \frac{\beta-\alpha t}{\alpha}\sum_{i=1}^m (I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i) \\
&\quad + \frac{(\alpha+\beta)(\beta-\alpha t)}{\alpha^2}\sum_{i=1}^m J_i(u(t_i),u(t_i)) + \frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2} \\
&= \int_0^t \left[\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\right]h(s)ds \\
&\quad + \int_t^1 \left[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\right]h(s)ds \\
&\quad + \sum_{i=1}^m \left[\left[\frac{\beta-\alpha t}{\alpha} + \frac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}\right]\right]J_i(u(t_i),u(t_i)) \\
&\quad + \sum_{i=1}^m \frac{\beta-\alpha t}{\alpha}I_i(u(t_i),u(t_i)) + \frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2} \\
&= \int_0^1 G_1(t,s)h(s)ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i),u(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i),u(t_i)) + G_4(t),
\end{aligned}
$$

where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are defined by (3.2)-(3.5).

Substituting (3.17) into (3.6), for $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we obtain

$$
\begin{aligned}
u(t) &= \int_0^t \frac{(t-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s + \frac{\beta-\alpha t}{\alpha}\int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s \\
&\quad + \frac{\beta^2-\alpha\beta t}{\alpha^2}\int_0^1 \frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s \\
&\quad + \frac{\beta-\alpha t}{\alpha}\sum_{i=1}^m (I_i(u(t_i),u(t_i)) - J_i(u(t_i),u(t_i))t_i)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\alpha+\beta)(\beta-\alpha t)}{\alpha^2}\sum_{i=1}^{m}J_i(u(t_i),u(t_i))\\
&+\sum_{i=1}^{k}(I_i(u(t_i),u(t_i))-J_i(u(t_i),u(t_i))t_i)\\
&+\sum_{i=1}^{k}J_i(u(t_i),u(t_i))t+\frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2}\\
=&\int_0^t\frac{(t-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s+\frac{\beta-\alpha t}{\alpha}\left(\int_0^{\mathrm{t}}+\int_{\mathrm{t}}^1\right)\frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\mathrm{d}s\\
&+\frac{\beta^2-\alpha\beta t}{\alpha^2}\left(\int_0^{\mathrm{t}}+\int_{\mathrm{t}}^1\right)\frac{(1-s)^{q-2}h(s)}{\Gamma(q-1)}\mathrm{d}s\\
&+\frac{\beta-\alpha t}{\alpha}\left(\sum_{i=1}^{k}+\sum_{i=k+1}^{m}\right)(I_i(u(t_i),u(t_i))-J_i(u(t_i),u(t_i))t_i)\\
&+\frac{(\alpha+\beta)(\beta-\alpha t)}{\alpha^2}\left(\sum_{i=1}^{k}+\sum_{i=k+1}^{m}\right)J_i(u(t_i),u(t_i))\\
&+\sum_{i=1}^{k}(I_i(u(t_i),u(t_i))-J_i(u(t_i),u(t_i))t_i)\\
&+\sum_{i=1}^{k}J_i(u(t_i),u(t_i))t+\frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2}\\
=&\int_0^t\left[\frac{(t-s)^{q-1}}{\Gamma(q)}+\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\right]h(s)ds\\
&+\int_t^1\left[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\right]h(s)ds\\
&+\sum_{i=1}^{k}\left[\frac{\beta-\alpha t_i}{\alpha}+\frac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}\right]J_i(u(t_i),u(t_i))\\
&+\sum_{i=k+1}^{m}\left[\frac{\beta-\alpha t}{\alpha}+\frac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}\right]J_i(u(t_i),u(t_i))\\
&+\sum_{i=1}^{k}\frac{\alpha+\beta-\alpha t}{\alpha}I_i(u(t_i),u(t_i))+\sum_{i=k+1}^{m}\frac{\beta-\alpha t}{\alpha}I_i(u(t_i),u(t_i))\\
&+\frac{\alpha\eta_1+(\beta-\alpha t)(\eta_1-\eta_2)}{\alpha^2}\\
=&\int_0^1G_1(t,s)h(s)ds+\sum_{i=1}^{m}G_2(t,t_i)J_i(u(t_i),u(t_i))\\
&+\sum_{i=1}^{m}G_3(t,t_i)I_i(u(t_i),u(t_i))+G_4(t),
\end{aligned}
$$

where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are defined by (3.2)-(3.5). The proof is complete. □

Remark 3.2. *From Lemma 3.1, it is easy to known that the solution of fractional impulsive differential equations can be expressed by using Green's function under the view of Wang et al. [17] and this expression of the solution of the*

problem (3) *is simpler than those of [16, 17, 23].*

When $I_k(u(t_k), u(t_k)) \equiv 0$ and $J_k(u(t_k), u(t_k)) \equiv 0, k = 1, 2, \cdots, m$, the corresponding special case of the problem (3) has been investigated. It is easy to obtain the following corollary.

Corollary 3.3. *Given $h(t) \in C(J, \mathbb{R}^+), 1 < q < 2$, the unique solution of*

$$\begin{cases} {}^{C}D^{q}_{0+}u(t) = h(t), t \in J = [0, 1], \\ \alpha u(0) + \beta u'(0) = \eta_1, \alpha u(1) + \beta u'(1) = \eta_2, \beta \geq \alpha > 0, \eta_1 \geq \eta_2 \geq 0, \end{cases}$$

is formulated by

$$u(t) = \int_0^1 G_1(t, s)h(s)ds + G_4(t), t \in J,$$

where

$$G_1(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{\Gamma(q)} + \dfrac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} \\ + \dfrac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, 0 \leq s \leq t \leq 1, \\ \dfrac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \dfrac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_4(t) = \frac{\alpha\eta_1 + (\beta - \alpha t)(\eta_1 - \eta_2)}{\alpha^2}, 0 \leq t \leq 1.$$

Remark 3.4. *From Corollary 3.3, one can know that when $I_k(u(t_k), u(t_k)) \equiv 0, J_k(u(t_k), u(t_k)) \equiv 0, k = 1, 2, \cdots, m$, the solution of fractional impulsive differential equations which can be expressed by using Green's function is that of the corresponding fractional differential equations. But in [2, 3, 19, 24–26], it is easy to known that this result can't hold.*

Lemma 3.5. *Let $\beta \geq \alpha > 0$, $\eta_1 \geq \eta_2 \geq 0$, then Green's functions $G_1(t, s)$, $G_2(t, t_i)$, $G_3(t, t_i)$ and $G_4(t)$ defined by (3.2)-(3.5) satisfy the following:*

(i) $G_1(t, s) \in C(J \times J, \mathbb{R}^+)$, $G_2(t, t_i)$, $G_3(t, t_i) \in C(J \times J, \mathbb{R}^+)$, $G_4(t) \in C(J, \mathbb{R}^+)$, and $G_1(t, s)$, $G_2(t, t_i)$, $G_3(t, t_i)$, $G_4(t) > 0$ for all $t, t_i, s \in (0, 1)$, where $J = [0, 1]$.

(ii) Green's functions $G_2(t, t_i)$, $G_3(t, t_i)$, $G_4(t)$ have the following properties:

$$\frac{\beta(\beta - \alpha)}{\alpha^2} \leq G_2(t, t_i) \leq \frac{\beta(\alpha + \beta)}{\alpha^2}, \forall t, t_i \in J,$$
$$\frac{\beta - \alpha}{\alpha} \leq G_3(t, t_i) \leq \frac{\alpha + \beta}{\alpha}, \forall t, t_i \in J,$$
$$\frac{\alpha\eta_2 + \beta(\eta_1 - \eta_2)}{\alpha^2} \leq G_4(t) \leq \frac{\alpha\eta_1 + \beta(\eta_1 - \eta_2)}{\alpha^2}, \forall t \in J.$$

Proof. From the expressions of $G_1(t, s)$, $G_2(t, t_i)$, $G_3(t, t_i)$, $G_4(t)$, it is obvious that (i) hold. Next,we will prove (ii). We can know from the definition of $G_2(t, t_i)$ that, for given $t_i \in (0, 1)(i = 1, 2, \cdots, m)$, $G_2(t, t_i)$ is decreasing with respect to t for $t \in J$. We let

$$g_1(t, t_i) = \frac{\beta - \alpha t_i}{\alpha} + \frac{(\beta - \alpha t_i)(\beta - \alpha t)}{\alpha^2}, 0 \leq t_i < t \leq 1, i = 1, 2, \cdots, m,$$

$$g_2(t,t_i)=\frac{\beta-\alpha t}{\alpha}+\frac{(\beta-\alpha t_i)(\beta-\alpha t)}{\alpha^2}, 0\le t<t_i\le 1, i=1,2,\cdots,m.$$

Hence, we derive

$$\begin{aligned}\min_{t\in[0,1]} G_2(t,t_i)&=\min\{\min_{t\in[t_i,1]} g_1(t,t_i),\min_{t\in[0,t_i]} g_2(t,t_i)\}\\&=\min\{g_1(1,t_i),g_2(t_i,t_i)\}=g_1(1,t_i)\\&=\frac{\beta(\beta-\alpha t_i)}{\alpha^2}\ge\frac{\beta(\beta-\alpha)}{\alpha^2},\end{aligned}$$

$$\begin{aligned}\max_{t\in[0,1]} G_2(t,t_i)&=\max\{\max_{t\in[t_i,1]} g_1(t,t_i),\max_{t\in[0,t_i]} g_2(t,t_i)\}\\&=\max\{g_1(t_i,t_i),g_2(0,t_i)\}=g_2(0,t_i)\\&=\frac{\alpha\beta+\beta(\beta-\alpha t_i)}{\alpha^2}\le\frac{\beta(\alpha+\beta)}{\alpha^2}.\end{aligned}$$

Thus, we have

$$\frac{\beta(\beta-\alpha)}{\alpha^2}\le G_2(t,t_i)\le\frac{\beta(\alpha+\beta)}{\alpha^2},\forall t,t_i\in J.$$

Similarly, we have

$$\begin{aligned}\frac{\beta-\alpha}{\alpha}\le G_3(t,t_i)\le\frac{\alpha+\beta}{\alpha},\forall t,t_i\in J.\\\frac{\alpha\eta_2+\beta(\eta_1-\eta_2)}{\alpha^2}\le G_4(t)\le\frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2},\forall t,t_i\in J.\end{aligned}$$

The proof is completed. □

In the following, we need assumptions as follows:

(H1) $f(t,u,v):J\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and increasing in u and v decreasing. $f(t,c_1,c_2)>0$ with $c_1=\min_{t\in[0,1]}w(t)$, $c_2=\max_{t\in[0,1]}w(t)$, where $w(t)=\int_0^1 G_1(t,s)ds>0, t\in[0,1]$, and there exists a constant $\beta_1\in(0,1)$ such that

$$f(t,\gamma u,\gamma^{-1}v)\ge\gamma^{\beta_1}f(t,u,v),\forall\gamma\in(0,1),t\in[0,1],u,v\in[0,\infty).$$

(H2) $g(t,u):J\times\mathbb{R}^+\to\mathbb{R}^+$ is increasing in u. $g(t,c_1)>0$ with $c_1=\min_{t\in[0,1]}w(t)$, where $w(t)=\int_0^1 G_1(t,s)ds>0, t\in[0,1]$, and

$$g(t,\mu u)\ge\mu g(t,u),\forall\mu\in(0,1),t\in[0,1],u\in[0,\infty).$$

(H3) $I_k(u,v),J_k(u,v)(k=1,2,\cdots,m)$ are increasing in $u\in[0,\infty)$ for fixed $v\in[0,\infty)$, decreasing in $v\in[0,\infty)$ for fixed $u\in[0,\infty)$. For all $\gamma\in(0,1),t\in[0,1],u,v\in[0,\infty)$, there exist $\beta_2,\beta_3\in(0,1)$ such that

$$I_k(\gamma u,\gamma^{-1}v)\ge\gamma^{\beta_2}I_k(u,v),J_k(\gamma u,\gamma^{-1}v)\ge\gamma^{\beta_3}J_k(u,v).$$

(H4) There exists a constant $\delta_0>0$ such that

$$f(t,u,v)\ge\delta_0 g(t,u),\forall t\in[0,1],u,v\in[0,\infty).$$

(H5) $f(t,u): J\times\mathbb{R}^+\to\mathbb{R}^+$ is increasing in u. $f(t,c_1)>0$ with $c_1=\min_{t\in[0,1]}w(t)$, $w(t)=\int_0^1 G_1(t,s)ds>0, t\in[0,1]$, and there exists a constant $\beta_1\in(0,1)$ such that

$$f(t,\gamma u)\geq\gamma^{\beta_1}f(t,u),\forall\gamma\in(0,1),t\in[0,1],u\in[0,\infty).$$

(H6) $I_k, J_k\in C(\mathbb{R}^+,\mathbb{R}^+), I_k(u), J_k(u)(k=1,2,\cdots,m)$ are increasing in $u\in[0,\infty)$. There exist $\beta_2,\beta_3\in(0,1)$ such that

$$I_k(\gamma u)\geq\gamma^{\beta_2}I_k(u), J_k(\gamma u)\geq\gamma^{\beta_3}J_k(u),\forall\gamma\in(0,1),t\in[0,1],u\in[0,\infty)$$

(H7) There exists a constant $\delta_0>0$ such that

$$f(t,u)\geq\delta_0 g(t,u),\forall t\in[0,1],u\in[0,\infty).$$

Theorem 3.6. *Assume that* (H1)-(H4) *hold. Then the problem* (1) *has a unique positive solution u^* in P_w. Moreover for any initial values $u_0,v_0\in P_w$, constructing successively the sequences*

$$\begin{aligned}
u_n(t)&=\int_0^1 G_1(t,s)(f(s,u_{n-1}(s),v_{n-1}(s))+g(s,u_{n-1}(s)))ds\\
&\quad+\sum_{i=1}^m G_2(t,t_i)J_i(u_{n-1}(t_i),v_{n-1}(t_i))\\
&\quad+\sum_{i=1}^m G_3(t,t_i)I_i(u_{n-1}(t_i),v_{n-1}(t_i))+G_4(t),\\
v_n(t)&=\int_0^1 G_1(t,s)(f(s,v_{n-1}(s),u_{n-1}(s))+g(s,v_{n-1}(s)))ds\\
&\quad+\sum_{i=1}^m G_2(t,t_i)J_i(v_{n-1}(t_i),u_{n-1}(t_i))\\
&\quad+\sum_{i=1}^m G_3(t,t_i)I_i(v_{n-1}(t_i),u_{n-1}(t_i))+G_4(t),n=1,2,\cdots
\end{aligned}$$

we have $u_n(t)\to u^*(t), v_n(t)\to u^*(t)$ as $n\to\infty$, where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$ and $G_4(t)$ are given as (3.2)-(3.5).

Proof. To begin with, by Lemma 3.1, the problem (1) has an integral formulation given by

$$\begin{aligned}
u(t)&=\int_0^1 G_1(t,s)(f(s,u(s),u(s))+g(s,u(s)))ds\\
&\quad+\sum_{i=1}^m G_2(t,t_i)J_i(u(t_i),u(t_i))\\
&\quad+\sum_{i=1}^m G_3(t,t_i)I_i(u(t_i),u(t_i))+G_4(t),t\in J.
\end{aligned}$$

Define two operators $A: P\times P\to E$ and $B: P\to E$ by

$$\begin{aligned}
A(u,v)(t)&=\int_0^1 G_1(t,s)f(s,u(s),v(s))ds+\sum_{i=1}^m G_2(t,t_i)J_i(u(t_i),v(t_i))\\
&\quad+\sum_{i=1}^m G_3(t,t_i)I_i(u(t_i),v(t_i))+G_4(t),
\end{aligned}$$

$$Bu(t) = \int_0^1 G_1(t,s)g(s,u(s))ds.$$

It is obvious that u is the solution of the problem (1) if and only if $u = A(u,u) + Bu$. We know from (H1),(H2) and (H3) that $A : P \times P \to P$ and $B : P \to P$. We check that A, B satisfy all assumptions of Lemma 2.7 in the sequel.

Firstly,we prove that A is a mixed monotone operator and satisfies $A(\gamma u, \gamma^{-1}v) \geq \gamma^{\beta}A(u,v), \forall \gamma \in (0,1), u, v \in P$. For $u_i, v_i (i = 1,2) \in P, u_1 \leq u_2, v_1 \geq v_2$, from (H1) and (H3), we obtain

$$\begin{aligned}
A(u_1,v_1)(t) &= \int_0^1 G_1(t,s)f(s,u_1(s),v_1(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(u_1(t_i),v_1(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u_1(t_i),v_1(t_i)) + G_4(t) \\
&\leq \int_0^1 G_1(t,s)f(s,u_2(s),v_2(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(u_2(t_i),v_2(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u_2(t_i),v_2(t_i)) + G_4(t) = A(u_2,v_2)(t).
\end{aligned}$$

For any $u, v \in P$ and $\gamma \in (0,1)$, let $\gamma^{\beta} = \min\{\gamma^{\beta_1}, \gamma^{\beta_2}, \gamma^{\beta_3}\}$, then $\gamma^{\beta} \in (0,1)$. From (H1) and (H3), we know that

$$\begin{aligned}
&A(\gamma u, \gamma^{-1}v)(t) \\
&= \int_0^1 G_1(t,s)f(s,\gamma u(s),\gamma^{-1}v(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(\gamma u(t_i),\gamma^{-1}v(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(\gamma u(t_i),\gamma^{-1}v(t_i)) + G_4(t) \\
&\geq \gamma^{\beta}[\int_0^1 G_1(t,s)f(s,u(s),v(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i),v(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i),v(t_i)) + G_4(t)] = \gamma^{\beta}A(u,v)(t).
\end{aligned}$$

Further, we show that the operator B is an increasing sub-homogeneous operator. From (H2) that B is increasing. For any $\mu \in (0,1)$ and $u \in P$, from (H2) we have

$$B(\mu u)(t) = \int_0^1 G_1(t,s)g(s,\mu u(s))ds \geq \mu \int_0^1 G_1(t,s)g(s,u(s))ds = \mu Bu(t).$$

Next we verify the conditions (i) and (ii) of Lemma 2.7. Let $r_1 = \min_{t\in[0,1]} f(t,c_1,c_2), r_2 = \max_{t\in[0,1]} f(t,c_2,c_1)$. From (H1),(H3) and lemma 3.5, we get

$$\begin{aligned}
A(w,w)(t) &= \int_0^1 G_1(t,s)f(s,w(s),w(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(w(t_i),w(t_i)) \\
&\quad + \sum_{i=1}^m G_3(t,t_i)I_i(w(t_i),w(t_i)) + G_4(t) \\
&\geq \int_0^1 G_1(t,s)f(s,c_1,c_2)ds \geq r_1 \int_0^1 G_1(t,s)ds = r_1 w(t).
\end{aligned}$$

and

$$\begin{aligned}
A(w,w)(t) &= \int_0^1 G_1(t,s)f(s,w(s),w(s))ds + \sum_{i=1}^{m} G_2(t,t_i)J_i(w(t_i),w(t_i)) \\
&\quad + \sum_{i=1}^{m} G_3(t,t_i)I_i(w(t_i),w(t_i)) + G_4(t) \\
&\leq \int_0^1 G_1(t,s)f(s,c_2,c_1)ds + \sum_{i=1}^{m} \frac{\beta(\alpha+\beta)}{\alpha^2} J_i(c_2,c_1) \\
&\quad + \sum_{i=1}^{m} \frac{\alpha+\beta}{\alpha} I_i(c_2,c_1) + \frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2} \\
&\leq r_2 \int_0^1 G_1(t,s)ds + \frac{1}{c_1}[\sum_{i=1}^{m} \frac{\beta(\alpha+\beta)}{\alpha^2} J_i(c_2,c_1) \\
&\quad + \sum_{i=1}^{m} \frac{\alpha+\beta}{\alpha} I_i(c_2,c_1) + \frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2}]w(t) \\
&= (r_2 + \frac{1}{c_1}[\sum_{i=1}^{m} \frac{\beta(\alpha+\beta)}{\alpha^2} J_i(c_2,c_1) + \sum_{i=1}^{m} \frac{\alpha+\beta}{\alpha} I_i(c_2,c_1) \\
&\quad + \frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2}])w(t).
\end{aligned}$$

That is

$$\begin{aligned}
r_1 w(t) &\leq A(w,w)(t) \\
&\leq (r_2 + \frac{1}{c_1}[\sum_{i=1}^{m} \frac{\beta(\alpha+\beta)}{\alpha^2} J_i(c_2,c_1) + \sum_{i=1}^{m} \frac{\alpha+\beta}{\alpha} I_i(c_2,c_1) \\
&\quad + \frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2}])w(t).
\end{aligned}$$

Similarly, let $r_3 = \min_{t\in[0,1]} g(t,c_1)$, $r_4 = \max_{t\in[0,1]} g(t,c_2)$. From (H2) and lemma 3.5, we get

$$r_3 w(t) \leq Bw(t) \leq r_4 w(t).$$

From (H1) and (H2), we have

$$r_1 = \min_{t\in[0,1]} f(t,c_1,c_2) > 0, r_3 = \min_{t\in[0,1]} g(t,c_1) > 0,$$

and in consequence,

$$\begin{aligned}
&r_2 + \frac{1}{c_1}[\sum_{i=1}^{m} \frac{\beta(\alpha+\beta)}{\alpha^2} J_i(c_2,c_1) + \sum_{i=1}^{m} \frac{\alpha+\beta}{\alpha} I_i(c_2,c_1) \\
&+ \frac{\alpha\eta_1+\beta(\eta_1-\eta_2)}{\alpha^2}] > r_1 > 0, \\
&r_4 > r_3 > 0.
\end{aligned}$$

So we prove that $A(w,w) \in P_w$, $Bw \in P_w$. Hence the condition (i) of Lemma 2.7 is satisfied.

For any $u, v \in P$ and $t \in (0,1)$, taking (H4) into consideration, we get

$$A(u,v)(t) = \int_0^1 G_1(t,s)f(s,u(s),v(s))ds + \sum_{i=1}^{m} G_2(t,t_i)J_i(u(t_i),v(t_i))$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} G_3(t,t_i)I_i(u(t_i),v(t_i)) + G_4(t),\\
&\geq \int_0^1 G_1(t,s)f(s,u(s),v(s))ds\\
&\geq \delta_0 \int_0^1 G_1(t,s)g(s,u(s))ds = \delta_0 Bu(t).
\end{aligned}
$$

Then we get $A(u,v) \geq \delta_0 Bu$, for all $u, v \in P$. So the condition (ii) of Lemma 2.7 is satisfied.

Finally, by means of Lemma 2.7, the problem (1) has a unique positive solution u^* in P_w. Moreover for any initial values $u_0, v_0 \in P_w$, constructing successively the sequences

$$
\begin{aligned}
u_n(t) &= \int_0^1 G_1(t,s)(f(s,u_{n-1}(s),v_{n-1}(s)) + g(s,u_{n-1}(s)))ds\\
&\quad + \sum_{i=1}^{m} G_2(t,t_i)J_i(u_{n-1}(t_i),v_{n-1}(t_i))\\
&\quad + \sum_{i=1}^{m} G_3(t,t_i)I_i(u_{n-1}(t_i),v_{n-1}(t_i)) + G_4(t),\\
v_n(t) &= \int_0^1 G_1(t,s)(f(s,v_{n-1}(s),u_{n-1}(s)) + g(s,v_{n-1}(s)))ds\\
&\quad + \sum_{i=1}^{m} G_2(t,t_i)J_i(v_{n-1}(t_i),u_{n-1}(t_i))\\
&\quad + \sum_{i=1}^{m} G_3(t,t_i)I_i(v_{n-1}(t_i),u_{n-1}(t_i)) + G_4(t), n = 1,2,\cdots
\end{aligned}
$$

we have $u_n(t) \to u^*(t), v_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s), G_2(t,t_i), G_3(t,t_i), G_4(t)$ are given as (3.2)-(3.5). □

Corollary 3.7. *Assume that* (H1), (H2) *and* (H4) *hold. Then the problem* (2) *has a unique positive solution u^* in P_w. Moreover for any initial values $u_0, v_0 \in P_w$, constructing successively the sequences*

$$
\begin{aligned}
u_n(t) &= \int_0^1 G_1(t,s)(f(s,u_{n-1}(s),v_{n-1}(s)) + g(s,u_{n-1}(s)))ds + G_4(t),\\
v_n(t) &= \int_0^1 G_1(t,s)(f(s,v_{n-1}(s),u_{n-1}(s)) + g(s,v_{n-1}(s)))ds\\
&\quad + G_4(t), n = 1,2,\cdots
\end{aligned}
$$

we have $u_n(t) \to u^*(t), v_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s)$, $G_4(t)$ are given as (3.2),(3.5).

If $g(t,u(t)) \equiv 0$, by means of Remark 2.8, we have the following two corollaries.

Corollary 3.8. *Assume that* (H1) *and* (H3) *hold. Then the problem* (1) *has a unique positive solution u^* in P_w. Moreover for any initial values $u_0, v_0 \in P_w$, constructing successively the sequences*

$$
\begin{aligned}
u_n(t) &= \int_0^1 G_1(t,s)f(s,u_{n-1}(s),v_{n-1}(s))ds\\
&\quad + \sum_{i=1}^{m} G_2(t,t_i)J_i(u_{n-1}(t_i),v_{n-1}(t_i))
\end{aligned}
$$

$$+\sum_{i=1}^{m} G_3(t,t_i)I_i(u_{n-1}(t_i),v_{n-1}(t_i))+G_4(t),$$

$$v_n(t)=\int_0^1 G_1(t,s)f(s,v_{n-1}(s),u_{n-1}(s))ds$$
$$+\sum_{i=1}^{m} G_2(t,t_i)J_i(v_{n-1}(t_i),u_{n-1}(t_i))$$
$$+\sum_{i=1}^{m} G_3(t,t_i)I_i(v_{n-1}(t_i),u_{n-1}(t_i))+G_4(t), n=1,2,\cdots$$

we have $u_n(t)\to u^*(t), v_n(t)\to u^*(t)$ as $n\to\infty$, where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are given as (3.2)-(3.5).

Corollary 3.9. *Assume that* (H1) *hold. Then the problem* (2) *has a unique positive solution u^* in P_w. Moreover for any initial values $u_0, v_0\in P_w$, constructing successively the sequences*

$$u_n(t)=\int_0^1 G_1(t,s)f(s,u_{n-1}(s),v_{n-1}(s))ds+G_4(t),$$
$$v_n(t)=\int_0^1 G_1(t,s)f(s,v_{n-1}(s),u_{n-1}(s))ds+G_4(t), n=1,2,\cdots$$

we have $u_n(t)\to u^*(t), v_n(t)\to u^*(t)$ as $n\to\infty$, where $G_1(t,s)$, $G_4(t)$ are given as (3.2),(3.5).

If $f(t,u(t),u(t))$, $I_k(u(t_k),u(t_k))$ and $J_k(u(t_k),u(t_k))$ are replaced by $f(t,u(t))$, $I_k(u(t_k))$ and $J_k(u(t_k))$ respectively, we can obtain the following new results.

Theorem 3.10. *Assume that* (H2) *and* (H5)-(H7) *hold. Then the problem* (1) *has a unique positive solution u^* in P_w. Moreover for any initial value $u_0\in P_w$, constructing successively the sequence*

$$u_n(t)=\int_0^1 G_1(t,s)(f(s,u_{n-1}(s))+g(s,u_{n-1}(s)))ds$$
$$+\sum_{i=1}^{m} G_2(t,t_i)J_i(u_{n-1}(t_i))+\sum_{i=1}^{m} G_3(t,t_i)I_i(u_{n-1}(t_i))$$
$$+G_4(t), n=1,2,\cdots$$

we have $u_n(t)\to u^*(t)$ as $n\to\infty$, where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are given as (3.2)-(3.5).

Proof. To begin with, by Lemma 3.1, the problem (1) has an integral formulation given by

$$u(t)=\int_0^1 G_1(t,s)(f(s,u(s))+g(s,u(s)))ds+\sum_{i=1}^{m} G_2(t,t_i)J_i(u(t_i))$$
$$+\sum_{i=1}^{m} G_3(t,t_i)I_i(u(t_i))+G_4(t), t\in J,$$

Define two operators $A: P\to E$ and $B: P\to E$ by

$$Au(t)=\int_0^1 G_1(t,s)f(s,u(s))ds+\sum_{i=1}^{m} G_2(t,t_i)J_i(u(t_i))$$
$$+\sum_{i=1}^{m} G_3(t,t_i)I_i(u(t_i))+G_4(t),$$

$$Bu(t) = \int_0^1 G_1(t,s)g(s,u(s))ds.$$

It is obvious that u is the solution of the problem (1) if and only if $u = Au + Bu$. We know from (H2),(H5) and(H6) that $A : P \to P$ and $B : P \to P$. We check that A, B satisfy all assumptions of Lemma 2.9 in the sequel.

Firstly, we prove that $A : P \to P$ be an increasing γ-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. For any $u \in P$ and $\gamma \in (0,1)$, let $\gamma^\beta = \min\{\gamma^{\beta_1}, \gamma^{\beta_2}, \gamma^{\beta_3}\}$, then $\gamma^\beta \in (0,1)$. From (H5) and (H6), we know that

$$\begin{aligned} A(\gamma u)(t) &= \int_0^1 G_1(t,s)f(s,\gamma u(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(\gamma u(t_i)) \\ &\quad + \sum_{i=1}^m G_3(t,t_i)I_i(\gamma u(t_i)) + G_4(t) \\ &\geq \gamma^\beta [\int_0^1 G_1(t,s)f(s,u(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) \\ &\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)) + G_4(t)] = \gamma^\beta Au(t). \end{aligned}$$

Hence the operator A is γ-concave operator. From (H5) that A is increasing. So $A : P \to P$ is an increasing γ-concave operator. We know from the proof of Theorem 3.6 that the operator B is an increasing sub-homogeneous operator.

Next we verify that conditions (i) and (ii) of Lemma 2.9. Similarly to the proof of Theorem 3.6. We prove that $Aw \in P_w$, $Bw \in P_w$. Hence the condition (i) of Lemma 2.9 is satisfied. For any $u, v \in P$ and $t \in (0,1)$, taking (H7) into consideration, we get

$$\begin{aligned} Au(t) &= \int_0^1 G_1(t,s)f(s,u(s))ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) \\ &\quad + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)) + G_4(t), \\ &\geq \int_0^1 G_1(t,s)f(s,u(s))ds \geq \delta_0 \int_0^1 G_1(t,s)g(s,u(s))ds = \delta_0 Bu(t). \end{aligned}$$

Then we get $Au \geq \delta_0 Bu$, for all $u \in P$. So the condition (ii) of Lemma 2.7 is satisfied.

Finally, by means of Lemma 2.9, the problem (1) has a unique positive solution u^* in P_w. Moreover for any initial value $u_0 \in P_w$, constructing successively the sequence

$$\begin{aligned} u_n(t) &= \int_0^1 G_1(t,s)(f(s,u_{n-1}(s)) + g(s,u_{n-1}(s)))ds \\ &\quad + \sum_{i=1}^m G_2(t,t_i)J_i(u_{n-1}(t_i)) + \sum_{i=1}^m G_3(t,t_i)I_i(u_{n-1}(t_i)) \\ &\quad + G_4(t), n = 1,2,\cdots \end{aligned}$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are given as (3.2)-(3.5). □

Corollary 3.11. *Assume that* (H2), (H5) *and* (H7) *hold. Then the problem* (2) *has a unique positive solution u^* in P_w. Moreover for any initial value $u_0 \in P_w$, constructing successively the sequence*

$$u_n(t) = \int_0^1 G_1(t,s)(f(s,u_{n-1}(s)) + g(s,u_{n-1}(s)))ds + G_4(t), n = 1,2,\cdots$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s)$, $G_4(t)$ are given as (3.2) and (3.5).

If $g(t,u(t)) \equiv 0$, by means of Remark 2.10, we have following two corollaries.

Corollary 3.12. *Assume that* (H5) *and* (H6) *hold. Then the problem* (1) *has a unique positive solution u^* in P_w. Moreover for any initial value $u_0 \in P_w$, constructing successively the sequence*

$$u_n(t) = \int_0^1 G_1(t,s)f(s,u_{n-1}(s))ds + \sum_{i=1}^{m} G_2(t,t_i)J_i(u_{n-1}(t_i)) + \sum_{i=1}^{m} G_3(t,t_i)I_i(u_{n-1}(t_i)) + G_4(t), n = 1,2,\cdots$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $G_4(t)$ are given as (3.2)-(3.5).

Corollary 3.13. *Assume that* (H5) *hold. Then the problem* (2) *has a unique positive solution u^* in P_w. Moreover for any initial value $u_0 \in P_w$, constructing successively the sequence*

$$u_n(t) = \int_0^1 G_1(t,s)f(s,u_{n-1}(s))ds + G_4(t), n = 1,2,\cdots$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G_1(t,s)$, $G_4(t)$ are given as (3.2) and (3.5).

Remark 3.14. *Comparing Theorems 3.6 and 3.10 with the main results in [2, 3, 16, 17, 19, 23–26], our new results do not demand that the nonlinear term and the impulse functions are bounded or satisfy Lipschitz conditions, clearly, these conditions are very strong. Moreover, our new results can guarantee the existence of a unique positive solution without assuming operators to be completely continuous or compactness condition and an iterative sequence is constructed to approximate it. Thus, the results of the above-mentioned works are generalized and significantly improved.*

4. Illustrative examples

Example 4.1. *Consider the BVPs of impulsive fractional differential equations:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2}}u(t) = (u(t))^{\frac{1}{3}} + (u(t))^{-\frac{1}{3}} + \arctan u(t) + t^2 + t + \dfrac{\pi}{2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0,1], t \neq \dfrac{1}{2}, \\ \Delta u(\dfrac{1}{2}) = (u(\dfrac{1}{2}))^{\frac{1}{4}} + (u(\dfrac{1}{2}))^{-\frac{1}{4}}, \Delta u'(\dfrac{1}{2}) = (u(\dfrac{1}{2}))^{\frac{1}{5}} + (u(\dfrac{1}{2}))^{-\frac{1}{5}}, \\ u(0) + 2u'(0) = 3, u(1) + 2u'(1) = 2. \end{cases} \tag{20}$$

In this case,$q = \frac{3}{2}$,$t_1 = \frac{1}{2}$, $\alpha = 1, \beta = 2$, $\eta_1 = 3, \eta_2 = 2$, and

$$f(t,u(t),v(t)) = (u(t))^{\frac{1}{3}} + (v(t))^{-\frac{1}{3}} + t + \frac{\pi}{2},$$
$$g(t,u(t)) = \arctan u(t) + t^2,$$
$$I_1(u(t_1),v(t_1)) = (u(\frac{1}{2}))^{\frac{1}{4}} + (v(\frac{1}{2}))^{-\frac{1}{4}},$$
$$J_1(u(t_1),v(t_1)) = (u(\frac{1}{2}))^{\frac{1}{5}} + (v(\frac{1}{2}))^{-\frac{1}{5}}.$$

Now we verify that conditions (H1)-(H4) of Theorem 3.6 are satisfied. By a simple computation, we have

$$w(t) = \int_0^1 G_1(t,s)ds = \frac{1}{3\Gamma(\frac{3}{2})}(2t^{\frac{3}{2}} - 8t + 16),$$
$$c_1 = \min_{t\in[0,1]} w(t) = \frac{10}{3\Gamma(\frac{3}{2})}, c_2 = \max_{t\in[0,1]} w(t) = \frac{16}{3\Gamma(\frac{3}{2})}.$$

Then, $f(t,c_1,c_2) > 0$, $g(t,c_1) > 0$. Moreover, for any $\mu \in (0,1)$, $\gamma \in (0,1)$, $t \in [0,1]$, $u,v \in [0,\infty)$, we get

$$\begin{aligned} f(t,\gamma u,\gamma^{-1}v) &= \gamma^{\frac{1}{3}}u^{\frac{1}{3}} + \gamma^{\frac{1}{3}}v^{-\frac{1}{3}} + t + \frac{\pi}{2} \\ &\geq \gamma^{\frac{5}{12}}(u^{\frac{1}{3}} + v^{-\frac{1}{3}} + t + \frac{\pi}{2}) = \gamma^{\frac{5}{12}} f(t,u,v), \\ I_1(\gamma u,\gamma^{-1}v) &= \gamma^{\frac{1}{4}}u^{\frac{1}{4}} + \gamma^{\frac{1}{4}}v^{-\frac{1}{4}} \geq \gamma^{\frac{1}{3}}(u^{\frac{1}{4}} + v^{-\frac{1}{4}}) = \gamma^{\frac{1}{3}} I_1(u,v), \\ J_1(\gamma u,\gamma^{-1}v) &= \gamma^{\frac{1}{5}}u^{\frac{1}{5}} + \gamma^{\frac{1}{5}}v^{-\frac{1}{5}} \geq \gamma^{\frac{4}{15}}(u^{\frac{1}{5}} + v^{-\frac{1}{5}}) = \gamma^{\frac{4}{15}} J_1(u,v), \\ g(t,\mu u) &= \arctan \mu u + t^2 \geq \mu \arctan u + t^2 \geq \mu(\arctan u + t^2) = \mu g(t,u). \end{aligned}$$

From the expressions of $f(t,u(t),v(t))$, $g(t,u(t))$, $I_1(u(t_1),v(t_1))$ and $J_1(u(t_1),v(t_1))$, it is obvious that (H1)-(H3) hold. For $t \in [0,1]$, $u,v \in [0,\infty)$, there exists a constant $\delta_0 > 0$ such that

$$\begin{aligned} f(t,u(t),v(t)) &= (u(t))^{\frac{1}{3}} + (v(t))^{-\frac{1}{3}} + t + \frac{\pi}{2} \geq t + \frac{\pi}{2} \\ &\geq \arctan u(t) + t^2 \geq \delta_0(\arctan u(t) + t^2) = \delta_0 g(t,u(t)). \end{aligned}$$

Thus (H4) is proved. We know from Theorem 3.6 that the problem (20) has a unique positive solution in P_w.

Example 4.2. *Consider the BVPs of impulsive fractional differential equations:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2}}u(t) = (u(t))^{\frac{1}{3}} + \arctan u(t) + t^2 + t + \frac{\pi}{2}, t \in [0,1], t \neq \frac{1}{2}, \\ \Delta u(\frac{1}{2}) = (u(\frac{1}{2}))^{\frac{1}{4}}, \Delta u'(\frac{1}{2}) = (u(\frac{1}{2}))^{\frac{1}{5}}, \\ u(0) + 2u'(0) = 3, u(1) + 2u'(1) = 2. \end{cases} \tag{21}$$

In this case,$q = \frac{3}{2}$,$t_1 = \frac{1}{2}$, $\alpha = 1, \beta = 2$, $\eta_1 = 3, \eta_2 = 2$, and

$$f(t,u(t)) = (u(t))^{\frac{1}{3}} + t + \frac{\pi}{2}, g(t,u(t)) = \arctan u(t) + t^2,$$
$$I_1(u(t_1)) = (u(\frac{1}{2}))^{\frac{1}{4}}, J_1(u(t_1)) = (u(\frac{1}{2}))^{\frac{1}{5}}.$$

Now we verify that conditions (H2) and (H5)-(H7) of Theorem 3.10 are satisfied. From the proof of Example 4.1, we know that $f(t,c_1) > 0$. Moreover, for any $\mu \in (0,1)$, $\gamma \in (0,1)$, $t \in [0,1]$, $u \in [0,\infty)$, we get

$$f(t,\gamma u) = \gamma^{\frac{1}{3}}u^{\frac{1}{3}} \geq \gamma^{\frac{5}{12}} f(t,u), g(t,\mu u) = \arctan \mu u + t^2 \geq \mu g(t,u).$$
$$I_1(\gamma u) = \gamma^{\frac{1}{4}}u^{\frac{1}{4}} \geq \gamma^{\frac{1}{3}} I_1(u), J_1(\gamma u) = \gamma^{\frac{1}{5}}u^{\frac{1}{5}} \geq \gamma^{\frac{4}{15}} J_1(u).$$

From the expressions of $f(t,u(t))$, $g(t,u(t))$, $I_1(u(t_1)))$, $J_1(u(t_1))$, it is obvious that (H2) and (H5)-(H6) hold. For $t \in [0,1]$, $u \in [0,\infty)$, there exists a constant $\delta_0 > 0$ such that

$$\begin{aligned} f(t,u(t)) &= (u(t))^{\frac{1}{3}} + t + \frac{\pi}{2} \geq t + \frac{\pi}{2} \\ &\geq \arctan u(t) + t^2 \geq \delta_0(\arctan u(t) + t^2) = \delta_0 g(t,u(t)). \end{aligned}$$

Thus (H7) is proved. We know from Theorem 3.10 that the problem (21) has a unique positive solution in P_w.

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