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Chen's Improved Inequality for Pointwise Hemi-Slant Warped Products in Kaehler Manifolds

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Abstract. Recently, B.-Y. Chen discovered a technique to find the relation between second fundamental form and the warping function of warped product submanifolds. In this paper, we extend our further study of [24] by giving non-trivial examples of warped product pointwise hemi-slant submanifolds. Finally, we establish a sharp estimation for the squared norm of the second fundamental form $||h||^2$ in terms of the warping function f. The equality case is also investigated.

1. Introduction

Hemi-slant submanifolds of Kaehler manifolds were studied by Sahin [16] as a generalized class of CR-submanifolds studied by Bejancu [2] and B.-Y. Chen [3, 4]. In the same paper, Sahin studied their warped products in Kaehler manifolds. He proved that the warped products of the form $M_\perp \times_f M_\theta$ in a Kaehler manifold $\tilde{M}$ do not exist and then he introduced hemi-slant warped products of the form $M_\theta \times_f M_\perp$, where $M_\perp$ and M_θ are totally real and proper slant submanifold of $\tilde{M}$. He has given many examples and proved a characterization theorem. Moreover, he established a relationship for the squared norm of the second fundamental form and the warping function in terms of the slant angle.

Recently, Srivastava et al. [18] studied pointwise hemi-slant warped products of the form $M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$ in a Kaehler manifold and they proved several interesting results including characterisation theorems and inequalities. In fact, the results of second kind of pointwise hemi-slant warped products derived from [16], while the first case is interesting because in case of general slant the first kind of warped products do not exist in Kaehler manifolds.

In this paper, by giving an example of the existence of pointwise hemi-slant warped products of the form $M_\perp \times_f M_\theta$ we proved the modified Chen's inequality. In our earlier paper we proved that if the pointwise hemi-slant warped product submanifold $M = M_\perp \times_f M_\theta$ of a Kaehler manifold $\tilde{M}$ is mixed totally geodesic then M is Riemannian product of a totally real submanifold $M_\perp$ and a pointwise slant submanifold M_θ.

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2. Preliminaries

Let $\tilde{M}^m$ be a complex m-dimensional Kaehler manifold with complex structure J. Then, we have $\tilde{\nabla}J = 0$. Let M^n be a real n-dimensional Riemannian manifold isometrical immersed in $\tilde{M}$. We denote the metric tensor of $\tilde{M}$ as well as the induced metric on M by the same symbol g. Let ∇ and $\nabla^\perp$ be the induced connections on TM and $T^\perp M$, respectively. Then, the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{2}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$, where $\Gamma(TM)$ is the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ is the set of all vector normal to M. Moreover, $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\tilde{M}$ and A_ξ is the shape operator of M with respect to ξ.

For any X tanget to M, we write

$$JX = PX + FX, \tag{3}$$

where PX and FX are the tangential and normal components of JX, respectively. Then P is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM.

For any $p \in M$ and $\{e_1, \cdots, e_n, \cdots, e_{2m}\}$ is an orthonormal frame of $T_p\tilde{M}$ such that $e_1, \cdots, e_n$ are tangent to M at p. Then, we define the length of the second fundamental from by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} \Big(g(h(e_i, e_j), e_r)\Big)^2, \ \ \forall\, i, j \in \{1, \cdots, n\}, \ \ r \in \{n+1, \cdots, 2m\}. \tag{4}$$

A submanifold M is said to be totally geodesic if $h = 0$.

A (differentiable) distribution $\mathcal{D}$ defined on a submanifold M of $(\tilde{M}, J, g)$ is called pointwise *θ-slant* if, for each point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and $\mathcal{D}$ is independent of the choice of the nonzero vector $X \in \mathcal{D}$ (cf. [5, 6, 11]). A pointwise θ-slant distribution is called slant if θ is globally constant. Also, it is holomorphic or complex if $\theta = 0$; and it is called totally real if $\theta = \frac{\pi}{2}$, globally. A poitwise θ-slant distribution is called proper pointwise slant whenever $\theta \neq 0, \frac{\pi}{2}$ and θ is not a constant.

From Chen's result (Lemma 2.1) of [11], it is known that M is a pointwise slant submanifold of an almost Hermitian manifold $\tilde{M}$ if and only if

$$P^2 = -(\cos^2\theta)I, \tag{5}$$

for some real-valued function θ defined on M, where I denotes the identity transformation of the tangent bundle TM of M. The following relations are the consequences of (5) as

$$g(PX, PY) = \cos^2\theta\, g(X, Y), \tag{6}$$

$$g(FX, FY) = \sin^2\theta\, g(X, Y) \tag{7}$$

for any $X, Y \in \Gamma(TM)$.

Now, we define pointwise hemi-slant submanifolds of a Kaehler manifold.

Definition 2.1. *Let $\tilde{M}$ be a Kaehler manifold and M a real submanifold of $\tilde{M}$. Then, we say that M is a pointwise hemi-slant submanifold if there exists a pair of orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ on M such that*

(i) The tangent space TM admits the orthogonal direct decomposition $TM = \mathcal{D}^\perp \oplus \mathcal{D}^\theta$.

(ii) The distribution $\mathcal{D}^\perp$ is totally real, i.e. $J(\mathcal{D}^\perp) \subset T^\perp M$.

(iii) The distribution $\mathcal{D}^\theta$ is pointwise slant with slant function θ.

The normal bundle $T^\perp M$ of a poitwise hemi-slant submanifold M is decomposed by

$$T^\perp M = J\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu, \quad J\mathcal{D}^\perp \perp F\mathcal{D}^\theta \tag{8}$$

where μ is the invariant normal subbundle of $T^\perp M$.

We give the following example of pointwise hemi-slant submanifolds in the Euclidean space.

Let $\mathbb{R}^{2m}$ be the Euclidean $2m$-space with the standard Euclidean metric and let $\mathbb{C}^m$ denote the complex Euclidean m-space $(\mathbb{R}^{2m}, J)$ equipped with the canonical complex structure J defined by

$$J(x_1, y_1, \ldots, x_m, y_m) = (-y_1, x_1, \ldots, -y_m, x_m).$$

Thus we have

$$J\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad i, j = 1, \ldots, m. \tag{9}$$

Example 2.2. *Consider a submanifold M of $\mathbb{C}^4$ given by the immersion*

$$\chi(u, v, \theta, \phi) = ((u^2 + v^2)/2,\ u,\ (u^2 - v^2)/2,\ v,\ \cos(\theta + \phi),\ \sin(\theta + \phi),\ \cos(\theta - \phi),\ \sin(\theta - \phi))$$

for non-vanishing u, $v\,(u \neq v)$. Then, the tangent space of M is spanned by the following orthogonal frame fields $\{X_1, Y_1, X_2, Y_2\}$, where

$$\begin{aligned}
X_1 &= u\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + u\frac{\partial}{\partial x_2}, \quad Y_1 = v\frac{\partial}{\partial x_1} - v\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},\\
X_2 &= -\sin(\theta + \phi)\frac{\partial}{\partial x_3} + \cos(\theta + \phi)\frac{\partial}{\partial y_3} - \sin(\theta - \phi)\frac{\partial}{\partial x_4} + \cos(\theta - \phi)\frac{\partial}{\partial y_4},\\
Y_2 &= -\sin(\theta + \phi)\frac{\partial}{\partial x_3} + \cos(\theta + \phi)\frac{\partial}{\partial y_3} + \sin(\theta - \phi)\frac{\partial}{\partial x_4} - \cos(\theta - \phi)\frac{\partial}{\partial y_4}.
\end{aligned}$$

Clearly, JX_2 and JY_2 are orthogonal to TM and hence $\mathcal{D}^\perp = \mathrm{Span}\{X_2, Y_2\}$ is a totally real distribution and $\mathcal{D}^\theta = \mathrm{Span}\{X_1, Y_1\}$ is a proper pointwise slant distribution with slant angle $\Theta = \cos^{-1}\left(\frac{u-v}{\sqrt{(1+2u^2)(1+2v^2)}}\right)$. Hence, M is a proper pointwise hemi-slant submanifold of $\mathbb{C}^4$.

3. Pointwise hemi-slant warped products

In this section, we study pointwise hemi-slant warped products of the form $M_\perp \times_f M_\theta$ in a Kaehler manifold. We define these submanifolds as follows:

Definition 3.1. A warped product $M_\perp \times_f M_\theta$ of a totally real submanifold $M_\perp$ and a pointwise slant submanifold M_θ of a Kaehler manifold $(\tilde{M}, J, g)$ is called a *warped product pointwise hemi-slant submanifold*.

A warped product pointwise hemi-slant submanifold $M_\perp \times_f M_\theta$ is called *proper* if M_θ is proper pointwise slant and $M_\perp$ is totally real in $\tilde{M}$. Otherwise, $M_\perp \times_f M_\theta$ is called *non-proper*.

First, we recall the following lemmas.

Lemma 3.2. *[1] For $U, V \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that*

(i) $\nabla_U V \in \Gamma(TB)$,

(ii) $\nabla_U Z = \nabla_Z U = U(lnf)Z$,

(iii) $\nabla_Z W = \nabla'_Z W - \frac{g(Z,W)}{f}\vec{\nabla} f$,

where ∇ and ∇' denote the Levi-Civita connections on M and F, respectively and $\vec{\nabla} f$ is the gradient of f defined by $g(\vec{\nabla} f, U) = U(f)$.

Remark 3.3. *It is also important to note that for a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [1, 7].*

Lemma 3.4. *[24] Let $M = M_\perp \times {}_f M_\theta$ be a warped product pointwise hemi-slant submanifold of a Kaehler manifold $\tilde{M}$. Then*

(i) $g(h(X_1, Y_1), FX_2) = g(h(X_2, X_1), JY_1)$;
(ii) $g(h(X_2, Y_2), JX_1) = X_1(\ln f)\, g(X_2, PY_2) + g(h(X_1, X_2), FY_2)$

for any $X_1, Y_1 \in \Gamma(TM_\perp)$ and $X_2, Y_2 \in \Gamma(TM_\theta)$.

It is easy to obtain the following relation by interchanging X_2 by PX_2 and Y_2 by PY_2 in Lemma 3.4 (ii).

$$g(h(PX_2, Y_2), JX_1) = \cos^2\theta\, X_1(lnf)\, g(X_2, Y_2) + g(h(PX_2, X_1), FY_2), \tag{10}$$

$$g(h(X_2, PY_2), JX_1) = -\cos^2\theta\, X_1(lnf)\, g(X_2, Y_2) + g(h(X_1, X_2), FPY_2) \tag{11}$$

and

$$g(h(PX_2, PY_2), JX_1) = \cos^2\theta\, X_1(lnf)\, g(X_2, PY_2) + g(h(PX_2, X_1), FPY_2). \tag{12}$$

Theorem 3.5. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise hemi-slant sunmanifold of a Kaehler manifold $\tilde{M}$. Then, if $\theta = 0$, then M is a CR-product of $M_\perp$ and M_0.*

Proof. The proof follows in two ways: If $\theta = 0$, then M_0 be a complex submanifold and then by Theorem 3.1 of [7], the result follows.

On the other hand, from Lemma 3.4 (ii), we have

$$g(h(X_2, Y_2), JX_1) = X_1(\ln f)\, g(X_2, JY_2) \tag{13}$$

for any $X_1 \in \Gamma(TM_\perp)$ and $X_2, Y_2 \in \Gamma(TM_0)$. By polarization identity, we find

$$g(h(X_2, Y_2), JX_1) = -X_1(\ln f)\, g(X_2, JY_2). \tag{14}$$

Then, from (13) and (14), we find that

$$X_1(\ln f)\, g(X_2, JY_2) = 0.$$

Since, g is a Rieannian metric, then we find $X_1(\ln f) = 0$, that is f is constant. □

In [18], Srivastava et al. established the following inequality.

Theorem 3.6. *Let $M = M_\perp \times_f M_\theta$ be a mixed totally geodesic warped product submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}$ such that $M_\perp$ is a q-dimensional totally real submanifold and M_θ is a $2p$-dimensional proper pointwise slant submanifold of $\tilde{M}$. Then, the squared norm of the second fundamental form $||h||^2$ of M satisfies*

$$||h||^2 \geq 2p \cos^2\theta ||\nabla(\ln f)||^2, \tag{15}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

From the following corollary of [24], we find that the above theorem is not valid due to the non-existence of mixed totally geodesic warped products. On the other hand, the equality case of the above inequality was not discussed.

Theorem 3.7. *(Corollary 4.5 of [24]) There does not exist any proper warped product mixed totally geodesic submanifold of the form $M = M_\perp \times_f M_\theta$ of a Kaehler manifold $\tilde{M}$ such that $M_\perp$ is a totally real submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$.*

Thus, we have given the following remark.

Remark 3.8. *[24] The inequality for second fundamental form of these kind of warped products may not be evaluated.*

On the contrary statement in the above remark, in this paper, we establish the inequality when the warped product is not mixed totally geodesic.

Let $M = M_\perp \times_f M_\theta$ be a n-dimensional warped product submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}$. We denote the tangent bundles of $M_\perp$ and M_θ by $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ with their real dimensions q and $2p$, respectively. Then, $\{e_1, \cdots, e_q\}$ and $\{e_{q+1} = e_1^*, \cdots, e_{q+p} = e_p^*,\ e_{q+p+1} = e_{p+1}^* = \sec\theta\, Pe_1^*, \cdots, e_{q+2p} = e_{2p}^* = \sec\theta\, Pe_p^*\}$ are the orthonormal frame fields of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. Clearly, we have

$$J\mathcal{D}^\perp = \mathrm{Span}\{\mathrm{Je_1}, \cdots \mathrm{Je_q}\},\quad \mathrm{F}\mathcal{D}^\theta = \mathrm{Span}\{\csc\theta\,\mathrm{Fe_1^*}, \cdots \csc\theta\mathrm{Fe_p^*},\ \csc\theta\sec\theta\mathrm{FPe_1^*}, \cdots, \csc\theta\sec\theta\mathrm{FPe_p^*}\},$$
$$\mu = \mathrm{Span}\{\tilde{\mathrm{e}}_1, \cdots, \tilde{\mathrm{e}}_{2(\mathrm{m-n})}\}.$$

Now, we prove the main theorem of this paper.

Theorem 3.9. *Let $M = M_\perp \times_f M_\theta$ be a warped product submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}$ such that $h(\mathcal{D}^\theta, \mathcal{D}^\theta)$ is orthogonal to $J\mathcal{D}^\perp$ where $\mathcal{D}^\perp$ is a q-dimensional totally real distribution corresponding to $M_\perp$ and $\mathcal{D}^\theta$ is a $2p$-dimensional proper pointwise slant distribution corresponding to M_θ in $\tilde{M}$. Then,*

(i) The squared norm of the second fundamental form $\|h\|^2$ of M satisfies

$$\|h\|^2 \geq 4p\cot^2\theta\, \|\nabla(\ln f)\|^2, \tag{16}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

(ii) If the equality sign in (16) holds identically, then $M_\perp$ and M_θ are totally geodesic and totally umbilical proper pointwise slant submanifolds of $\tilde{M}$, respectively. Moreover, M never be a mixed totally geodesic submanifold of $\tilde{M}$.

Proof. For a pointwise hemi-slant submanifold, we have

$$\|h\|^2 = \|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 + 2\|h(\mathcal{D}^\perp, \mathcal{D}^\theta)\|^2 + \|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\|^2. \tag{17}$$

Then, using (4), we find the each term in the right hand side of (17) as follows:

$$\|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 = \sum_{i,j=1}^{q} g(h(e_i, e_j),\ h(e_i, e_j)) = \sum_{k=1}^{2m}\sum_{i,j=1}^{q} \Big(g(h(e_i, e_j),\ e_k)\Big)^2.$$

Using the constructed frame fields, we derive

$$\|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 = \sum_{k=1}^{q}\sum_{i,j=1}^{q} \Big(g(h(e_i, e_j),\ Je_r)\Big)^2 + \sum_{k=1}^{2p}\sum_{i,j=1}^{q} \Big(g(h(e_i, e_j),\ Fe_k^*)\Big)^2 + \sum_{k=1}^{2(m-n)}\sum_{i,j=1}^{q} \Big(g(h(e_i, e_j),\ \tilde{e}_r)\Big)^2. \tag{18}$$

As we have no relations for the warped products for the first and third terms, by leaving these positive terms, we find

$$\|h(\mathcal{D}^\perp, \mathcal{D}^\perp)\|^2 \geq \csc^2\theta \sum_{k=1}^{p}\sum_{i,j=1}^{q} \Big\{\Big(g(h(e_i, e_j),\ Fe_r^*)\Big)^2 + \sec^2\theta\Big(g(h(e_i, e_j),\ FPe_k^*)\Big)^2\Big\}.$$

Using Lemma 3.4 (i), we have

$$\|h(\mathcal{D}^\perp,\mathcal{D}^\perp)\|^2 \geq \csc^2\theta \sum_{k=1}^{p}\sum_{i,j=1}^{q}\left\{\left(g(h(e_i,e_k^*),\ Je_j)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_k^*),\ Je_j)\right)^2\right\}. \tag{19}$$

From the leaving terms of (18), we find

$$h(\mathcal{D}^\perp,\mathcal{D}^\perp) \perp J\mathcal{D}^\perp, \qquad h(\mathcal{D}^\perp,\mathcal{D}^\perp) \perp \mu. \tag{20}$$

Similarly, we obtain

$$\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)\|^2 = \sum_{i,k=1}^{q}\sum_{j=1}^{2p}\left(g(h(e_i,e_j^*),Je_k)\right)^2 + \sum_{j,k=1}^{2p}\sum_{i=1}^{q}\left(g(h(e_i,e_j^*),Fe_k^*)\right)^2 + \sum_{k=1}^{2(m-n)}\sum_{i=1}^{q}\sum_{j=1}^{2p}\left(g(h(e_i,e_j^*),\tilde{e}_k)\right)^2. \tag{21}$$

Leaving the third positive term in the right hand side and using the adopted frame fields, we derive

$$\begin{aligned}\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)\|^2 \geq &\sum_{i,k=1}^{q}\sum_{j=1}^{p}\left\{\left(g(h(e_i,e_j^*),Je_k)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_j^*),Je_k)\right)^2\right\}\\ &+ \csc^2\theta\sum_{j,k=1}^{p}\sum_{i=1}^{q}\left\{\left(g(h(e_i,e_j^*),Fe_k^*)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_j^*),Fe_k^*)\right)^2\right\}\\ &+ \csc^2\theta\sec^2\theta\sum_{j,k=1}^{p}\sum_{i=1}^{q}\left\{\left(g(h(e_i,e_j^*),FPe_k^*)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_j^*),FPe_k^*)\right)^2\right\}\end{aligned}$$

and by the leaving term, we find

$$h(\mathcal{D}^\perp,\mathcal{D}^\theta) \perp \mu. \tag{22}$$

Using Lemma 3.4 (ii) and relations (10)-(12), finally we get

$$\|h(\mathcal{D}^\perp,\mathcal{D}^\theta)\|^2 \geq \sum_{i,k=1}^{q}\sum_{j=1}^{p}\left\{\left(g(h(e_i,e_j^*),Je_k)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_j^*),Je_k)\right)^2\right\} + 2p\cot^2\theta\sum_{i}^{q}\left(e_i(\ln f)\right)^2. \tag{23}$$

Furthermore, we find

$$\|h(\mathcal{D}^\theta,\mathcal{D}^\theta)\|^2 = \sum_{k=1}^{q}\sum_{i,j=1}^{2p}\left(g(h(e_i^*,e_j^*),Je_k)\right)^2 + \sum_{k=1}^{2p}\sum_{i=1,j}^{2p}\left(g(h(e_i^*,e_j^*),Fe_k^*)\right)^2 + \sum_{k=1}^{2(m-n)}\sum_{i,j=1}^{2p}\left(g(h(e_i^*,e_j^*),\tilde{e}_k)\right)^2 = 0. \tag{24}$$

Since, we have no relation for the second and the third terms in the middle equation of (24). Whenever, the first term vanishes identically by the hypothesis of the theorem. Then, from leaving these positive terms, we find

$$h(\mathcal{D}^\theta,\mathcal{D}^\theta) \perp F\mathcal{D}^\theta \quad h(\mathcal{D}^\theta,\mathcal{D}^\theta) \perp J\mathcal{D}^\perp, \quad h(\mathcal{D}^\theta,\mathcal{D}^\theta) \perp \mu. \tag{25}$$

Then, using (19), (23) and (24) in (17), we derive

$$\|h\|^2 \geq \left(2+\csc^2\theta\right)\sum_{i,k=1}^{q}\sum_{j=1}^{p}\left\{\left(g(h(e_i,e_j^*),Je_k)\right)^2 + \sec^2\theta\left(g(h(e_i,Pe_j^*),Je_k)\right)^2\right\} + 4p\cot^2\theta\|\nabla(\ln f)\|^2. \tag{26}$$

Hence, the inequality (16) follows from (26) and from the leaving fist positive terms, we have either $\csc^2\theta = -2$, which is not possible or $g(h(\mathcal{D}^\perp, \mathcal{D}^\theta), J\mathcal{D}^\perp) = 0$, i.e.,

$$h(\mathcal{D}^\perp, \mathcal{D}^\theta) \perp J\mathcal{D}^\perp. \tag{27}$$

Then, from (20), (22), (25) and (27), we conclude that

$$h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \;\; h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\theta) \subset F\mathcal{D}^\theta. \tag{28}$$

If the equality holds in (16), then from Remark 3.3 with (28) we conclude that $M_\perp$ is totally geodesic submanifold and M_θ is a totally umbilical submanifold of $\tilde{M}$. Moreover, by Corollary 3.7 M never be a mixed totally geodesic submanifold of $\tilde{M}$. Hence, the theorem is proved completely. $\square$

Now, we provide a non-trivial example of warped product pointwise hemi-slant submanifolds in Euclidean space.

Example 3.10. *Consider a submanifold M of $\mathbb{C}^6$ with cartesian coordinates $(x_1, y_1, \cdots, x_6, y_6)$ and the complex structure J is defined in* (9)*. Let M is defined by the following immersion*

$$\phi(u, v, w) = (ku,\ 0,\ u\cos v,\ u\sin v,\ kw,\ kv,\ -u\sin w,\ -u\cos w,\ -u\sin v,\ -u\cos v,\ u\cos w,\ u\sin w)$$

for non-zero constant k and non-vanishing functions u, v and w on M. Then, the tangent space of M is spanned by the following vector fields

$$\begin{aligned}
X_1 &= k\frac{\partial}{\partial x_1} + \cos v\frac{\partial}{\partial x_2} + \sin v\frac{\partial}{\partial y_2} - \sin w\frac{\partial}{\partial x_4} - \cos w\frac{\partial}{\partial y_4} - \sin v\frac{\partial}{\partial x_5} - \cos v\frac{\partial}{\partial y_5} + \cos w\frac{\partial}{\partial x_6} + \sin w\frac{\partial}{\partial y_6},\\
X_2 &= -u\sin v\frac{\partial}{\partial x_2} + u\cos v\frac{\partial}{\partial y_2} + k\frac{\partial}{\partial y_3} - u\cos v\frac{\partial}{\partial x_5} + u\sin v\frac{\partial}{\partial y_5},\\
Y_2 &= k\frac{\partial}{\partial x_3} - u\cos w\frac{\partial}{\partial x_4} + u\sin w\frac{\partial}{\partial y_4} - u\sin w\frac{\partial}{\partial x_6} + u\cos w\frac{\partial}{\partial y_6}.
\end{aligned}$$

Clearly, JX_1 is orthogonal to $TM = \mathrm{Span}\{X_1, X_2, Y_2\}$ and hence $\mathcal{D}^\perp = \mathrm{Span}\{X_1\}$ is a totally real distribution, while $\mathcal{D}^\theta = \mathrm{Span}\{X_2, Y_2\}$ is a proper pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{k^2}{\sqrt{2u^2+k^2}}\right)$. Hence, M is a proper pointwise hemi-slant submanifold. It is easy to verify that both the distributions are integrable. If $M_\perp$ and M_θ are their integral leaves corresponding to $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively, then the metric tensor of the product manifold $M = M_\perp \times M_\theta$ is given by

$$g = \left(4 + k^2\right)du^2 + \left(2u^2 + k^2\right)\left(dv^2 + dw^2\right) = g_{M_\perp} + f^2 g_{M_\theta}$$

where $f = \sqrt{k^2 + 2u^2}$ is the warping function on M. Hence, M is a warped product hemi-slant submanifold of $\mathbb{C}^6$.

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