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Generalized Simpson Type Inequalities Involving Riemann–Liouville Fractional Integrals and Their Applications

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Abstract. We establish a Simpson type identity and several Simpson type inequalities for Riemann–Liouville fractional integrals. As applications, we apply the obtained results to special means of real numbers, an error estimate for Simpson type quadrature formula, and q-digamma function, respectively.

1. Introduction

The following inequality is well known the Simpson's integral inequality:

$$\left|\frac{1}{6}\left[g(\varphi_1)+4g\left(\frac{\varphi_1+\varphi_2}{2}\right)+g(\varphi_2)\right]-\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} g(t)\mathrm{d}t\right| \le \frac{1}{2880}\left\|g^{(4)}\right\|_\infty(\varphi_2-\varphi_1)^4, \tag{1}$$

where $g:[\varphi_1,\varphi_2]\to\mathbb{R}$ is a four-order differentiable mapping on (φ_1,φ_2) and $\left\|g^{(4)}\right\|_\infty=\sup_{\tau\in(\varphi_1,\varphi_2)}\left|g^{(4)}(\tau)\right|<\infty$.

For recent results about the inequality (1), we refer to some articles by Ertŭgral and Sarikaya [7], Hsu et al. [9], Noor et al. [13], Sarikaya and Bardak [14], Shuang and Qi [16] and Set et al. [15].

Let us introduce that Eftekhari [6] proposed a class of (s,m)-convex mappings, as follows: A mapping $g:[0,\infty)\to\mathbb{R}$ is called (s,m)-convex in the second sense, for certain fixed $(s,m)\in(0,1]^2$, if

$$g\big(\lambda\varphi_1+m(1-\lambda)\varphi_2\big)\le\lambda^s g(\varphi_1)+m(1-\lambda)^s g(\varphi_2)$$

holds for all $\varphi_1,\varphi_2\in[0,\infty)$ and $\lambda\in[0,1]$.

For certain related results involving such kinds mappings, we refer to study by Du et al [2] and [3].

Next, we retrospect the space of all complex-valued Lebesgue measurable functions, which will be used subsequently.

Let $\chi_\kappa^p(a,b)$ ($\kappa\in\mathbb{R},1\le p\le\infty$) be the space of all complex-valued Lebesgue measurable functions g on (a,b) for which $\|g\|_{\chi_\kappa^p}<\infty$, where the norm $\|\cdot\|_{\chi_\kappa^p}$ is defined with the following expression:

$$\|g\|_{\chi_\kappa^p}=\left(\int_a^b\left|\tau^\kappa g(\tau)\right|^p\frac{\mathrm{d}\tau}{\tau}\right)^{1/p},\quad(1\le p<\infty)$$

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and

$$\|g\|_{X_\kappa^\infty} = \operatorname*{ess\,sup}_{a<\tau<b} \left[\tau^\kappa |g(\tau)|\right], \quad (p = \infty),$$

where ess sup stands for essential supremum.

The definition of the Riemann–Liouville fractional integrals below is well known in the literature.

Definition 1.1. *Let $g \in L([a,b])$. The Riemann–Liouville fractional integrals $\mathcal{J}_{a^+}^{\alpha} g$ and $\mathcal{J}_{b^-}^{\alpha} g$ of order $\alpha > 0$ with $a \geq 0$ are defined as*

$$\mathcal{J}_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} g(\tau) \mathrm{d}\tau$$

and

$$\mathcal{J}_{b^-}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} g(\tau) \mathrm{d}\tau$$

with $a < t < b$ and $\Gamma(\alpha) = \int_0^\infty e^{-\tau}\tau^{\alpha-1}\mathrm{d}\tau$.

For more results related to such kinds fractional integrals, we refer to recent papers [1, 4, 5, 8, 10–12, 17] and the references therein.

Motivated by the results mentioned above, especially the results developed in [14] and [16], we aim to present several integral inequalities of Simpson type for mappings whose first derivatives belong to the Lebesgue L_q spaces, and also for mappings whose first derivatives at some powers are (s,m)-convex through Riemann–Liouville fractional integrals.

2. Preliminary Lemma

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $0 < m \leq 1$ and let $g : \left[ma, b\right] \to \mathbb{R}$ be a differentiable mapping with $a < b$ and $a \in \mathbb{R}$, $b > 0$. If $g \in L_1\left(\left[ma, b\right]\right)$ and $\alpha > 0$, then the following equation holds:*

$$\begin{aligned}\mathcal{G}_g(a,b;m,\alpha,w) = {} & \frac{(w-ma)^2}{2(b-ma)} \int_0^1 \Big(\frac{t^\alpha}{2} - \frac{1}{5}\Big) g'\Big(\frac{1-t}{2}ma + \frac{1+t}{2}w\Big)\mathrm{d}t \\ & + \frac{(b-w)^2}{2(b-ma)} \int_0^1 \Big(\frac{1}{5} - \frac{t^\alpha}{2}\Big) g'\Big(\frac{1+t}{2}w + \frac{1-t}{2}b\Big)\mathrm{d}t,\end{aligned} \tag{2}$$

where

$$\begin{aligned}\mathcal{G}_g(a,b;m,\alpha,w) := {} & \frac{3}{10} g(w) + \frac{1}{5(b-ma)}\Big[(b-w)g\Big(\frac{w+b}{2}\Big) + (w-ma)g\Big(\frac{ma+w}{2}\Big)\Big] \\ & - \frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-ma)(w-ma)^{\alpha-1}} \mathcal{J}_{w^-}^{\alpha} g\Big(\frac{ma+w}{2}\Big) - \frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-ma)(b-w)^{\alpha-1}} \mathcal{J}_{w^+}^{\alpha} g\Big(\frac{w+b}{2}\Big)\end{aligned}$$

and $w = (1-\nu)ma + \nu b$, for $\nu \in [0,1]$.

Proof. For $\nu \in (0,1)$, by integration by parts and changing the variable, we state that

$$\begin{aligned}& \int_0^1 \Big(\frac{t^\alpha}{2} - \frac{1}{5}\Big) g'\Big(\frac{1-t}{2}ma + \frac{1+t}{2}w\Big)\mathrm{d}t \\ = {} & \frac{2}{w-ma}\Big(\frac{t^\alpha}{2} - \frac{1}{5}\Big) g\Big(\frac{1-t}{2}ma + \frac{1+t}{2}w\Big)\Big|_0^1 - \frac{1}{w-ma}\int_0^1 \alpha t^{\alpha-1} g\Big(\frac{1-t}{2}ma + \frac{1+t}{2}w\Big)\mathrm{d}t \\ = {} & \frac{2}{w-ma}\Big[\frac{3}{10}g(w) + \frac{1}{5}g\Big(\frac{ma+w}{2}\Big)\Big] - \frac{2^\alpha \alpha}{(w-ma)^{\alpha+1}} \int_{\frac{ma+w}{2}}^{w} \Big(x - \frac{ma+w}{2}\Big)^{\alpha-1} g(x)\mathrm{d}x \\ = {} & \frac{2}{w-ma}\Big[\frac{3}{10}g(w) + \frac{1}{5}g\Big(\frac{ma+w}{2}\Big)\Big] - \frac{2^\alpha\Gamma(1+\alpha)}{(w-ma)^{\alpha+1}} \mathcal{J}_{w^-}^{\alpha} g\Big(\frac{ma+w}{2}\Big).\end{aligned} \tag{3}$$

Similarly, we get that

$$\int_0^1 \Big(\frac{1}{5} - \frac{t^\alpha}{2}\Big) g'\Big(\frac{1+t}{2} w + \frac{1-t}{2} b\Big) \mathrm{d}t = \frac{2}{b-w}\Big[\frac{3}{10} g(w) + \frac{1}{5} g\Big(\frac{w+b}{2}\Big)\Big] - \frac{2^\alpha \Gamma(1+\alpha)}{(b-w)^{\alpha+1}} \mathcal{J}^\alpha_{w^+} g\Big(\frac{w+b}{2}\Big). \tag{4}$$

Multiplying both sides of (3) and (4) by $\frac{(w-ma)^2}{2(b-ma)}$ and $\frac{(b-w)^2}{2(b-ma)}$, respectively, and adding the resulting identities, we obtain the desired result (2).

For $\nu = 0$, we have that

$$\mathcal{G}_g(a, b; m, \alpha, ma) = \frac{b - ma}{2} \int_0^1 \Big(\frac{1}{5} - \frac{t^\alpha}{2}\Big) g'\Big(\frac{1+t}{2} ma + \frac{1-t}{2} b\Big) \mathrm{d}t,$$

where

$$\mathcal{G}_g(a, b; m, \alpha, ma) := \frac{3}{10} g(ma) + \frac{1}{5} g\Big(\frac{ma+b}{2}\Big) - \frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-ma)^\alpha} \mathcal{J}^\alpha_{(ma)^+} g\Big(\frac{ma+b}{2}\Big).$$

For $\nu = 1$, we have that

$$\mathcal{G}_g(a, b; m, \alpha, b) = \frac{b - ma}{2} \int_0^1 \Big(\frac{t^\alpha}{2} - \frac{1}{5}\Big) g'\Big(\frac{1-t}{2} ma + \frac{1+t}{2} b\Big) \mathrm{d}t,$$

where

$$\mathcal{G}_g(a, b; m, \alpha, b) := \frac{3}{10} g(b) + \frac{1}{5} g\Big(\frac{ma+b}{2}\Big) - \frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-ma)^\alpha} \mathcal{J}^\alpha_{b^-} g\Big(\frac{ma+b}{2}\Big).$$

A simple proof of the above identities can be done by conducting integration by parts in the integrals from the right side and changing the variable. This ends the proof. □

3. Main Results

Theorem 3.1. *Under all assumptions of Lemma 2.1, suppose that g' is bounded, i.e., $\|g'\|_\infty = \sup_{\tau \in [ma,b]} |g'(\tau)| < \infty$. Then for $w \in [ma, b]$, the following inequality holds:*

$$\Big|\mathcal{G}_g(a, b; m, \alpha, w)\Big| \le \frac{(w-ma)^2 + (b-w)^2}{2(b-ma)} \Phi(\alpha) \|g'\|_\infty, \tag{5}$$

where

$$\Phi(\alpha) = \frac{\alpha}{\alpha+1}\Big(\frac{2}{5}\Big)^{1+\frac{1}{\alpha}} + \frac{3-2\alpha}{10(\alpha+1)}.$$

Proof. If we utilize Lemma 2.1 and properties of the absolute value, then we have

$$\begin{aligned}
&\Big|\mathcal{G}_g(a, b; m, \alpha, w)\Big| \\
&\le \frac{(w-ma)^2}{2(b-ma)} \int_0^1 \Big|\frac{t^\alpha}{2} - \frac{1}{5}\Big| \Big|g'\Big(\frac{1-t}{2} ma + \frac{1+t}{2} w\Big)\Big| \mathrm{d}t \\
&\quad + \frac{(b-w)^2}{2(b-ma)} \int_0^1 \Big|\frac{1}{5} - \frac{t^\alpha}{2}\Big| \Big|g'\Big(\frac{1+t}{2} w + \frac{1-t}{2} b\Big)\Big| \mathrm{d}t \\
&\le \frac{(w-ma)^2 + (b-w)^2}{2(b-ma)} \|g'\|_\infty \int_0^1 \Big|\frac{t^\alpha}{2} - \frac{1}{5}\Big| \mathrm{d}t.
\end{aligned}$$

The desired inequality yields from the above by noting that

$$\int_0^1 \left|\frac{t^\alpha}{2}-\frac{1}{5}\right|\mathrm{d}t=\frac{\alpha}{\alpha+1}\Big(\frac{2}{5}\Big)^{1+\frac{1}{\alpha}}+\frac{3-2\alpha}{10(\alpha+1)}.$$

The proof is completed. □

Theorem 3.2. *Under all assumptions of Lemma 2.1, suppose that* $g'\in L_1\Big(\big[ma,b\big]\Big)$. *Then for* $w\in[ma,b]$, *the following inequality holds:*

$$\Big|\mathcal{G}_g(a,b;m,\alpha,w)\Big|\le\frac{3}{10}\|g'\|_1, \tag{6}$$

where $\|g'\|_1=\int_{ma}^b |g'(\tau)|\mathrm{d}\tau<\infty$.

Proof. Utilizing Lemma 2.1 and changing the variable, one has

$$\begin{aligned}
&\Big|\mathcal{G}_g(a,b;m,\alpha,w)\Big|\\
&\le\frac{(w-ma)^2}{2(b-ma)}\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{5}\right|\left|g'\Big(\frac{1-t}{2}ma+\frac{1+t}{2}w\Big)\right|\mathrm{d}t\\
&\quad+\frac{(b-w)^2}{2(b-ma)}\int_0^1\left|\frac{1}{5}-\frac{t^\alpha}{2}\right|\left|g'\Big(\frac{1+t}{2}w+\frac{1-t}{2}b\Big)\right|\mathrm{d}t\\
&\le\frac{3}{10}\left[\frac{(w-ma)}{(b-ma)}\int_{\frac{ma+w}{2}}^{w}\Big|g'(u)\Big|\mathrm{d}u-\frac{(b-w)}{(b-ma)}\int_{\frac{w+b}{2}}^{w}\Big|g'(u)\Big|\mathrm{d}u\right]\\
&\le\frac{3}{10}\Big[\int_{\frac{ma+w}{2}}^{w}\Big|g'(u)\Big|\mathrm{d}u+\int_{w}^{\frac{w+b}{2}}\Big|g'(u)\Big|\mathrm{d}u\Big]\\
&=\frac{3}{10}\int_{\frac{ma+w}{2}}^{\frac{w+b}{2}}\Big|g'(u)\Big|\mathrm{d}u\\
&\le\frac{3}{10}\int_{ma}^{b}\Big|g'(u)\Big|\mathrm{d}u=\frac{3}{10}\|g'\|_1.
\end{aligned}$$

Here, we use the fact that $\left|\frac{1}{5}-\frac{t^\alpha}{2}\right|\le\frac{1}{2}-\frac{1}{5}=\frac{3}{10}$, for $t\in[0,1]$ with some fixed $\alpha>0$, $0\le\frac{w-ma}{b-ma}\le1$ and $0\le\frac{b-w}{b-ma}\le1$. This ends the proof. □

Theorem 3.3. *Under all assumptions of Lemma 2.1, suppose that* $g'\in L_q([ma,b])$ *with* $1<q<\infty$. *Then for* $w\in[ma,b]$ *and* $\alpha\in(0,1]$, *the following inequality holds:*

$$\Big|\mathcal{G}_g(a,b;m,\alpha,w)\Big|\le\mathcal{D}^{\frac{1}{p}}\Big(2^{1/q}(b-ma)^{1/p}\Big)\|g'\|_q, \tag{7}$$

where

$$\mathcal{D}=\frac{2^{\frac{1}{\alpha}}\cdot5^{-p-\frac{1}{\alpha}}}{p\alpha+1}+\frac{2^{\frac{1}{\alpha}}(2^{-\frac{1}{\alpha}}-5^{-\frac{1}{\alpha}})^{p\alpha+1}}{p\alpha+1},$$

and $\|g'\|_q=\Big(\int_{ma}^b|g'(\tau)|^q\mathrm{d}\tau\Big)^{\frac{1}{q}}<\infty$ *with* $p^{-1}+q^{-1}=1$.

Proof. By using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\mathcal{G}_g(a,b;m,\alpha,w)\right| \\
&\leq \frac{(w-ma)^2}{2(b-ma)}\Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|g'\Big(\frac{1-t}{2}ma+\frac{1+t}{2}w\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad+\frac{(b-w)^2}{2(b-ma)}\Big(\int_0^1\Big|\frac{1}{5}-\frac{t^\alpha}{2}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|g'\Big(\frac{1+t}{2}w+\frac{1-t}{2}b\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}.
\end{aligned} \tag{8}$$

Using suitable substitutions, we obtain that

$$\begin{aligned}
&\left|\mathcal{G}_g(a,b;m,\alpha,w)\right| \\
&\leq \Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big\{\frac{(w-ma)^2}{2(b-ma)}\Big[\frac{2}{(w-ma)}\int_{\frac{ma+w}{2}}^{w}\big|g'(u)\big|^q\mathrm{d}u\Big]^{\frac{1}{q}} \\
&\quad+\frac{(b-w)^2}{2(b-ma)}\Big[\frac{2}{(b-w)}\int_w^{\frac{w+b}{2}}\big|g'(u)\big|^q\mathrm{d}u\Big]^{\frac{1}{q}}\Big\} \\
&=\Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big\{\frac{(w-ma)^{2-1/q}}{2^{1-1/q}(b-ma)}\Big(\int_{\frac{ma+w}{2}}^{w}\big|g'(u)\big|^q\mathrm{d}u\Big)^{\frac{1}{q}} \\
&\quad+\frac{(b-w)^{2-1/q}}{2^{1-1/q}(b-ma)}\Big(\int_w^{\frac{w+b}{2}}\big|g'(u)\big|^q\mathrm{d}u\Big)^{\frac{1}{q}}\Big\} \\
&\leq\Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[2^{1/q-1}(b-ma)^{1-1/q}\Big]\Big\{\Big(\int_{\frac{ma+w}{2}}^{w}\big|g'(u)\big|^q\mathrm{d}u\Big)^{\frac{1}{q}}+\Big(\int_w^{\frac{w+b}{2}}\big|g'(u)\big|^q\mathrm{d}u\Big)^{\frac{1}{q}}\Big\} \\
&\leq\Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[2^{1/q}(b-ma)^{1/p}\Big]\Big(\int_{ma}^{b}\big|g'(u)\big|^q\mathrm{d}u\Big)^{\frac{1}{q}} \\
&=\Big(\int_0^1\Big|\frac{t^\alpha}{2}-\frac{1}{5}\Big|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(2^{1/q}(b-ma)^{1/p}\Big)\|g'\|_q.
\end{aligned}$$

Also

$$\int_0^1\Big|\frac{1}{5}-\frac{t^\alpha}{2}\Big|^p\mathrm{d}t\leq\int_0^1\Big|\frac{1}{5^{\frac{1}{\alpha}}}-\frac{t}{2^{\frac{1}{\alpha}}}\Big|^{p\alpha}\mathrm{d}t=\frac{2^{\frac{1}{\alpha}}\cdot 5^{-p-\frac{1}{\alpha}}}{p\alpha+1}+\frac{2^{\frac{1}{\alpha}}(2^{-\frac{1}{\alpha}}-5^{-\frac{1}{\alpha}})^{p\alpha+1}}{p\alpha+1},$$

which follows from $|\mathbb{A}^\alpha-\mathbb{B}^\alpha|\leq|\mathbb{A}-\mathbb{B}|^\alpha$ for any $\mathbb{A},\mathbb{B}\geq 0$ and $\alpha\in(0,1]$. The proof is complete. □

Theorem 3.4. *Let $0<m\leq 1$ and let $g:\big[ma,\frac{b}{m}\big]\to\mathbb{R}$ be a differentiable mapping with $0\leq a<b$, such that $g\in L_1\big(\big[ma,\frac{b}{m}\big]\big)$. If $|g'|^q$ is (s,m)-convex in the second sense with $(s,m)\in(0,1]^2$, for $q>1$ with $p^{-1}+q^{-1}=1$, then for $w\in[ma,b]$ and $\alpha\in(0,1]$, the following inequality holds:*

$$\begin{aligned}
\left|\mathcal{G}_g(a,b;m,\alpha,w)\right|\leq\mathcal{D}^{\frac{1}{p}}\Big\{&\frac{(w-ma)^2}{2(b-ma)}\Big[\frac{m}{2^s(s+1)}\big|g'(a)\big|^q+\frac{2^{s+1}-1}{2^s(s+1)}\big|g'(w)\big|^q\Big]^{\frac{1}{q}} \\
&+\frac{(b-w)^2}{2(b-ma)}\Big[\frac{2^{s+1}-1}{2^s(s+1)}\big|g'(w)\big|^q+\frac{m}{2^s(s+1)}\Big|g'\Big(\frac{b}{m}\Big)\Big|^q\Big]^{\frac{1}{q}}\Big\},
\end{aligned} \tag{9}$$

where $\mathcal{D}$ is defined in Theorem 3.3.

Proof. Continuing from inequality (8) and utilizing the (s,m)-convexity of $|g'|^q$, we get the desired result. □

Corollary 3.5. *Consider Theorem 3.4.*
(*i*) *Putting $w = ma$, we get that*

$$\left|\mathcal{G}_g(a,b;m,\alpha,ma)\right| \leq \frac{b-ma}{2}\mathcal{D}^{\frac{1}{p}}\left[\frac{2^{s+1}-1}{2^s(s+1)}\left|g'(ma)\right|^q + \frac{m}{2^s(s+1)}\left|g'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}. \tag{10}$$

(*ii*) *Putting $w = \frac{ma+b}{2}$, we get that*

$$\begin{aligned}&\left|\mathcal{G}_g\left(a,b;m,\alpha,\frac{ma+b}{2}\right)\right| \\ &\leq \frac{b-ma}{8}\mathcal{D}^{\frac{1}{p}}\left\{\left[\frac{m}{2^s(s+1)}\left|g'(a)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|g'\left(\frac{ma+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ &\quad\left. + \left[\frac{2^{s+1}-1}{2^s(s+1)}\left|g'\left(\frac{ma+b}{2}\right)\right|^q + \frac{m}{2^s(s+1)}\left|g'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\} \\ &\leq \frac{b-ma}{8}\mathcal{D}^{\frac{1}{p}}\left\{\left(\frac{m}{2^s(s+1)}\right)^{\frac{1}{q}}\left|g'(a)\right| + 2\left(\frac{2^{s+1}-1}{2^s(s+1)}\right)^{\frac{1}{q}}\left|g'\left(\frac{ma+b}{2}\right)\right| + \left(\frac{m}{2^s(s+1)}\right)^{\frac{1}{q}}\left|g'\left(\frac{b}{m}\right)\right|\right\}.\end{aligned} \tag{11}$$

The second above inequality is obtained by utilizing the fact that

$$\sum_{j=1}^{n}(\rho+\xi)^{\sigma} \leq \sum_{j=1}^{n}(\rho)^{\sigma} + \sum_{j=1}^{n}(\xi)^{\sigma}, \quad \rho,\xi > 0, \ \ 0 \leq \sigma < 1. \tag{12}$$

(*iii*) *Putting $\omega = b$, we get that*

$$\left|\mathcal{G}_g(a,b;m,\alpha,b)\right| \leq \frac{b-ma}{2}\mathcal{D}^{\frac{1}{p}}\left[\frac{m}{2^s(s+1)}\left|g'(a)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|g'(b)\right|^q\right]^{\frac{1}{q}}. \tag{13}$$

Corollary 3.6. *Combining the inequalities* (10) *and* (13)*, one has*

$$\begin{aligned}&\left|\frac{1}{10}\left[3g(ma) + 4g\left(\frac{ma+b}{2}\right) + 3g(b)\right] - \frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-ma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{(ma)^+}g\left(\frac{ma+b}{2}\right) + \mathcal{J}^{\alpha}_{b^-}g\left(\frac{ma+b}{2}\right)\right]\right| \\ &\leq \frac{b-ma}{2}\mathcal{D}^{\frac{1}{p}}\left\{\left[\frac{2^{s+1}-1}{2^s(s+1)}\left|g'(ma)\right|^q + \frac{m}{2^s(s+1)}\left|g'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ &\quad\left. + \left[\frac{m}{2^s(s+1)}\left|g'(a)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}\left|g'(b)\right|^q\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{14}$$

Specially, putting $\alpha = 1 = m$ and using inequality (12)*, we get*

$$\begin{aligned}&\left|\frac{b-a}{10}\left[3g(a) + 4g\left(\frac{a+b}{2}\right) + 3g(b)\right] - \int_a^b g(u)\mathrm{d}u\right| \\ &\leq \frac{(b-a)^2}{2}\left[\frac{2(2^{p+1}+3^{p+1})}{10^{p+1}(p+1)}\right]^{\frac{1}{p}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}\right)^{\frac{1}{q}} + \left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}}\right]\left[\left|g'(a)\right| + \left|g'(b)\right|\right].\end{aligned} \tag{15}$$

Theorem 3.7. *Let $0 < m \leq 1$ and let $g : \left[ma, \frac{b}{m}\right] \to \mathbb{R}$ be a differentiable mapping with $0 \leq a < b$, such that $g \in L_1\left(\left[ma, \frac{b}{m}\right]\right)$. If $|g'|$ is (s,m)-convex in the second sense with $(s,m) \in (0,1]^2$, for $p^{-1} + q^{-1} = 1$ with $p > 1$, then for $w \in [ma, b]$ and $\alpha \in (0,1]$, the following inequality holds:*

$$\begin{aligned}\left|\mathcal{G}_g(a,b;m,\alpha,w)\right| \leq \mathcal{D}^{\frac{1}{p}}&\left\{\frac{(w-ma)^2}{2(b-ma)}\left[\left(\frac{1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} m\left|g'(a)\right| + \left(\frac{2^{qs+1}-1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}}\left|g'(w)\right|\right]\right. \\ &\left. + \frac{(b-w)^2}{2(b-ma)}\left[\left(\frac{2^{qs+1}-1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}}\left|g'(w)\right| + \left(\frac{1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} m\left|g'\left(\frac{b}{m}\right)\right|\right]\right\},\end{aligned} \tag{16}$$

where $\mathcal{D}$ is defined in Theorem 3.3.

Proof. Utilizing Lemma 2.1 and using the (s,m)-convexity of $|g'|$, one has

$$\begin{aligned}
&\left|\mathcal{G}_g(a,b;m,\alpha,w)\right| \\
&\le \frac{(w-ma)^2}{2(b-ma)}\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{5}\right|\left|g'\Big(\frac{1-t}{2}ma+\frac{1+t}{2}w\Big)\right|\mathrm{d}t \\
&\quad+\frac{(b-w)^2}{2(b-ma)}\int_0^1\left|\frac{1}{5}-\frac{t^\alpha}{2}\right|\left|g'\Big(\frac{1+t}{2}w+\frac{1-t}{2}b\Big)\right|\mathrm{d}t \\
&\le \frac{(w-ma)^2}{2(b-ma)}\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{5}\right|\left[\Big(\frac{1-t}{2}\Big)^s m|g'(a)|+\Big(\frac{1+t}{2}\Big)^s|g'(w)|\right]\mathrm{d}t \\
&\quad+\frac{(b-w)^2}{2(b-ma)}\int_0^1\left|\frac{1}{5}-\frac{t^\alpha}{2}\right|\left[\Big(\frac{1+t}{2}\Big)^s|g'(w)|+\Big(\frac{1-t}{2}\Big)^s m\left|g'\Big(\frac{b}{m}\Big)\right|\right]\mathrm{d}t.
\end{aligned}$$

Using Hölder's inequality, we have that

$$\begin{aligned}
&\left|\mathcal{G}_g(a,b;m,\alpha,w)\right| \\
&\le \frac{(w-ma)^2}{2(b-ma)}\left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{5}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left\{\left[\int_0^1\Big(\frac{1-t}{2}\Big)^{qs}\mathrm{d}t\right]^{\frac{1}{q}}m\left|g'(a)\right|+\left[\int_0^1\Big(\frac{1+t}{2}\Big)^{qs}\mathrm{d}t\right]^{\frac{1}{q}}\left|g'(w)\right|\right\} \\
&\quad+\frac{(b-w)^2}{2(b-ma)}\left(\int_0^1\left|\frac{1}{5}-\frac{t^\alpha}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left\{\left[\int_0^1\Big(\frac{1+t}{2}\Big)^{qs}\mathrm{d}t\right]^{\frac{1}{q}}\left|g'(w)\right|+\left[\int_0^1\Big(\frac{1-t}{2}\Big)^{qs}\mathrm{d}t\right]^{\frac{1}{q}}m\left|g'\Big(\frac{b}{m}\Big)\right|\right\}.
\end{aligned}$$

The desired inequality yields from the above by noting that

$$\int_0^1\Big(\frac{1+t}{2}\Big)^{qs}\mathrm{d}t=\frac{2^{qs+1}-1}{2^{qs}(qs+1)}$$

and

$$\int_0^1\Big(\frac{1-t}{2}\Big)^{qs}\mathrm{d}t=\frac{1}{2^{qs}(qs+1)}.$$

Thus, the proof is completed. □

Corollary 3.8. *If we put $w=\frac{ma+b}{2}$ in Theorem 3.7, then we get that*

$$\begin{aligned}
\left|\mathcal{G}_g\Big(a,b;m,\alpha,\frac{ma+b}{2}\Big)\right| \le \frac{b-ma}{8}\mathcal{D}^{\frac{1}{p}}&\left\{\Big(\frac{1}{2^{qs}(qs+1)}\Big)^{\frac{1}{q}}m\left|g'(a)\right|+2\Big(\frac{2^{qs+1}-1}{2^{qs}(qs+1)}\Big)^{\frac{1}{q}}\left|g'\Big(\frac{ma+b}{2}\Big)\right|\right. \\
&\left.+\Big(\frac{1}{2^{qs}(qs+1)}\Big)^{\frac{1}{q}}m\left|g'\Big(\frac{b}{m}\Big)\right|\right\}.
\end{aligned} \tag{17}$$

4. Applications

4.1. Special Means

For $0<a<b$, we review the well-known mean values as follows:
(i) The arithmetic mean: $A(a,b)=\frac{a+b}{2}$.

(ii) The ϑ-logarithmic mean: $L_\vartheta(a,b) = \left[\frac{b^{\vartheta+1}-a^{\vartheta+1}}{(\vartheta+1)(b-a)}\right]^{\frac{1}{\vartheta}}$, $\vartheta \in \mathbb{Z}\backslash\{0,-1\}$.

Considering the mapping $g(x) = \frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$, $x > 0$ with $s \in (0,1)$, if we apply it to the second inequality of (11) in Corollary 3.5 and Corollary 3.8 with $\alpha = 1 = m$, respectively, then we have the following results.

Proposition 4.1. *Let $0 < a < b$, $0 < s < 1$ and $p^{-1} + q^{-1} = 1$ with $q > 1$. Then one has*

$$\left|\frac{q}{10(s+q)}\left[A^{\frac{s}{q}+1}\left(\frac{3a}{2},\frac{b}{2}\right) + 3A^{\frac{s}{q}+1}(a,b) + A^{\frac{s}{q}+1}\left(\frac{a}{2},\frac{3b}{2}\right)\right] - \frac{q}{2(s+q)}L^{\frac{s}{q}+1}_{\frac{s}{q}+1}\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)\right|$$
$$\leq \frac{(b-a)}{4}\left[\frac{2(2^{p+1}+3^{p+1})}{10^{p+1}(p+1)}\right]^{\frac{1}{p}}\left[\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} A\left(a^{\frac{s}{q}},b^{\frac{s}{q}}\right) + \left(\frac{2^{s+1}-1}{2^s(s+1)}\right)^{\frac{1}{q}} A^{\frac{s}{q}}(a,b)\right].$$

Proposition 4.2. *Under all assumptions of Proposition 4.1, we have*

$$\left|\frac{q}{10(s+q)}\left[A^{\frac{s}{q}+1}\left(\frac{3a}{2},\frac{b}{2}\right) + 3A^{\frac{s}{q}+1}(a,b) + A^{\frac{s}{q}+1}\left(\frac{a}{2},\frac{3b}{2}\right)\right] - \frac{q}{2(s+q)}L^{\frac{s}{q}+1}_{\frac{s}{q}+1}\left(\frac{3a+b}{4},\frac{a+3b}{4}\right)\right|$$
$$\leq \frac{(b-a)}{4}\left[\frac{2(2^{p+1}+3^{p+1})}{10^{p+1}(p+1)}\right]^{\frac{1}{p}}\left[\left(\frac{1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} A\left(a^{\frac{s}{q}},b^{\frac{s}{q}}\right) + \left(\frac{2^{qs+1}-1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} A^{\frac{s}{q}}(a,b)\right].$$

Let the bounds in Proposition 4.1 and Proposition 4.2, respectively, be denoted by $G_1(s,q)$ and $G_2(s,q)$, that is

$$G_1(s,q) = \left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} A\left(a^{\frac{s}{q}},b^{\frac{s}{q}}\right) + \left(\frac{2^{s+1}-1}{2^s(s+1)}\right)^{\frac{1}{q}} A^{\frac{s}{q}}(a,b)$$

and

$$G_2(s,q) = \left(\frac{1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} A\left(a^{\frac{s}{q}},b^{\frac{s}{q}}\right) + \left(\frac{2^{qs+1}-1}{2^{qs}(qs+1)}\right)^{\frac{1}{q}} A^{\frac{s}{q}}(a,b),$$

where we omit $\frac{(b-a)}{4}\left[\frac{2(2^{p+1}+3^{p+1})}{10^{p+1}(p+1)}\right]^{\frac{1}{p}}$, since they are identical in the two error bounds above.

Next, we compare the bounds above. For $a = 2$ and $b = 5$, from Figure 1 (a) and (b), show that $G_2(s,q)$ is a better error bound than $G_1(s,q)$. Therefore, it reveals that the result of Proposition 4.2 is better than that of Proposition 4.1.

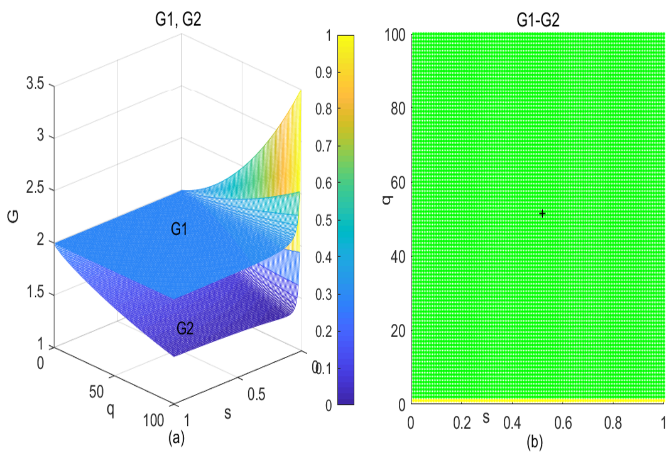

Figure 1: (**a**) Error surface of G_1 and G_2 on the variables s and q; (**b**) the positive and negative distribution corresponding to $G_1 - G_2$ (Here, the green part stands for $G_1 - G_2 \geq 0$).

4.2. Simpson Type Quadrature Formula

Let U be a partition of the interval $[a, b]$, i.e., $U : a = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = b$ and consider the quadrature formula

$$\int_a^b g(\mu)\mathrm{d}\mu = \mathcal{A}_S(g, U) + \mathcal{R}_S(g, U),$$

where

$$\mathcal{A}_S(g, U) = \frac{3}{10}\sum_{j=0}^{n-1}\Big[g(\mu_j) + g(\mu_{j+1})\Big](\mu_{j+1} - \mu_j) + \frac{2}{5}\sum_{j=0}^{n-1} g\Big(\frac{\mu_j + \mu_{j+1}}{2}\Big)(\mu_{j+1} - \mu_j)$$

for the Simpson type formula and $\mathcal{R}_S(g, U)$ denotes the related approximation error of the integral $\int_a^b g(\mu)\mathrm{d}\mu$.

Proposition 4.3. *Suppose that all assumptions of Theorem 3.4 are satisfied. Then the following Simpson error estimate satisfies*

$$\begin{aligned}&\Big|\mathcal{R}_S(g, U)\Big|\\ &\leq \Big[\frac{2(2^{p+1} + 3^{p+1})}{10^{p+1}(p + 1)}\Big]^{\frac{1}{p}}\Big[\Big(\frac{2^{s+1} - 1}{2^s(s + 1)}\Big)^{\frac{1}{q}} + \Big(\frac{1}{2^s(s + 1)}\Big)^{\frac{1}{q}}\Big]\sum_{j=0}^{n-1}\frac{(\mu_{j+1} - \mu_j)^2}{2}\Big[\big|g'(\mu_j)\big| + \big|g'(\mu_{j+1})\big|\Big].\end{aligned}$$

Proof. Considering the subintervals of $[a, b]$, i.e., $[\mu_j, \mu_{j+1}] \subseteq [a, b]$, using inequality (15) in Corollary 3.6, we obtain

$$\begin{aligned}&\left|\frac{\mu_{j+1} - \mu_j}{10}\Big[3g(\mu_j) + 4g\Big(\frac{\mu_j + \mu_{j+1}}{2}\Big) + 3g(\mu_{j+1})\Big] - \int_{\mu_j}^{\mu_{j+1}} g(u)\mathrm{d}u\right|\\ &\leq \frac{(\mu_{j+1} - \mu_j)^2}{2}\Big[\frac{2(2^{p+1} + 3^{p+1})}{10^{p+1}(p + 1)}\Big]^{\frac{1}{p}}\Big[\Big(\frac{2^{s+1} - 1}{2^s(s + 1)}\Big)^{\frac{1}{q}} + \Big(\frac{1}{2^s(s + 1)}\Big)^{\frac{1}{q}}\Big]\Big[\big|g'(\mu_j)\big| + \big|g'(\mu_{j+1})\big|\Big].\end{aligned}$$

Summing over j from 0 to $n - 1$ and utilizing the triangle inequality, we get that

$$\begin{aligned}&\Big|\mathcal{R}_S(g, U)\Big|\\ &= \left|\mathcal{A}_S(g, U) - \int_a^b g(\mu)\mathrm{d}\mu\right|\\ &= \sum_{j=0}^{n-1}\left|\frac{1}{10}(\mu_{j+1} - \mu_j)\Big[3g(\mu_j) + 4g\Big(\frac{\mu_j + \mu_{j+1}}{2}\Big) + 3g(\mu_{j+1})\Big] - \int_{\mu_j}^{\mu_{j+1}} g(u)\mathrm{d}u\right|\\ &\leq \Big[\frac{2(2^{p+1} + 3^{p+1})}{10^{p+1}(p + 1)}\Big]^{\frac{1}{p}}\Big[\Big(\frac{2^{s+1} - 1}{2^s(s + 1)}\Big)^{\frac{1}{q}} + \Big(\frac{1}{2^s(s + 1)}\Big)^{\frac{1}{q}}\Big]\sum_{j=0}^{n-1}\frac{(\mu_{j+1} - \mu_j)^2}{2}\Big[\big|g'(\mu_j)\big| + \big|g'(\mu_{j+1})\big|\Big],\end{aligned}$$

which is the required result. □

4.3. q-digamma Function

Let $q > 1$ and $\tau > 0$, the q-digamma mapping ϕ_q is defined by

$$\begin{aligned}\phi_q(\tau) &= -\ln(q - 1) + \ln q\Big(\tau - \frac{1}{2} - \sum_{i=0}^{\infty}\frac{q^{-(i+\tau)}}{1 - q^{-(i+\tau)}}\Big)\\ &= -\ln(q - 1) + \ln q\Big(\tau - \frac{1}{2} - \sum_{i=1}^{\infty}\frac{q^{-i\tau}}{1 - q^{-i\tau}}\Big).\end{aligned}$$

Proposition 4.4. *Let $0 < a < b$, and $p^{-1} + q^{-1} = 1$ with $q > 1$. Then one has*

$$\begin{aligned}&\left|\frac{1}{10}\left[\phi_q\left(\frac{3a+b}{4}\right) + 3\phi_q\left(\frac{a+b}{2}\right) + \phi_q\left(\frac{a+3b}{4}\right)\right]\right.\\&\left.\quad - \frac{2^{2\alpha-2}\Gamma(1+\alpha)}{(b-a)^\alpha}\mathcal{J}^\alpha_{(\frac{a+b}{2})^-}\phi_q\left(\frac{3a+b}{4}\right) - \frac{2^{2\alpha-2}\Gamma(1+\alpha)}{(b-a)^\alpha}\mathcal{J}^\alpha_{(\frac{a+b}{2})^+}\phi_q\left(\frac{a+3b}{4}\right)\right|\\&\le \frac{b-a}{8}\mathcal{D}^{\frac{1}{p}}\left\{\left(\frac{1}{2^q(q+1)}\right)^{\frac{1}{q}}\left|\phi_q'(a)\right| + 2\left(\frac{2^{q+1}-1}{2^q(q+1)}\right)^{\frac{1}{q}}\left|\phi_q'\left(\frac{a+b}{2}\right)\right| + \left(\frac{1}{2^q(q+1)}\right)^{\frac{1}{q}}\left|\phi_q'(b)\right|\right\},\end{aligned}$$

where $\mathcal{D}$ is defined in Theorem 3.3.

Proof. Note that the mapping $\tau \mapsto \phi_q'(\tau)$, $\forall\, q > 1$ is completely monotonic on $(0, \infty)$. Consequently, the mapping $\tau \mapsto |\phi_q'(\tau)|$ is convex on $(0, \infty)$. Applying the mapping $g(\tau) = \phi_q(\tau)$, $\tau > 0$ with $s = 1 = m$ to Corollary 3.8, we obtain the desired inequality. □

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