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Weak Solutions for a $(p(z), q(z))$-Laplacian Dirichlet Problem

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Abstract. We establish the existence of a nontrivial and nonnegative solution for a double phase Dirichlet problem driven by a $(p(z), q(z))$-Laplacian operator plus a potential term. Our approach is variational, but the reaction term f need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition.

1. Introduction

In this paper we are interested in the existence of a nontrivial and nonnegative solution for the following class of double phase problems:

$$\begin{cases} -\operatorname{div}\,(a(z)|\nabla u|^{p(z)-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(z)-2}\nabla u) + b(z)|u|^{p(z)-2}u = f(z, u(z)) \quad \text{in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where

(a) $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary;

(b) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is

$$z \to f(z, \xi) \text{ is measurable for each } \xi \in \mathbb{R},$$

$$\xi \to f(z, \xi) \text{ is continuous for a.a. } z \in \Omega;$$

(c) $p, q \in C(\overline{\Omega})$ are such that $q(z) < p(z)$ for all $z \in \overline{\Omega}$ and

$$1 < q^- := \inf_{z\in\Omega} q(z) \le q(z) \le q^+ := \sup_{z\in\Omega} q(z) < +\infty,$$
$$1 < p^- := \inf_{z\in\Omega} p(z) \le p(z) \le p^+ := \sup_{z\in\Omega} p(z) < +\infty;$$

(d) $a, b \in L^\infty(\Omega)$ are such that $0 < a_0 \le a(z)$ and $0 \le b_0 < b(z)$ for all $z \in \Omega$.

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The study of double phase problems involving variable growth conditions is motivated by their applications in mathematical physics. For example, they are useful tools to model non-Newtonian fluids changing their viscosity when electro-magnetic fields interfer. Several authors have given their contributions to the study of nonlinear problems with unbalanced growth. We start pointing out that Marcellini in [11] established regularity results of minimizers in the abstract setting of quasiconvex integrals. These kind of problems have a key role in modelling elastic body deformation and nonlinear elasticity phenomena. In this direction we recall two Zhikov's papers [22, 23], that provide models for strongly anisotropic materials in the framework of homogenization. The associated functionals also demonstrated their importance in studying duality theory and Lavrentiev phenomenon [21]. In this direction, several results can be found in different papers by Mingione et al. [1, 2, 5, 6], which are linked to Zhikov's papers [22, 23]. Also, Papageorgiou et al. in [15] consider a double phase eigenvalue problem driven by the (p, q)-Laplacian plus an indefinite and unbounded potential, with a Robin boundary condition. For other remarkable papers dealing with regularity and existence of solutions of elliptic double phase problems involving variable exponents see, for example, [3, 10, 14, 19, 20]. For some results with constant exponents see [13, 17, 18].

The motivation behind this study is given by some recent papers dealing with nonlinear problems with unbalanced growth whose main results are briefly collected in what follows. Let

$$\mathcal{F}(u) = \int_\Omega a(z)|\nabla u|^{p(z)}dz + \int_\Omega c(z)|\nabla u|^{q(z)}dz + \int_\Omega b(z)|u|^{p(z)}dz, \tag{2}$$

where $1 < q(z) < p(z)$ and $a(z)$, $b(z)$, $c(z) \geq 0$ for all $z \in \Omega$.

Regularity results for minimizers of (2) with $a(z) \geq 0$, $b(z) = 0$, $c(z) = 1$ for all $z \in \Omega$ can be found in [5].

The case $c \equiv 0$ has been studied by Chabrowski and Fu in [4]. In fact, they established existence of a nontrivial and nonnegative weak solution for the following $p(z)$-Laplacian Dirichlet problem

$$\begin{cases} -\operatorname{div}\,(a(z)|\nabla u|^{p(z)-2}\nabla u) + b(z)|u|^{p(z)-2}u = f(z, u(z)) \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

In [14], Papageorgiou and Vetro have proved the existence of one and three non trivial weak solutions for Dirichlet boundary value problems driven by a $(p(z), q(z))$-Laplacian operator, with $a(z) = c(z) = 1$ and $b(z) = 0$ for all $z \in \Omega$, that is

$$\begin{cases} -\operatorname{div}\,(|\nabla u|^{p(z)-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(z)-2}\nabla u) = f(z, u(z)) \quad \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

The aim of this paper is to extend these results to the case $a(z), b(z) > 0$ and $c(z) = 1$ for all $z \in \Omega$, that is Problem (1), in the setting of superlinear (see Section 3) and sublinear (see Section 4) growth of f. We point out that we do not employ the Ambrosetti-Rabinowitz condition, which is common in the literature when dealing with superlinear problems. In the last section (namely Section 5), we consider the parametrical problem

$$\begin{cases} -\operatorname{div}\,(a(z)|\nabla u|^{p(z)-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(z)-2}\nabla u) + b(z)|u|^{p(z)-2}u = \lambda f(z, u(z)) \quad \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda > 0$. In the parametric setting, using the results obtained in Section 3, we deduce the existence of a nontrivial and nonnegative weak solution u_λ for all $\lambda > 0$. Furthermore, we show that for the solution u_λ, we have $\|u_\lambda\| \to +\infty$ as $\lambda \to 0^+$.

2. Mathematical background

In this section, we collect some basic properties of Lebesgue and Sobolev spaces with variable exponent. We recall that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary. We set

$$\mathcal{M}_\Omega = \{u : \Omega \to \mathbb{R} : \text{ u is measurable}\}.$$

Let $\rho_p : \mathcal{M}_\Omega \to \mathbb{R} \cup \{+\infty\}$ be the mapping defined by

$$\rho_p(u) := \int_\Omega |u(z)|^{p(z)} dz. \tag{3}$$

We consider the variable exponent Lebesgue space $L^{p(z)}(\Omega)$ given as

$$L^{p(z)}(\Omega) = \left\{ u \in \mathcal{M}_\Omega : \rho_p(u) < +\infty \right\},$$

equipped with the Luxemburg norm, that is

$$\|u\|_{L^{p(z)}(\Omega)} := \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(z)}{\lambda} \right|^{p(z)} dz \leq 1 \right\}.$$

Consequently, the generalized Lebesgue-Sobolev space $W^{1,p(z)}(\Omega)$ is given by

$$W^{1,p(z)}(\Omega) := \{u \in L^{p(z)}(\Omega) : |\nabla u| \in L^{p(z)}(\Omega)\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(z)}(\Omega)} = \|u\|_{L^{p(z)}(\Omega)} + \|\, |\nabla u|\, \|_{L^{p(z)}(\Omega)}. \tag{4}$$

We define $W_0^{1,p(z)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(z)}(\Omega)$.

From [8] we have that $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$ endowed with the above norms, are separable, reflexive and uniformly convex Banach spaces. Let $p \in C(\overline{\Omega})$, we recall that the critical Sobolev exponent p^* of p is given by

$$p^*(z) = \frac{Np(z)}{N - p(z)} \text{ if } p(z) < N \quad \text{and} \quad p^*(z) = +\infty \text{ if } p(z) \geq N.$$

We recall the following embedding theorem.

Proposition 2.1 ([9]). *Assume that $p \in C(\overline{\Omega})$ with $p(z) > 1$ for each $z \in \overline{\Omega}$. If $\beta \in C(\overline{\Omega})$ and $1 < \beta(z) < p^*(z)$ for all $z \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(z)}(\Omega) \hookrightarrow L^{\beta(z)}(\Omega)$.*

Throughout the paper the embedding constant of $W^{1,p(z)}(\Omega) \hookrightarrow L^{\beta(z)}(\Omega)$ is denoted by C_β. In addition, from Theorem 1.11 of [9], we deduce that the embedding $L^{p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$ is continuous, whenever $q, p \in C(\overline{\Omega})$ and $1 < q(z) < p(z)$ for all $z \in \Omega$.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular of the $L^{p(z)}(\Omega)$ space, which is the mapping ρ_p defined in (3).

Theorem 2.2 ([9]). *Let $u \in L^{p(z)}(\Omega)$. Then we have that*

(i) $\|u\|_{L^{p(z)}(\Omega)} < 1\ (= 1, > 1) \Leftrightarrow \rho_p(u) < 1\ (= 1, > 1)$;

(ii) *if* $\|u\|_{L^{p(z)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(z)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^+}_{L^{p(z)}(\Omega)}$;

(iii) *if* $\|u\|_{L^{p(z)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(z)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^-}_{L^{p(z)}(\Omega)}$.

It is well known that the norm $\|u\|_{W^{1,p(z)}(\Omega)}$ is equivalent to the norm $\|\, |\nabla u|\, \|_{L^{p(z)}(\Omega)}$ on $W_0^{1,p(z)}(\Omega)$, in virtue of the following Poincaré inequality ([7], Theorem 8.2.18)

$$\|u\|_{L^{p(z)}(\Omega)} \leq c\, \||\nabla u|\|_{L^{p(z)}(\Omega)} \quad \text{for some } c > 0, \text{ all } u \in W_0^{1,p(z)}(\Omega).$$

As a consequence, from now on, we will consider the norm $\|u\| = \| |\nabla u| \|_{L^{p(z)}(\Omega)}$ on $W_0^{1,p(z)}(\Omega)$ instead of the one given in (4).

A function $u \in W_0^{1,p(z)}(\Omega)$ is a weak solution of problem (1) if

$$\int_\Omega a(z)|\nabla u|^{p(z)-2}\nabla u \nabla w dz + \int_\Omega |\nabla u|^{q(z)-2}\nabla u \nabla w dz + \int_\Omega b(z)|u|^{p(z)-2}uw dz = \int_\Omega f(z,u)w dz, \tag{5}$$

for each $w \in W_0^{1,p(z)}(\Omega)$.

Now, we consider the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$F(z,t) = \int_0^t f(z,\xi)d\xi \quad \text{for all } t \in \mathbb{R},\ z \in \Omega,$$

and the functional $I : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ given as

$$I(u) = \int_\Omega F(z,u)\,dz, \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Suitable assumptions in the sequel (namely (H_1), (H_5)) ensure that $I \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$ and the embedding given by Proposition 2.1 implies that I admits the following compact derivative

$$\langle I'(u), w\rangle = \int_\Omega f(z,u)w\,dz, \quad \text{for all } u, w \in W_0^{1,p(z)}(\Omega).$$

To problem (1) we associate the functional $J : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \frac{a(z)}{p(z)}|\nabla u|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla u|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|u|^{p(z)}dz - I(u) \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

We say that u is a critical point of J if it satisfies

$$\langle J'(u), w\rangle = \int_\Omega a(z)|\nabla u|^{p(z)-2}\nabla u \nabla w dz + \int_\Omega |\nabla u|^{q(z)-2}\nabla u \nabla w dz + \int_\Omega b(z)|u|^{p(z)-2}uw dz - \int_\Omega f(z,u)w dz = 0$$

for all $w \in W_0^{1,p(z)}(\Omega)$. So, from the definition of weak solutions of problem (1), we deduce that they coincide with the critical points of J.

3. Supercritical case

In this section, we prove that problem (1) has at least one nontrivial and nonnegative weak solution. Later on, we denote with $\mathbb{R}^+$ the set of positive real numbers. We consider the following set of hypotheses:

(H_0) $f \in C(\bar{\Omega} \times \mathbb{R})$, $f(z,\xi) = 0$ for all $z \in \Omega$ and $\xi \leq 0$;

(H_1) there exist $\alpha \in C(\overline{\Omega})$ such that $p^+ < \alpha^- \leq \alpha^+ < p^*(z)$ for all $z \in \overline{\Omega}$ and $a_1, a_2 \in [0, +\infty[$ such that

$$|f(z,\xi)| \leq a_1 + a_2\xi^{\alpha(z)-1} \quad \text{for all } (z,\xi) \in \Omega \times \mathbb{R}^+;$$

(H_2) there exists $\epsilon \in \left]0, \frac{a_0}{C_{p^+}^{p^+}}\right[$ e $\delta > 0$ such that $F(z,t) \leq \frac{\epsilon}{p^+}t^{p^+}$ for a.a. $z \in \Omega$, all $0 < t < \delta$, where C_{p^+} denotes the embedding constant of $W^{1,p(z)}(\Omega) \hookrightarrow L^{p^+}(\Omega)$;

(H_3) $\lim\limits_{t\to+\infty} \dfrac{F(z,t)}{t^{p^+}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(H_4) there exists $d \in L^1(\Omega)$ such that

$$e(z,t) \le e(z,s) + d(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 < t < s \text{ , where } e(z,t) = f(z,t)t - p^+F(z,t).$$

We need the following notion of $(C)_c$ condition. Let X be a Banach space and X^* its topological dual.

Definition 3.1. *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$. We say that J satisfies the $(C)_c$ condition if any sequence $\{u_n\} \subset X$ such that*

(*i*) $J(u_n) \to c \in \mathbb{R}$ *as* $n \to +\infty$

(*ii*) $(1 + \|u_n\|)J'(u_n) \to 0$ *in* X^* *as* $n \to +\infty$

has a convergent subsequence. A sequence satisfying conditions (i) and (ii) is said $(C)_c$ sequence.

For the following Hölder inequality see [16], p. 8.

Proposition 3.2 (Hölder inequality). *Let $L^{p'(z)}(\Omega)$ the conjugate space of $L^{p(z)}(\Omega)$, where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$. For any $u \in L^{p(z)}(\Omega)$ and $v \in L^{p'(z)}(\Omega)$ the Hölder type inequality holds, that is*

$$\left| \int_\Omega uv\, dz \right| \le 2\, \|u\|_{L^{p(z)}(\Omega)} \|v\|_{L^{p'(z)}(\Omega)}. \tag{6}$$

Remark 3.3 (see [12], p. 25). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a bounded domain, $1 < p(z) < +\infty$ for all $z \in \Omega$. Then the following inequalities hold for all $u, v \in \mathbb{R}^N$:*

(i) $|u - v|^2 \le c_1(u - v)(|u|^{p(z)-2}u - |v|^{p(z)-2}v)(|u| + |v|)^{2-p(z)}$ *if* $1 < p(z) < 2$;

(ii) $|u - v|^{p(z)} \le c_2(|u|^{p(z)-2}u - |v|^{p(z)-2}v)(u - v)$ *if* $p(z) \ge 2$.

Lemma 3.4. *Let (H_1) hold and $\{u_n\}$ be a bounded $(C)_c$ sequence. Then $\{u_n\}$ admits a convergent subsequence.*

Proof. Let $\{u_n\}$ be a bounded sequence. The reflexivity of $W_0^{1,p(z)}(\Omega)$ ensures that, eventually passing to a subsequence still denoted with $\{u_n\}$, there exists $u \in W_0^{1,p(z)}(\Omega)$ such that $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$.

We consider the following partition of $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{z \in \Omega : p(z) < 2\} \quad \text{and} \quad \Omega_2 = \{z \in \Omega : p(z) \ge 2\}.$$

We consider

$$\begin{aligned}
&\int_\Omega a(z)(|\nabla u_i|^{p(z)-2}\nabla u_i - |\nabla u_j|^{p(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j)dz \\
&\quad + \int_\Omega (|\nabla u_i|^{q(z)-2}\nabla u_i - |\nabla u_j|^{q(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j)dz \\
&\quad + \int_\Omega b(z)(|u_i|^{p(z)-2}u_i - |u_j|^{p(z)-2}u_j)(u_i - u_j)dz \\
&\le |\langle J'(u_i), u_i - u_j\rangle| + |\langle J'(u_j), u_i - u_j\rangle| + \left|\int_\Omega (f(z,u_i) - f(z,u_j))(u_i - u_j)dz\right| \\
&\le C(\|J'(u_i)\|_{W^{1,p(z)}(\Omega)^*} + \|J'(u_j)\|_{W^{1,p(z)}(\Omega)^*} + \|I'(u_i) - I'(u_j)\|_{W^{1,p(z)}(\Omega)^*}) \to 0.
\end{aligned} \tag{7}$$

On the one hand, using Proposition 3.3 (i) and Hölder inequality (6), we obtain

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(z)} dz \\
&\leq C_1 \int_{\Omega_1} \left((|\nabla u_i|^{p(z)-2}\nabla u_i - |\nabla u_j|^{p(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(z)}{2}} (|\nabla u_i|^{p(z)} + |\nabla u_j|^{p(z)})^{\frac{2-p(z)}{2}} dz \\
&\leq 2C_1 \left\| \left((|\nabla u_i|^{p(z)-2}\nabla u_i - |\nabla u_j|^{p(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(z)}{2}} \right\|_{L^{\frac{2}{p(z)}}(\Omega_1)} \|(|\nabla u_i|^{p(z)} + |\nabla u_j|^{p(z)})^{\frac{2-p(z)}{2}}\|_{L^{\frac{2}{2-p(z)}}(\Omega_1)}.
\end{aligned}$$

By (7) we deduce

$$\left\| \left((|\nabla u_i|^{p(z)-2}\nabla u_i - |\nabla u_j|^{p(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j) \right)^{\frac{p(z)}{2}} \right\|_{L^{\frac{2}{p(z)}}(\Omega_1)} \to 0. \tag{8}$$

Since $\int_{\Omega_1} (|\nabla u_i|^{p(z)} + |\nabla u_j|^{p(z)})^{\frac{2-p(z)}{2}\cdot\frac{2}{2-p(z)}} dz$ is bounded, by (8),

$$\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(z)} dz \to 0. \tag{9}$$

On the other hand, by Proposition 3.3 (ii) and (7), we have

$$\int_{\Omega_2} |\nabla u_i - \nabla u_j|^{p(z)} dz \leq c_2 \int_{\Omega_2} (|\nabla u_i|^{p(z)-2}\nabla u_i - |\nabla u_j|^{p(z)-2}\nabla u_j)(\nabla u_i - \nabla u_j) dz \to 0. \tag{10}$$

From (9) and (10), we infer that $\||\nabla u_i - \nabla u_j|\|_{L^{p(z)}(\Omega)} \to 0$ and hence $\|u_i - u_j\| \to 0$. That is $\{u_n\}$ is a Cauchy sequence, so it is convergent. This ends our proof. □

Lemma 3.5. *Let $(H_1), (H_3), (H_4)$ hold and let $\{u_n\}$ be a $(C)_c$ sequence such that*

$$\|u_n\| \to +\infty \text{ and } v_n := \frac{u_n}{\|u_n\|} \to v \in L^{p^+}(\Omega) \text{ and } L^{\alpha(z)}(\Omega) \quad \text{as } n \to +\infty.$$

Then the Lebesgue measure of the set $\Omega_0 := \{z \in \Omega : v(z) > 0\}$ is equal to zero.

Proof. Since by hypothesis $\|u_n\| \to +\infty$ as $n \to +\infty$, we can suppose that $\|u_n\| \geq 1$ for all $n \in \mathbb{N}$. Proceeding by contradiction we assume that $|\Omega_0| > 0$. Then for a.a. $z \in \Omega_0$ we have that $u_n(z) \to +\infty$ as $n \to +\infty$. By (H_3), we deduce that

$$\lim_{n\to+\infty} \frac{F(z, u_n)}{\|u_n\|^{p^+}} = \lim_{n\to+\infty} \frac{F(z, u_n)}{u_n^{p^+}} v_n^{p^+} = +\infty \quad \text{for a.a. } z \in \Omega_0. \tag{11}$$

By Fatou's lemma and (11), we get

$$\lim_{n\to+\infty} \int_{\Omega_0} \frac{F(z, u_n)}{\|u_n\|^{p^+}} dz = +\infty.$$

Thus,

$$\lim_{n\to+\infty} \int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^{p^+}} dz \geq \lim_{n\to+\infty} \int_{\Omega_0} \frac{F(z, u_n)}{\|u_n\|^{p^+}} dz = +\infty. \tag{12}$$

Since by hypothesis $J(u_n) \to c$, there exists a sequence $\{c_n\}$ with $c_n \to 0$ such that

$$\begin{aligned} c &= J(u_n) + c_n \\ &= \int_\Omega \frac{a(z)}{p(z)} |\nabla u_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla u_n|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} u_n^{p(z)} dz - \int_\Omega F(z, u_n) dz + c_n \\ &\geq \frac{a_0}{p^+} \|u_n\|^{p^-} - \int_\Omega F(z, u_n) dz + c_n, \end{aligned}$$

for all $n \in \mathbb{N}$. Then, we obtain

$$\int_\Omega F(z, u_n) dz \geq \frac{a_0}{p^+} \|u_n\|^{p^-} - c + c_n \to +\infty \quad \text{as } n \to +\infty. \tag{13}$$

Also, we have that

$$\begin{aligned} c &= J(u_n) + c_n \\ &= \int_\Omega \frac{a(z)}{p(z)} |\nabla u_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla u_n|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} u_n^{p(z)} dz - \int_\Omega F(z, u_n) dz + c_n \\ &\leq \frac{\|a\|_\infty}{p^-} \|u_n\|^{p^+} + \frac{1}{q^-} \max\left\{ \|\nabla u_n\|^{q^+}_{L^{q(z)}(\Omega)}, \|\nabla u_n\|^{q^-}_{L^{q(z)}(\Omega)} \right\} + C_2 \|u_n\|^{p^+} - \int_\Omega F(z, u_n) dz + c_n \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Theorem 2.2, for some } C_2 > 0) \\ &\leq C_3 \|u_n\|^{p^+} - \int_\Omega F(z, u_n) dz + c_n \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

where $C_3 = \frac{\|a\|_\infty}{p^-} + \frac{1}{q^-} \max\{C_q^{q^-}, C_q^{q^+}\} + C_2$ with C_q to denote the constant of the continuous embedding $L^{p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$. Thus, by (13), there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n\|^{p^+} \geq \frac{c}{C_3} + \frac{1}{C_3} \int_\Omega F(z, u_n) dz - \frac{c_n}{C_3} > 0 \quad \text{for all } n \geq n_0.$$

Therefore

$$\lim_{n \to +\infty} \int_\Omega \frac{F(z, u_n)}{\|u_n\|^{p^+}} dz \leq \lim_{n \to +\infty} \frac{\int_\Omega F(z, u_n) dz}{\frac{c}{C_3} + \frac{1}{C_3} \int_\Omega F(z, u_n) dz - \frac{c_n}{C_3}} = C_3,$$

which leads to contradiction with (12) and hence $|\Omega_0| = 0$. □

Remark 3.6. *Let $Z = \{u \in W_0 : u(z) \leq 0$ for all $z \in \Omega\}$. Let $\{u_n\} \subset Z$ be a $(C)_c$ sequence. We note that if $u_n \leq 0$ for all $n \in \mathbb{N}$, hypothesis (H_0) implies that $F(z, u_n) = 0$ for all $n \in \mathbb{N}$. Coercivity of functional*

$$J_{|Z}(u) = \int_\Omega \frac{a(z)}{p(z)} |\nabla u|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla u|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} |u|^{p(z)} dz,$$

ensures that $\{u_n\}$ is bounded.

Proposition 3.7. *If $(H_1), (H_3), (H_4)$ hold, then the functional J satisfies the $(C)_c$ condition for each $c > 0$.*

Proof. Let $\{u_n\}$ be a $(C)_c$ sequence in $W_0^{1,p(z)}(\Omega)$. We want to prove that $\{u_n\}$ is bounded. Proceeding by absurd, we assume that $\{u_n\}$ is unbounded. So it is not restrictive to suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$. We consider

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{for all } n \in \mathbb{N}.$$

Then, we assume that there exists $v \in W_0^{1,p(z)}(\Omega)$ such that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad v_n \to v \quad \text{in } L^{p^+}(\Omega) \text{ and } L^{\alpha(z)}(\Omega),$$

since $\|v_n\| = 1$ for all $n \in \mathbb{N}$. By Lemma 3.5 we have $v(z) \leq 0$ for a.a. $z \in \Omega$.

Now, for all u_n, the function $J(tu_n)$ is continuous in $[0,1]$ with respect to the variable t. Consequently, there exists $t_n \in [0,1]$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(tu_n).$$

Let $r_n = r^{\frac{1}{p^-}} v_n$ for some $r > 1$, all $n \in \mathbb{N}$. By (H_1) and Krasnoselskii's theorem (see [12], p. 41), since $v_n \to v$ in $L^{\alpha(z)}(\Omega)$ and $v_n(z) \to v(z) \leq 0$ for a.a. $z \in \Omega$ as $n \to +\infty$, we obtain that

$$\lim_{n \to +\infty} \int_\Omega F(z, r_n) dz = 0. \tag{14}$$

Now, (14) and $\|u_n\| \to +\infty$ ensure that there exists $n_1 \in \mathbb{N}$ such that

$$\int_\Omega F(z, r_n) dz < \frac{a_0 r}{2p^+} \quad \text{and} \quad 0 < \frac{r^{\frac{1}{p^+}}}{\|u_n\|} \leq 1 \quad \text{for all } n \geq n_1.$$

Thus

$$\begin{aligned} J(t_n u_n) &\geq J(r_n) \\ &= \int_\Omega \frac{a(z)}{p(z)} |\nabla r_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla r_n|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} |r_n|^{p(z)} dz - \int_\Omega F(z, r_n) dz \\ &\geq \frac{a_0}{p^+} \|r_n\|^{p^-} - \int_\Omega F(z, r_n) dz \quad (\|r_n\| = r^{\frac{1}{p^-}} > 1) \\ &\geq \frac{a_0 r}{p^+} - \frac{a_0 r}{2p^+} = \frac{a_0 r}{2p^+} \quad \text{for all } n \geq n_1. \end{aligned}$$

The arbitrarity of $r > 1$ implies that

$$J(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \tag{15}$$

Clearly, there exists n_2 such that $t_n \in]0,1[$ for all $n \geq n_2$, since $J(0) = 0$ and $J(u_n) \to c$. Consequently,

$$\frac{d}{dt} J(tu_n)\Big|_{t=t_n} = 0 \quad \Rightarrow \quad \langle J'(t_n u_n), t_n u_n \rangle = 0 \quad \text{for all } n \geq n_2.$$

So,

$$
\begin{aligned}
J(t_nu_n) &= J(t_nu_n) - \frac{1}{p^+}\langle J'(t_nu_n), t_nu_n\rangle \\
&= \int_\Omega \frac{a(z)}{p(z)}|\nabla t_nu_n|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla t_nu_n|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|t_nu_n|^{p(z)}dz - \int_\Omega F(z,t_nu_n)dz \\
&\quad - \frac{1}{p^+}\int_\Omega a(z)|\nabla t_nu_n|^{p(z)}dz - \frac{1}{p^+}\int_\Omega |\nabla t_nu_n|^{q(z)}dz - \frac{1}{p^+}\int_\Omega b(z)|t_nu_n|^{p(z)}dz + \frac{1}{p^+}\int_\Omega f(z,t_nu_n)t_nu_n(z)dz \\
&= \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+}\right] a(z)t_n^{p(z)}|\nabla u_n|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p^+}\right] t_n^{q(z)}|\nabla u_n|^{q(z)}dz + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+}\right] b(z)t_n^{p(z)}|u_n|^{p(z)}dz \\
&\quad + \frac{1}{p^+}\int_\Omega [f(z,t_nu_n)t_nu_n(z) - p^+F(z,t_nu_n)]dz \\
&\le \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+}\right] a(z)|\nabla u_n|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p^+}\right] |\nabla u_n|^{q(z)}dz + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+}\right] b(z)|u_n|^{p(z)}dz \\
&\quad + \frac{1}{p^+}\int_\Omega ([f(z,u_n)u_n - p^+F(z,u_n)] + d(z))dz \quad \text{(by } (H_4))\\
&= J(u_n) - \frac{1}{p^+}\langle J'(u_n), u_n\rangle + \frac{1}{p^+}\|d\|_{L^1(\Omega)} \to c + \frac{1}{p^+}\|d\|_{L^1(\Omega)} \ \text{ as } n \to +\infty.
\end{aligned}
$$

This contradicts (15) and so $\{u_n\}$ is a bounded sequence in $W_0^{1,p(z)}(\Omega)$.

Then by Lemma 3.4, $\{u_n\}$ has a convergent subsequence. We conclude that the $(C)_c$ condition is satisfied. □

Lemma 3.8. *If (H_1) and (H_2) hold, then there exist $\rho > 0$ and $\delta > 0$ such that $J(u) \ge \delta$ for each $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| = \rho$.*

Proof. We recall that the embeddings $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ and $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$ are continuous and so there exist two constants $C_{p^+}, C_\alpha > 0$ such that

$$\|u\|_{L^{p^+}(\Omega)} \le C_{p^+}\|u\| \quad \text{and} \quad \|u\|_{L^{\alpha(z)}(\Omega)} \le C_\alpha\|u\|. \tag{16}$$

Combining (H_1) and (H_2), we can verify that, for each $\varepsilon > 0$, there exists a constant C_ε such that

$$F(z,t) \le \frac{\varepsilon}{p^+}t^{p^+} + C_\varepsilon t^{\alpha(z)} \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}^+. \tag{17}$$

If $u \in W_0^{1,p(z)}(\Omega)$ is such that $\|u\| < 1$, using (16) and (17), we obtain

$$
\begin{aligned}
J(u) &= \int_\Omega \frac{a(z)}{p(z)}|\nabla u|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla u|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|u|^{p(z)}dz - \int_\Omega F(z,u)dz \\
&\ge \frac{a_0}{p^+}\int_\Omega |\nabla u|^{p(z)}dz - \frac{\varepsilon}{p^+}\int_\Omega |u|^{p^+}dz - C_\varepsilon\int_\Omega |u|^{\alpha(z)}dz \\
&\ge \frac{a_0}{p^+}\|u\|^{p^+} - \frac{\varepsilon C_{p^+}^{p^+}}{p^+}\|u\|^{p^+} - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^-} \\
&= \frac{a_0 - \varepsilon C_{p^+}^{p^+}}{p^+}\|u\|^{p^+} - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^-} \\
&= \left[\frac{a_0 - \varepsilon C_{p^+}^{p^+}}{p^+} - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^- - p^+}\right]\|u\|^{p^+}.
\end{aligned}
$$

Now, we choose $\rho > 0$ such that

$$\sigma = \frac{a_0 - \varepsilon C_{p^+}^{p^+}}{p^+} - C_\varepsilon C_\alpha^{\alpha^-} \rho^{\alpha^- - p^+} > 0.$$

Then $J(u) \geq \sigma \rho^{p^+} = \delta > 0$ for every $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| = \rho$. □

Lemma 3.9. *If* (H_1) *and* (H_3) *hold, then there exists* $w \in W_0^{1,p(z)}(\Omega)$ *such that* $J(w) < 0$ *and* $\|w\| > \rho$.

Proof. Using (H_1) and (H_3), we deduce that, for all $M > 0$, there exists $C_M > 0$ such that

$$F(z,t) \geq M t^{p^+} - C_M \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}^+. \tag{18}$$

Let $\zeta \in W_0^{1,p(z)}(\Omega)$ such that $\zeta(z) > 0$ for all $z \in \Omega$. From (18), for all $t > 1$, we get

$$\begin{aligned} J(t\zeta) &= \int_\Omega \frac{a(z)t^{p(z)}}{p(z)} |\nabla \zeta|^{p(z)} dz + \int_\Omega \frac{t^{q(z)}}{q(z)} |\nabla \zeta|^{q(z)} dz + \int_\Omega \frac{b(z)t^{p(z)}}{p(z)} \zeta^{p(z)} dz - \int_\Omega F(z,t\zeta) dz \\ &\leq t^{p^+} \Big(\int_\Omega \frac{a(z)}{p(z)} |\nabla \zeta|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla \zeta|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} \zeta^{p(z)} dz - M \int_\Omega \zeta^{p^+} dz \Big) + C_M |\Omega|. \end{aligned}$$

If we choose $M > 0$ such that

$$\int_\Omega \frac{a(z)}{p(z)} |\nabla \zeta|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla \zeta|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} \zeta^{p(z)} dz - M \int_\Omega \zeta^{p^+} dz < 0,$$

we obtain that $\lim_{n \to +\infty} J(t\zeta) = -\infty$. It follows that there exists $w = t_0 \zeta \in W_0^{1,p(z)}(\Omega)$ such that $J(w) < 0$ and $\|w\| > \rho$. □

Now, we recall the following version of the Mountain Pass Theorem.

Theorem 3.10 ([12], Theorem 5.40). *If* $J \in C^1(X, \mathbb{R})$ *satisfies the* $(C)_c$ *condition, there exist* $u_0, u_1 \in X$ *and* $\rho > 0$ *such that*

$$\|u_1 - u_0\| > \rho, \quad \max\{J(u_0), J(u_1)\} < \inf\{J(u) : \|u - u_0\| = \rho\} = m_\rho \quad \textit{and}$$

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \textit{ with } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

then $c \geq m_\rho$ *and* c *is a critical value of* J *(i.e., there exists* $\widehat{u} \in X$ *such that* $J'(\widehat{u}) = 0$ *and* $J(\widehat{u}) = c$*).*

Now we are ready to state the following theorem.

Theorem 3.11. *If* $(H_1) - (H_4)$ *hold, then Problem* (1) *has at least one nontrivial and nonnegative weak solution in* $W_0^{1,p(z)}(\Omega)$.

Proof. Since the functional J satisfies the $(C)_c$ condition and the mountain pass geometry, Theorem 3.10 ensures the existence of a critical point $u \in W_0^{1,p(z)}(\Omega)$. Moreover $J(u) = c \geq \delta > 0 = J(0)$, so u is a nontrivial solution. Now we prove that u is nonnegative. Let $u^- = \max\{-u, 0\}$. We consider (5) written with $w = -u^-$. Since $\int_\Omega f(z,u)(-u^-) dz = 0$, we obtain

$$\int_\Omega a(z) |\nabla u^-|^{p(z)} dz + \int_\Omega |\nabla u^-|^{q(z)} dz + \int_\Omega b(z) |u^-|^{p(z)} dz = 0.$$

Then it must be

$$\int_\Omega a(z) |\nabla u^-|^{p(z)} dz = \int_\Omega |\nabla u^-|^{q(z)} dz = \int_\Omega b(z) |u^-|^{p(z)} dz = 0,$$

and so $u \geq 0$. □

4. Subcritical case

In this section we consider the following set of hypotheses:

(H_0) $f \in C(\bar{\Omega} \times \mathbb{R})$, $f(z,\xi) = 0$ for all $z \in \Omega$ and $\xi \leq 0$;
(H_5) there exist $b_1, b_2 \in [0, +\infty[$ and $\beta \in C(\overline{\Omega})$ with $1 \leq \beta^- \leq \beta(z) \leq \beta^+ < q^-$, satisfying

$$|f(z,\xi)| \leq b_1 + b_2 \xi^{\beta(z)-1} \quad \text{for all } (z,\xi) \in \Omega \times \mathbb{R}^+;$$

(H_6) there exists $b_3 \in]0, +\infty[$ such that $F(z,\xi) \geq b_3 \xi^{\beta^-}$ for all $\xi > 0$.

Theorem 4.1. *If* (H_0), (H_5) *and* (H_6) *hold, then Problem* (1) *has a weak nontrivial and nonnegative solution* $u \in W_0^{1,p(z)}(\Omega)$.

Proof. We prove that J is bounded from below. We have that

$$\begin{aligned} J(u) &\geq \int_\Omega \frac{a(z)}{p(z)} |\nabla u|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla u|^{q(z)} dz + \int_\Omega \frac{b(z)}{p(z)} |u|^{p(z)} dz - \int_\Omega b_1 |u| dz - \int_\Omega \frac{b_2}{\beta(z)} |u|^{\beta(z)} dz \quad (\text{by } (H_5)) \\ &\geq \frac{1}{p^+} \int_\Omega \left(a_0 |\nabla u|^{p(z)} dz + \frac{p^+}{q^+} |\nabla u|^{q(z)} dz + b_0 |u|^{p(z)} - p^+ b_1 |u| - p^+ b_2 |u|^{\beta(z)} \right) dz \\ &\geq \frac{1}{p^+} \int_\Omega (a_0 C_4 |u|^{p(z)} - p^+ b_1 |u| - p^+ b_2 |u|^{\beta(z)}) dz \\ &= \frac{1}{p^+} \int_\Omega |u| \left(\frac{a_0 C_4 |u|^{p(z)-1}}{2} - p^+ b_1 \right) + |u|^{\beta(z)} \left(\frac{a_0 C_4 |u|^{p(z)-\beta(z)}}{2} - p^+ b_2 \right) dz. \end{aligned}$$

We set

$$K := \max \left\{ 1, \left(\frac{2p^+ b_1}{a_0 C_4} \right)^{\frac{1}{p^- - 1}}, \left(\frac{2p^+ b_2}{a_0 C_4} \right)^{\frac{1}{p^- - \beta^+}} \right\}$$

and consider the following partition of $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{z \in \Omega : |u(z)| \geq K\} \quad \text{and} \quad \Omega_2 = \{z \in \Omega : |u(z)| < K\}.$$

We have

$$\int_{\Omega_1} a_0 C_4 |u|^{p(z)} - p^+ b_1 |u| - p^+ b_2 |u|^{\beta(z)} dz \geq 0. \tag{19}$$

On the other hand

$$\begin{aligned} \left| \int_{\Omega_2} a_0 C_4 |u|^{p(z)} - p^+ b_1 |u| - p^+ b_2 |u|^{\beta(z)} dz \right| &\leq \int_{\Omega_2} a_0 C_4 K^{p(z)} + p^+ b_1 K + p^+ b_2 K^{\beta(z)} dz \\ &\leq 2(a_0 C_4 K^{p^+} + p^+ b_1 K + p^+ b_2 K^{\beta^+}) |\Omega|, \end{aligned}$$

which implies

$$\int_{\Omega_2} a_0 C_4 |u|^{p(z)} - p^+ b_1 |u| - p^+ b_2 |u|^{\beta(z)} dz \geq -2(a_0 C_4 K^{p^+} + p^+ b_1 K + p^+ b_2 K^{\beta^+}) |\Omega|. \tag{20}$$

From (19) and (20), we get that J is bounded from below. Since J is weakly continuous and differentiable thanks to hypothesis (H_5), we get that J has a critical point u that is a weak solution of Problem (1).

Now we prove that u is nontrivial. Let $w \in W_0^{1,p}(\Omega)$ with $w(z) > 0$ for all $z \in \Omega$ and $t \in]0, 1[$. Then we have

$$\begin{aligned} J(u) &= \inf\{J(v) : v \in W_0^{1,p}(\Omega)\} \\ &\leq J(tw) \leq \int_\Omega \frac{a(z)t^{p(z)}}{p(z)}|\nabla w|^{p(z)}dz + \int_\Omega \frac{t^{q(z)}}{q(z)}|\nabla w|^{q(z)}dz + \int_\Omega \frac{b(z)t^{p(z)}}{p(z)}w^{p(z)}dz - \int_\Omega b_3 t^{\beta^-} w^{\beta^-} dz \quad \text{(by } (H_6)) \\ &\leq t^{q^-} \int_\Omega \left(\frac{a(z)}{p(z)}|\nabla w|^{p(z)} + \frac{1}{q(z)}|\nabla w|^{q(z)} + \frac{b(z)}{p(z)}w^{p(z)} \right) dz - b_3 t^{\beta^-} \int_\Omega w^{\beta^-} dz \\ &\leq t^{\beta^-} \left(t^{q^- - \beta^-} \int_\Omega \left(\frac{a(z)}{p(z)}|\nabla w|^{p(z)} + \frac{1}{q(z)}|\nabla w|^{q(z)} + \frac{b(z)}{p(z)}w^{p(z)} \right) dz - b_3 \int_\Omega w^{\beta^-} dz \right) < 0 \end{aligned}$$

for t sufficiently small. Consequently, from $J(u) < 0 = J(0)$, we conclude that u is a nontrivial weak solution. Proceeding as in the last lines of the proof developed for Theorem 3.11, we get that u is nonnegative. This concludes our proof. □

5. The parametric case

We consider the Problem

$$\begin{cases} -\operatorname{div}\,(a(z)|\nabla u|^{p(z)-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(z)-2}\nabla u) + b(z)|u|^{p(z)-2}u = \lambda f(z, u(z)) & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{21}$$

where $\lambda > 0$ is a real parameter. The associated functional to (21) is given by

$$J_\lambda(u) = \int_\Omega \frac{a(z)}{p(z)}|\nabla u|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla u|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|u|^{p(z)}dz - \lambda I(u) \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

As a consequence of Theorem 3.11 we deduce the following theorem.

Theorem 5.1. *Let* $(H_1) - (H_4)$ *hold. For all* $\lambda > 0$*, Problem* (21) *has at least one nontrivial and nonnegative weak solution* $u_\lambda \in W_0^{1,p(z)}(\Omega)$.

Remark 5.2. *We note that in the sublinear case the result of existence of a nontrivial and nonnegative weak solution for Problem* (21) *is a consequence of Theorem 4.1.*

Lemma 5.3. *If* (H_1) *holds, then there exist positive constants* σ_λ *and* r_λ *such that* $\lim_{\lambda\to 0^+} \sigma_\lambda = +\infty$ *and* $J_\lambda(u) \geq \sigma_\lambda > 0$ *for all* $u \in W_0^{1,p(z)}(\Omega)$ *such that* $\|u\| = r_\lambda$.

Proof. Let $w \in W_0^{1,p(z)}(\Omega)$ with $\|w\| > 1$. It follows from (H_1) that there exists $C_5 > 0$ such that

$$F(z, t) \leq C_5(t^{\alpha(z)} + 1) \qquad \text{for all } (z, t) \in \Omega \times \mathbb{R}^+. \tag{22}$$

Then

$$\begin{aligned} J_\lambda(w) &\geq \int_\Omega \frac{a(z)}{p(z)}|\nabla w|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla w|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|w|^{p(z)}dz - \lambda C_5 \int_\Omega (|w|^{\alpha(z)} + 1)dz \\ &\geq \frac{a_0}{p^+}\|w\|^{p^-} - \lambda C_6\|w\|^{\alpha^+} - \lambda C_5|\Omega|. \end{aligned} \tag{23}$$

From (H_1) we have that $p^- < \alpha^+$ and so we can choose $t \in]0, (\alpha^+ - p^-)^{-1}[$. Thus $r_\lambda := \lambda^{-t} > 1$ for λ small enough. Now, considering (23) for $\|w\| = r_\lambda = \lambda^{-t}$, we get

$$J_\lambda(u) \geq \frac{a_0}{p^+}\lambda^{-tp^-} - \lambda^{1-t\alpha^+}C_6 - \lambda C_5|\Omega|.$$

We put $\sigma_\lambda = \lambda^{-tp^-}\left(\frac{a_0}{p^+} - \lambda^{1-t(\alpha^+-p^-)}C_6\right) - \lambda C_5|\Omega|$. The choice of t ensures that there exists λ_0 sufficiently small such that $\sigma_\lambda > 0$ for all $0 < \lambda < \lambda_0$. Moreover $\sigma_\lambda \to +\infty$ as $\lambda \to 0^+$. □

Theorem 5.4. *If* (H_1), (H_3) *and* (H_4) *hold, then there exists* $\lambda_0 > 0$ *such that, for all* $0 < \lambda < \lambda_0$, *Problem* (21) *has at least a nontrivial and nonnegative weak solution* u_λ *and* $\lim_{\lambda \to 0^+} \|u_\lambda\| = +\infty$.

Proof. Clearly, J_λ satisfies the $(C)_c$ condition for all $\lambda > 0$. Moreover the hypotheses of Theorem 3.10 are satisfied in virtue of Lemma 3.7, Lemma 3.9 and Lemma 5.3. As a consequence, there exists a nontrivial critical point u_λ for J_λ such that

$$J_\lambda(u_\lambda) = c \geq \sigma_\lambda.$$

Using (22), we get

$$\begin{aligned} J_\lambda(u_\lambda) &\leq \int_\Omega \frac{a(z)}{p(z)}|\nabla u_\lambda|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla u_\lambda|^{q(z)}dz + \int_\Omega \frac{b(z)}{p(z)}|u_\lambda|^{p(z)}dz + \lambda C_5 \int_\Omega (|u_\lambda|^{\alpha^+} + 1)dz \\ &\leq C_7 \max\{\|u_\lambda\|^{p^+}, \|u_\lambda\|^{q^-}\} + \lambda C_8 \max\{\|u_\lambda\|^{\alpha^+}, \|u_\lambda\|^{\alpha^-}\} + \lambda C_5|\Omega|. \end{aligned}$$

To conclude, from $J_\lambda(u_\lambda) \to +\infty$ as $\lambda \to 0^+$ we infer that $\lim_{\lambda\to 0^+} \|u_\lambda\| = +\infty$. □

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