Filomat 34:4 (2020), 1187–1196
https://doi.org/10.2298/FIL2004187A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Perturbation Theory, M-Essential Spectra of Operator Matrices

Boulbeba Abdelmoumen[a], Sonia Yengui[a]

[a] *Département de Mathématiques, Institut préparatoire aux études d'ingénieurs de Sfax, Route de Manzel Chaker, Km 0.5, BP , 3000, Sfax, Tunisie.*

Abstract. In this paper, we will establish some results on perturbation theory of block operator matrices acting on X^n, where X is a Banach space. These results are exploited to investigate the M-essential spectra of a general class of operators defined by a 3×3 block operator matrix acting on a product of Banach spaces X^3.

1. Introduction

Let X be a Banach space. In this paper, we investigate the M-essential spectra of a general class of operators defined by a 3×3 block operator matrix acting on a product of Banach spaces X^3

$$L_0 = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix},$$

where the entries of the matrix are in general unbounded operators. Note that L_0 is neither a closed nor a closable operator, even if its entries are closed. We prove under some conditions, that L_0 is closable. We shall denote its closure by L. We denote by $\mathcal{L}(X)$ (respectively $\mathcal{C}(X)$) the set of all bounded (respectively closed, densely defined) linear operators acting on X and we denote by $\mathcal{K}(X)$ the subspace of compact operators. For $T \in \mathcal{C}(X)$, we write $\mathcal{D}(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset X$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$ in X.

We denote by $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$ the classes of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators. The sets of left and right Fredholm inverses are respectively, defined by:

$$\Phi_l(X) := \{T \in \mathcal{C}(X) \text{ such that } T \text{ has a left Fredholm inverse}\},$$
$$\Phi_r(X) := \{T \in \mathcal{C}(X) \text{ such that } T \text{ has a right Fredholm inverse}\}.$$

Let $\Phi_+^b(X)$, $\Phi_-^b(X)$, $\Phi^b(X)$, $\Phi_l^b(X)$ and $\Phi_r^b(X)$ denote respectively the sets

$$\Phi_+(X) \cap \mathcal{L}(X),\ \Phi_-(X) \cap \mathcal{L}(X),\ \Phi(X) \cap \mathcal{L}(X),\ \Phi_l(X) \cap \mathcal{L}(X) \text{ and } \Phi_r(X) \cap \mathcal{L}(X).$$

2010 *Mathematics Subject Classification*. Primary 47Bxx
Keywords. Operator matrices; Essential spectra; Fredholm perturbation.
Received: 14 September 2014; Accepted: 23 March 2020
Communicated by Dragan S. Djordjević
Email addresses: Boulbeba.Abdelmoumen@ipeis.rnu.tn (Boulbeba Abdelmoumen), sonia.yengui@ipeis.rnu.tn (Sonia Yengui)

It follows from [9, Theorem 14. and 15. p. 160] that

$$\Phi_l^b(X) = \{T \in \Phi_+^b(X) \text{ such that } R(T) \text{ is complemented}\}$$

and

$$\Phi_r^b(X) = \{T \in \Phi_-^b(X) \text{ such that } \ker(T) \text{ is complemented}\}.$$

Note that we have the following inclusions:

$$\Phi^b(X) \subset \Phi_l^b(X) \subset \Phi_+^b(X)$$

and

$$\Phi^b(X) \subset \Phi_r^b(X) \subset \Phi_-^b(X).$$

Definition 1.1. *Let X be a Banach space and let $F \in \mathcal{L}(X)$.*

(*i*) *The operator F is called Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$.*

(*ii*) *F is called a upper (resp. lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X)$ (resp. $U + F \in \Phi_-(X)$) whenever $U \in \Phi_+(X)$ (resp. $U \in \Phi_-(X)$).*

(*iii*) *F is called a left (resp. right) semi-Fredholm perturbation if $U + F \in \Phi_l(X)$ (resp. $U + F \in \Phi_r(X)$) whenever $U \in \Phi_l(X)$ (resp. $U \in \Phi_r(X)$).*

We denote by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, $\mathcal{F}_-(X)$, $\mathcal{F}_l(X)$, $\mathcal{F}_r(X)$, the sets of Fredholm, upper semi-Fredholm, lower semi-Fredholm, left semi-Fredholm and right semi-Fredholm respectively.

If in Definition 1.1 we replace $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_l(X)$ and $\Phi_r(X)$ by $\Phi^b(X)$, $\Phi_+^b(X)$, $\Phi_-^b(X)$, $\Phi_l^b(X)$ and $\Phi_r^b(X)$ we obtain the sets $\mathcal{F}^b(X)$, $\mathcal{F}_+^b(X)$, $\mathcal{F}_-^b(X)$, $\mathcal{F}_l^b(X)$ and $\mathcal{F}_r^b(X)$. These classes of operators were introduced and investigated in [7, 13]. In particular it is shown that $\mathcal{F}^b(X)$, $\mathcal{F}_+^b(X)$, $\mathcal{F}_-^b(X)$, $\mathcal{F}_l^b(X)$ and $\mathcal{F}_r^b(X)$ are closed two-sided ideals of $\mathcal{L}(X)$. Note that in general we have:

$$\mathcal{K}(X) \subset \mathcal{F}_+^b(X) \subset \mathcal{F}^b(X),$$
$$\mathcal{K}(X) \subset \mathcal{F}_-^b(X) \subset \mathcal{F}^b(X).$$

The following result was established in [3]

Lemma 1.2. *[3] Let X be a Banach space, then*

$$\mathcal{F}(X) = \mathcal{F}^b(X),\ \mathcal{F}_+(X) = \mathcal{F}_+^b(X) \text{ and } \mathcal{F}_-(X) = \mathcal{F}_-^b(X).$$

Let $S \in \mathcal{L}(X)$. For $T \in \mathcal{C}(X)$, we define the S-resolvent set by:

$$\rho_S(T) := \{\lambda \in \mathbb{C},\ \lambda S - T \text{ has a bounded inverse}\},$$

and the S-spectrum of T

$$\sigma_S(T) = \mathbb{C} \setminus \rho_S(T).$$

In this paper, for $S \in \mathcal{L}(X)$, we are concerned with the following S-essential spectra:

$$\begin{aligned}
&\sigma_{e_1,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_+(X)\},\\
&\sigma_{e_2,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_-(X)\},\\
&\sigma_{e_3,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_\pm(X)\},\\
&\sigma_{e_4,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi(X)\},\\
&\sigma_{e_5,S}(T) := \mathbb{C} \setminus \{\lambda \in \Phi_{T,S} \text{ such that } i(\lambda S - T) = 0\},\\
&\sigma_{e_6,S}(T) := \mathbb{C} \setminus \{\lambda \in \mathbb{C} \text{ such that all scalars near } \lambda \text{ are in } \rho_S(T) \text{ and that } i(\lambda S - T) = 0\}.\\
&\sigma_{le,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_l(X)\},\\
&\sigma_{re,S}(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \notin \Phi_r(X)\}.
\end{aligned}$$

Remark that

$$\sigma_{e_3,S}(T) := \sigma_{e_1,S}(T) \cap \sigma_{e_2,S}(T) \subset \sigma_{e_4,S}(T) \subset \sigma_{e_5,S}(T) \subset \sigma_{e_6,S}(T).$$

Note that if $S = I$, we recover the usual definition of the essential spectra of a closed densely defined operator T.

A complex number λ is in $\Phi_{T,S}$ if $\lambda S - T \in \Phi(X)$. The set $\Phi_{T,S}$ has very nice properties such as:

Proposition 1.3. *[15] Let $T \in \mathcal{C}(X)$ and S a non null bounded linear operator acting on X. Then we have the following results:*

(i) $\Phi_{T,S}$ is open.

(ii) $i(\lambda S - T)$ is constant on any component of $\Phi_{T,S}$.

(iii) $\alpha(\lambda S - T)$ and $\beta(\lambda S - T)$ are constant on any component of $\Phi_{T,S}$ except on a discrete set of points on which they have larger values.

In the following we will denote the complement of a subset $\Omega \subset \mathbb{C}$ by ${}^C\Omega$.

Proposition 1.4. *[15] Let $T \in \mathcal{C}(X)$ and $M \in \mathcal{L}(X)$.*
(i) If ${}^C\sigma_{e_4,M}(T)$ is connected and $\rho_M(T)$ is not empty, then

$$\sigma_{e_4,M}(T) = \sigma_{e_5,M}(T).$$

(ii) If ${}^C\sigma_{e_5,M}(T)$ is connected and $\rho_M(T)$ is not empty, then

$$\sigma_{e_5,M}(T) = \sigma_{e_6,M}(T).$$

The study of the essential spectra of block operator matrices has been arround for many years. Among the works in this subject we can quote, for example, [1, 4–6, 8, 14–19]. Note that the idea of studying the spectral characteristics of block operator matrices goes back to the classics of the spectral theoryfor the differential operator (see for instance [9–12]). Recently, C. Tretter gives in [16–18] an account research and presents a wide panorama of methods to investigate the spectral theory of block operator matrices. In the paper [6], M. Faierman, R. Mennicken and M. Möller propose a method for dailing with the spectral theory for pencils of the form $L_0 - \mu M$, where M is a bounded operator. The authors in [4], extend the obtained results in [19] and prove some localization results on the essential spectra of a general class of operators defined by a 2×2 block operator matrix. The analysis uses the concept of the measures of weak-noncompactness which possess some nice properties (cf [2]). Similarly, [15] study the M-essential spectra of 2×2 operator matrix. Whereas in the paper of [5], Aref and all investigate the essential spectra of a 3×3 blok operator matrix.

The purpose of this work is to pursue the analysis started in [4, 5, 8, 15, 19]. In Section 1, we establish some stability results on Fredholm theory. The main results of this section is Theorem 2.4. In Section 2, we apply the results of Section 1 to describe the M-essential spectra of a general class of operators defined by a 3×3 block operator matrix, where M is a bounded operator (see Theorem 3.3).

2. Some results on perturbation theory of matrix operator

In this section we will establish some results on perturbation theory of matrix operator that acts on X^n where X is a Banach space. We beguin with the following preparating results which are crucial for the purpose of our paper.

Proposition 2.1. *Let $A_{ij} \in \mathcal{L}(X)$, $(i, j) \in \{1, ..., n\}^2$ such that $A_{ij} = 0$ if $i > j$, and consider the matrix operator:*

$$T_u = (A_{ij})_{1 \le i,j \le n} \in \mathcal{L}(X^n).$$

(*i*) *If,* $\forall i \in \{1, ..., n\}$, $A_{ii} \in \Phi_*(X)$, *then* $T_u \in \Phi_*(X^n)$, *where* $_*$ *designs* $+, -, l$ *or* r.

(*ii*) *If* $T_u \in \Phi_+(X^n)$, *then* $A_{11} \in \Phi_+(X)$.

(*iii*) *If* $T_u \in \Phi_-(X^n)$, *then* $A_{nn} \in \Phi_-(X)$.

(*iv*) *If* $T_u \in \Phi_l(X^n)$, *then* $A_{11} \in \Phi_l(X)$.

(*v*) *If* $T_u \in \Phi_r(X^n)$, *then* $A_{nn} \in \Phi_r(X)$.

Proof. (*i*) We can write T_u in the following form:

$$T_u = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix} \begin{pmatrix} I & A_{12} & \cdots & A_{1n} \\ 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & I & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{pmatrix} \quad (1)$$

We use a reasoning by induction on $n \in \mathbb{N} \backslash \{0, 1\}$ and we apply [9, Theorem 5, p 156].

The results of (*ii*) and (*iv*) follow immediately from (1) and [9, Theorem 6, p 157].

The assertions (*iii*) and (*v*) can be checked if we write T_u in the following form:

$$T_u = \begin{pmatrix} I & 0 & \cdots & . & 0 \\ 0 & \ddots & 0 & . & 0 \\ \vdots & \ddots & . & . & \vdots \\ . & . & & I & 0 \\ 0 & \cdots & . & 0 & A_{nn} \end{pmatrix} \begin{pmatrix} I & 0 & \cdots & 0 & A_{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & & 0 & . \\ \vdots & \ddots & \ddots & \ddots & A_{n-1n} \\ 0 & \cdots & & 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1\,n-1} & 0 \\ 0 & \ddots & \vdots & \vdots \\ & & . & . \\ \vdots & \ddots & A_{n-1\,n-1} & 0 \\ 0 & \cdots & 0 & I \end{pmatrix}. \quad (2)$$

Using the same reasoning as the proof of the previous proposition, we can show the following:

Proposition 2.2. *Let* $A_{ij} \in \mathcal{L}(X)$, $(i, j) \in \{1, ..., n\}^2$ *such that* $A_{ij} = 0$ *if* $i < j$, *and consider the matrix operator:*

$$T_l = (A_{ij})_{1 \leq i,j \leq n} \in \mathcal{L}(X^n).$$

(*i*) *If,* $\forall i \in \{1, ..., n\}$, $A_{ii} \in \Phi_*(X)$ *then* $T_l \in \Phi_*(X^n)$, *where* $_*$ *designs* $+, -, l$ *or* r.

(*ii*) *If* $T_l \in \Phi_+(X^n)$, *then* $A_{nn} \in \Phi_+(X)$.

(*iii*) *If* $T_l \in \Phi_-(X^n)$, *then* $A_{11} \in \Phi_-(X)$.

(*iv*) *If* $T_l \in \Phi_l(X^n)$, *then* $A_{nn} \in \Phi_l(X)$.

(*v*) *If* $T_l \in \Phi_r(X^n)$, *then* $A_{11} \in \Phi_r(X)$.

As an immediate consequence of propositions 2.1 and 2.2 we have:

Corollary 2.3. *If* $T_u \in \Phi(X^n)$ *(resp.* $T_l \in \Phi(X^n)$*), then* $A_{11} \in \Phi_+(X)$ *and* $A_{nn} \in \Phi_-(X)$.
(resp. $A_{11} \in \Phi_-(X)$ *and* $A_{nn} \in \Phi_+(X)$*).*

The main result of this section is the following:

Theorem 2.4. *Let* $F := (F_{ij})_{1 \leq i,j \leq n}$ *where* $F_{ij} \in \mathcal{L}(X)$, $\forall (i, j) \in \{1, ..., n\}^2$. *Then*

(*i*) $F \in \mathcal{F}(X^n)$ *if and only if* $F_{ij} \in \mathcal{F}(X)$, $\forall (i, j) \in \{1, ..., n\}^2$.

(*ii*) $F \in \mathcal{F}_r(X^n)$ *if and only if* $F_{ij} \in \mathcal{F}_r(X)$, $\forall (i, j) \in \{1, ..., n\}^2$.

(*iii*) $F \in \mathcal{F}_l(X^n)$ *if and only if* $F_{ij} \in \mathcal{F}_l(X)$, $\forall (i, j) \in \{1, ..., n\}^2$.

Proof. (i) Suppose that $F := (F_{ij})_{1\le i,j\le n} \in \mathcal{F}(X^n)$. For $(i,j) \in \{1,...,n\}^2$, there exists P_{ij} and Q_{ij} two invertible matrix operators in $\mathcal{L}(X^n)$ such that:

$$P_{ij}FQ_{ij} = \begin{pmatrix} F_{ij} & & \cdots & \\ & & & \vdots \\ & & & \\ \vdots & \ddots & \ddots & \\ & \cdots & & \end{pmatrix} \in \mathcal{F}(X^n). \tag{3}$$

So, to prove that $F_{ij} \in \mathcal{F}(X)$, it suffice to prove that $F_{11} \in \mathcal{F}(X)$. Let A be in $\Phi(X)$ and consider

$$L_1 := \begin{pmatrix} A & -F_{12} & \cdots & . & -F_{1n} \\ 0 & I & -F_{23} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & . \\ & & & & -F_{n-1n} \\ 0 & \cdots & & 0 & I \end{pmatrix}.$$

It follows from Proposition 2.1(i) that $L_1 \in \Phi(X^n)$. Thus,

$$F + L_1 = \begin{pmatrix} F_{11}+A & 0 & \cdots & \cdots & 0 \\ F_{21} & I+F_{22} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ F_{n1} & \cdots & \cdots & F_{nn-1} & I+F_{nn} \end{pmatrix} \in \Phi(X^n).$$

Hence, by Corollary 2.3, $F_{11} + A \in \Phi_-(X)$.

Let $L_2 = \begin{pmatrix} A & 0 & \cdots & 0 \\ -F_{21} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -F_{n1} & \cdots & -F_{n\,n-1} & I \end{pmatrix}$. Then according to Proposition 2.1(i), $L_2 \in \Phi(X^n)$ and

$$F + L_2 = \begin{pmatrix} F_{11}+A & F_{12} & \cdots & F_{1n} \\ 0 & I+F_{22} & & \vdots \\ \vdots & \ddots & \ddots & F_{n-1\,nn} \\ 0 & \cdots & 0 & I+F_{nn} \end{pmatrix} \in \Phi(X^n).$$

Thus, by Corollary 2.3, $F_{11} + A \in \Phi_+(X)$ and therefore, $F_{11} \in \mathcal{F}(X)$.

Conversely, suppose that $F_{ij} \in \mathcal{F}(X)$, $\forall (i,j) \in \{1,...,n\}^2$. We can write:

$$F = \sum_{1\le i,j\le n} \widetilde{F}_{ij}, \text{ where } \widetilde{F}_{ij} = \begin{pmatrix} 0 & \cdots & & 0 & \cdots & \\ \vdots & & & \vdots & & \\ . & & & 0 & & \\ & \cdots & 0 & F_{ij} & 0\cdots & \\ \vdots & & & 0 & & \\ 0 & \cdots & & \vdots & \cdots & \end{pmatrix}.$$

So, it is sufficient to prove that if, for $(i, j) \in \{1, ..., n\}^2$, $F_{ij} \in \mathcal{F}(X)$, then $\widetilde{F}_{ij} \in \mathcal{F}(X^n)$. Using the same reasoning as (3):

$$P_{ij}\widetilde{F}_{ij}Q_{ij} = \begin{pmatrix} F_{ij} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & & & \\ 0 & \cdots & & 0 \end{pmatrix}. \tag{4}$$

So, to prove that $\widetilde{F}_{ij} \in \mathcal{F}(X^n)$, it suffice to prove that $\widetilde{F}_{11} \in \mathcal{F}(X^n)$.

Suppose that $F_{11} \in \mathcal{F}(X)$ and let $L := (L_{ij})_{1\le i,j\le n} \in \Phi(X^n)$. According to [9, Theorem 13. p. 159], there exists $L_0 := (L^0_{ij})_{1\le i,j\le n} \in \Phi(X^n)$ such that $LL_0 = I + K$, where $K \in \mathcal{K}(X^n)$. We have

$$(L + \widetilde{F}_{11})L_0 = I + K + \widetilde{F}_{11}L_0 = \begin{pmatrix} I + F_{11}L^0_{11} & F_{11}L^0_{12} & \cdots & F_{11}L^0_{1n} \\ 0 & I & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{pmatrix} + K.$$

Since $I + F_{11}L^0_{11} \in \Phi(X)$, then, by Proposition 2.1(*i*), $(L + \widetilde{F}_{11})L_0 \in \Phi(X^n)$. Thus $L + \widetilde{F}_{11} \in \Phi(X^n)$ and therefore $\widetilde{F}_{11} \in \mathcal{F}(X^n)$.

We prove the assertion (*ii*) in the same way as in (*i*).

To prove the assertion (*iii*), suppose that $F := (F_{ij})_{1\le i,j\le n} \in \mathcal{F}_l(X^n)$. Arguing as the proof of (*i*), we can deduce that $F_{ij} \in \mathcal{F}_l(X)$, $\forall (i, j) \in \{1, ..., n\}^2$. Conversely, Suppose that $F_{11} \in \mathcal{F}_l(X)$ and let $L := (L_{ij})_{1\le i,j\le n} \in \Phi_l(X^n)$. There exists $L_0 := (L^0_{ij})_{1\le i,j\le n} \in \Phi(X^n)$ such that $L_0L = I + K$, where $K \in \mathcal{K}(X^n)$. We have

$$L_0(L + \widetilde{F}_{11}) = I + K + L_0\widetilde{F}_{11} = \begin{pmatrix} I + F_{11}L^0_{11} & 0 & \cdots & 0 \\ L^0_{21}F_{21} & I & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L^0_{n1}F_{21} & \cdots & 0 & I \end{pmatrix} + K.$$

Since $I + F_{11}L^0_{11} \in \Phi_l(X)$, then, by Proposition 2.2, $L_0(L + \widetilde{F}_{11}) \in \Phi_l(X^n)$. Thus $L + \widetilde{F}_{11} \in \Phi_l(X^n)$ and therefore $\widetilde{F}_{11} \in \mathcal{F}_l(X^n)$.

3. The *M*-essential spectra of the 3 × 3 matrix operator *L*

The purpose of this section is to apply Theorem 2.4 to describe the M-essential spectra of the 3×3 matrix operator L, closure of L_0 that acts on the Banach space X^3 where M is a bounded operator formally defined on the product space X^3 by a matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

and L_0 is given by:

$$L_0 = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}.$$

Each of the entries operator acts on X and has its own domain.

In what follows, we will assume that the following hypotheses:

(H_1) The operator A is closed, densely defined linear operator on X with nonempty M_{11}-resolvent set $\rho_{M_{11}}(A)$.

(H_2) The operator D (resp. G) verifies that $\mathcal{D}(A) \subset \mathcal{D}(D)$ (resp. $\mathcal{D}(A) \subset \mathcal{D}(G)$) and for some (hence for all) $\mu \in \rho_{M_{11}}(A)$, the operator $D(A - \mu M_{11})^{-1}$ (resp. $G(A - \mu M_{11})^{-1}$) is bounded.

Set

$$F_1(\mu) = (D - \mu M_{21})(A - \mu M_{11})^{-1}.$$

and

$$F_2(\mu) = (G - \mu M_{31})(A - \mu M_{11})^{-1}.$$

(H_3) The operators B and C are densely defined on X and for some (hence for all) $\mu \in \rho_{M_{11}}(A)$, the operator $(A - \mu M_{11})^{-1}B$ (resp. $(A - \mu M_{11})^{-1}C$) is bounded on its domain.

Let

$$G_1(\mu) = \overline{(A - \mu M_{11})^{-1}(B - \mu M_{12})}$$

and

$$G_2(\mu) = (A - \mu M_{11})^{-1}(C - \mu M_{13}).$$

(H_4) The lineal $\mathcal{D}(B) \cap \mathcal{D}(E)$ is dense in X, and for some (hence for all) $\mu \in \rho_{M_{11}}(A)$, the operator $S_1(\mu) = E - (D - \mu M_{21})(A - \mu M_{11})^{-1}(B - \mu M_{12})$ is closed.

To explain this, let $\lambda, \mu \in \rho_{M_{11}}(A)$. We have:

$$S_1(\lambda) - S_1(\mu) = (\lambda - \mu)\Big(M_{21}G_1(\mu) + F_1(\lambda)M_{12} - F_1(\lambda)M_{11}G_1(\mu)\Big). \tag{5}$$

Since the operator on the right-hand side is bounded on its domain, then the operator $S_1(\mu)$ is closed for all $\mu \in \rho_{M_{11}}(A)$ if it is closed for some $\mu \in \rho_{M_{11}}(A)$.

(H_5) $\mathcal{D}(C) \subset \mathcal{D}(F)$ and the operator $F - D(A - \mu M_{11})^{-1}C$ is bounded on its domain for some (hence for all) $\mu \in \rho_{M_{11}}(A)$. We will suppose that there exist $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1)$ and we will denote by:

$$G_3(\mu) = \overline{(S_1(\mu) - \mu M_{22})^{-1}[(F - \mu M_{23}) - (D - \mu M_{21})(A - \mu M_{11})^{-1}(C - \mu M_{13})]}.$$

(H_6) The operator H satisfies that $\mathcal{D}(B) \subset \mathcal{D}(H)$, and for some (hence for all) $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1)$ the operator $H - G(A - \mu M_{11})^{-1}B(S_1(\mu) - \mu M_{22})^{-1}$ is bounded. Set

$$F_3(\mu) = [(H - \mu M_{32}) - (G - \mu M_{31})(A - \mu M_{11})^{-1}(B - \mu M_{12})](S_1(\mu) - \mu M_{22})^{-1}.$$

(H_7) For the operator K we will assume that $\mathcal{D}(C) \subset \mathcal{D}(K)$, and for some (hence for all) $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1)$ the operator $K - GG_2(\mu)H - F_2(\mu)BG_3(\mu)$ is closable. Denote by $S_2(\mu)$ the operator:

$$S_2(\mu) = K - (G - \mu M_{31})G_2(\mu)\Big[(H - \mu M_{32}) - F_2(\mu)(B - \mu M_{12})\Big]G_3(\mu).$$

and by $\overline{S_2(\mu)}$ its closure.

In the following theorem we establish the closedness of the operator L_0.

Theorem 3.1. *Let the conditions $(H_1) - (H_7)$ be satisfied. Then the operator L_0 is closable if and only if $S_2(\mu)$ is closable in X, for some $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$. Moreover, the closure L of L_0 can been written as follows:*

$$L = \mu M - U(\mu)\mathcal{D}(\mu)W(\mu), \tag{6}$$

where

$$U(\mu) = \begin{pmatrix} I & 0 & 0 \\ F_1(\mu) & I & 0 \\ F_2(\mu) & F_3(\mu) & I \end{pmatrix}, \; W(\mu) = \begin{pmatrix} I & G_1(\mu) & G_2(\mu) \\ 0 & I & G_3(\mu) \\ 0 & 0 & I \end{pmatrix}$$

and

$$\mathcal{D}(\mu) = \begin{pmatrix} \mu M_{11} - A & 0 & 0 \\ 0 & \mu M_{22} - S_1(\mu) & 0 \\ 0 & 0 & \mu M_{33} - \overline{S_2}(\mu) \end{pmatrix}.$$

Proof.
For $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, the operator L_0 can be factorized in the form:

$$L_0 = \mu M - U(\mu) \begin{pmatrix} \mu M_{11} - A & 0 & 0 \\ 0 & \mu M_{22} - S_1(\mu) & 0 \\ 0 & 0 & \mu M_{33} - S_2(\mu) \end{pmatrix} W(\mu). \tag{7}$$

The results follows the fact that the operators $U(\mu)$ and $W(\mu)$ are bounded and boundedly invertible.

Remark 3.2. *Let $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$ and set $\lambda \in \mathbb{C}$. While writing $L - \lambda M = L - \mu M + (\lambda - \mu)M$, we have*

$$L - \lambda M = U(\mu)\mathcal{D}_\lambda(\mu)W(\mu) - (\mu - \lambda)\mathcal{M}(\mu), \tag{8}$$

where

$$\mathcal{D}_\lambda(\mu) = \begin{pmatrix} A - \lambda M_{11} & 0 & 0 \\ 0 & S_1(\mu) - \lambda M_{22} & 0 \\ 0 & 0 & \overline{S_2(\mu)} - \lambda M_{33} \end{pmatrix}$$

and

$$\mathcal{M}(\mu) = \begin{pmatrix} 0 & M_{11}G_1(\mu) - M_{12} & M_{11}G_2(\mu) - M_{13} \\ F_1(\mu)M_{11} - M_{21} & F_1(\mu)M_{11}G_1(\mu) & F_1(\mu)M_{11}G_2(\mu) + M_{22}G_3(\mu) - M_{23} \\ F_2(\mu)M_{11} - M_{31} & F_2(\mu)M_{11}G_1(\mu) + F_3(\mu)M_{22} - M_{32} & F_2(\mu)M_{11}G_2(\mu) + F_3(\mu)M_{22}G_3(\mu) \end{pmatrix}.$$

Now, we are ready to state and prove the main result of this section.

Theorem 3.3. *Suppose that the assumptions $(H_1) - (H_7)$ are satisfied.*

(i) If, $\forall i \neq j$, $M_{ij} \in \mathcal{F}(X)$ and if, for some $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $F_k(\mu)$ and $G_k(\mu)$ are in $\mathcal{F}(X)$, $\forall k \in \{1, 2, 3\}$, then

$$\sigma_{e_4,M}(L) = \sigma_{e_4,M_{11}}(A) \cup \sigma_{e_4,M_{22}}(S_1(\mu)) \cup \sigma_{e_4,M_{33}}(\overline{S}_2(\mu)).$$

and

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_{11}}(A) \cup \sigma_{e_5,M_{22}}(S_1(\mu)) \cup \sigma_{e_5,M_{33}}(\overline{S}_2(\mu)).$$

Moreover, if the sets ${}^{C}\sigma_{e_4,M_{11}}(A)$ *and* ${}^{C}\sigma_{e_4,M_{22}}(S_1(\mu))$ *are connected, then*

$$\sigma_{e_5,M}(L) = \sigma_{e_5,M_{11}}(A) \cup \sigma_{e_5,M_{22}}(S_1(\mu)) \cup \sigma_{e_5,M_{33}}(\overline{S}_2(\mu)).$$

If in addition, ${}^{C}\sigma_{e_5,M_{11}}(A)$, ${}^{C}\sigma_{e_5,M_{22}}(\overline{S}_1(\mu))$ *are connected and* $\rho_{M_{33}}(\overline{S}_1(\mu)) \neq \emptyset$, *then*

$$\sigma_{e_6,M}(L) = \sigma_{e_6,M_{11}}(A) \cup \sigma_{e_6,M_{22}}(S_1(\mu)) \cup \sigma_{e_6,M_{33}}(\overline{S}_2(\mu)).$$

(*ii*) *If,* $\forall i \neq j$, $M_{ij} \in \mathcal{F}_l(X)$ *and if, for some* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $F_k(\mu)$ *and* $G_k(\mu)$ *are in* $\mathcal{F}_l(X)$, $\forall k \in \{1, 2, 3\}$, *then*

$$\sigma_{le,M}(L) = \sigma_{le,M_{11}}(A) \cup \sigma_{le,M_{22}}(S_1(\mu)) \cup \sigma_{le,M_{33}}(\overline{S}_2(\mu)).$$

(*iii*) *If* $\forall i \neq j$, $M_{ij} \in \mathcal{F}_r(X)$ *and if for some* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $F_k(\mu)$ *and* $G_k(\mu)$ *are in* $\mathcal{F}_r(X)$, $\forall k \in \{1, 2, 3\}$, *then*

$$\sigma_{re,M}(L) = \sigma_{re,M_{11}}(A) \cup \sigma_{re,M_{22}}(S_1(\mu)) \cup \sigma_{re,M_{33}}(\overline{S}_2(\mu)).$$

To prove Theorem 3.3 we shall need to the following lemma:

Lemma 3.4. (*i*) *Let* $\mu \in \rho_{M_{11}}(A)$.

If $F_1(\mu)$, $G_1(\mu)$ *and* M_{21} *are in* $\mathcal{F}(X)$ *then* $\sigma_{e_4,M_{22}}(S_1(\mu))$ *and* $\sigma_{e_5,M_{22}}(S_1(\mu))$ *do not depend on* μ.

(*ii*) *Let* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$.

If $G_1(\mu)$, $G_3(\mu)$ *and* M_{31} *are in* $\mathcal{F}(X)$, *then* $\sigma_{e_4,M_{22}}(\overline{S_2(\mu)})$ *and* $\sigma_{e_5,M_{22}}(\overline{S_2(\mu)})$ *do not depend on* μ.

Proof.

(*i*) Follows immediately from the equation (5).

(*ii*) Let $\lambda, \mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$. Then

$$\begin{aligned} S_2(\lambda) - S_2(\mu) = (G - \lambda M_{31})[G_2(\mu) - G_2(\lambda) - G_1(\mu)G_3(\lambda) + G_1(\lambda) - G_3(\mu)] + \\ (H - \lambda M_{32})(G_3(\mu) - G_3(\lambda)) + (\lambda - \mu)(M_{31}G_2(\mu) + M_{32}G_3(\mu) \\ - M_{31}G_1(\mu)G_3(\mu)). \end{aligned}$$

Proof of Theorem 3.1.

(*i*) According to the hypotheses and applying Theorem 2.4, the second operator in the right hand side of Eq. (8), $\mathcal{M}(\mu)$, is a Fredholm perturbation. Since $U(\mu)$ and $W(\mu)$ are boundlessly invertible, then

$$L - \lambda M \in \Phi(X^3) \Longleftrightarrow \mathcal{D}_\lambda(\mu) \in \Phi(X^3).$$

Moreover, we have

$$i(L - \lambda M) = i(\mathcal{D}_\lambda(\mu)) = i(A - \lambda M_{11}) + i(S_1(\mu) - \lambda M_{22}) + i(S_2(\mu) - \lambda M_{22}).$$

If $i(A - \lambda M_{11}) = i(S_1(\mu) - \lambda M_{22}) = i(S_2(\mu) - \lambda M_{22}) = 0$, then $i(L - \lambda M) = 0$. Hence

$$\sigma_{e_5,M}(L) \subseteq \sigma_{e_5,M_{11}}(A) \cup \sigma_{e_5,M_{22}}(S_1(\mu)) \cup \sigma_{e_5,M_{22}}(\overline{S}_2(\mu)).$$

Finally, the results of assertion (*i*) follow from Proposition 1.4

We can prove easily (*ii*) and (*iii*) by using the relation (8).

Theorem 3.5. *Suppose that the assumptions* $(H_1)-(H_7)$ *are satisfied.*

(i) *If, for some* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $G_k(\mu) \in \mathcal{F}_+(X)$, $\forall k \in \{1,2,3\}$ *and* $\mathcal{M}(\mu) \in \mathcal{F}_+(X)$, *then*

$$\sigma_{e_1,M}(L) = \sigma_{e_1,M_{11}}(A) \cup \sigma_{e_1,M_{22}}(S_1(\mu)) \cup \sigma_{e_1,M_{33}}(\overline{S}_2(\mu)).$$

(ii) *If, for some* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $G_k(\mu) \in \mathcal{F}_-(X)$, $\forall k \in \{1,2,3\}$ *and* $\mathcal{M}(\mu) \in \mathcal{F}_-(X)$, *then*

$$\sigma_{e_2,M}(L) = \sigma_{e_2,M_{11}}(A) \cup \sigma_{e_2,M_{22}}(S_1(\mu)) \cup \sigma_{e_2,M_{33}}(\overline{S}_2(\mu)).$$

(iii) *If, for some* $\mu \in \rho_{M_{11}}(A) \cap \rho_{M_{22}}(S_1(\mu))$, $G_k(\mu) \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, $\forall k \in \{1,2,3\}$ *and* $\mathcal{M}(\mu) \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, *then*

$$\sigma_{e_3,M}(L) = \sigma_{e_3,M_{11}}(A) \cup \sigma_{e_3,M_{22}}(S_1(\mu)) \cup \sigma_{e_3,M_{33}}(\overline{S}_2(\mu)) \cup$$

$$\sigma_{e_1,M_{11}}(A) \cap \Big[\sigma_{e_2,M_{22}}(S_1(\mu)) \cap \sigma_{e_2,M_{33}}(\overline{S}_2(\mu))\Big] \cup \sigma_{e_1,M_{22}}(S_1(\mu)) \cap \Big[\sigma_{e_2,M_{11}}(A) \cup \sigma_{e_2,M_{33}}(\overline{S}_2(\mu))\Big] \cup \sigma_{e_1,M_{33}}(\overline{S}_2(\mu)) \cap \Big[\sigma_{e_2,M_{11}}(A) \cup \sigma_{e_2,M_{22}}(S_1(\mu))\Big].$$

Proof.

The results follow immediately from (8).

References

[1] F. V. Atkinson, H. Langer, R. Mennicken and A. A. Shkalikov, *The essential spectrum of some matrix operators,* Math. Nachr., 167 5-20 (1994).
[2] J. Banaś and Rivero, J. *On measures of weak noncompactness*, Ann. Mat. Pura Appl, **151** (1988), 213-262.
[3] A. Ben Amar, A. Jeribi, M. Mnif, *Some results on Fredholm and semi-Fredholm Perturbations and applications.*, preprint, 2010.
[4] B. Abdelmoumen, *Essential Spectra of Some Matrix Operators by Means of Measures of weak noncompactness.*, Oper and Matr. 2014
[5] A. Ben Amar, A.Jeribi and B. Krichen, *Essential Spectra of a* 3×3 *Operator Matrices and application to Three-Group Transport Equations.*, Integr.Equ. Oper. Theory. 68(2010), 1-12.
[6] M. Faierman, R. Mennicken and M. Moller, *A boundary Eigenvalue Problem for a system of partial Differential Operators Occuring in Magnetohydrodynamics Math. Nachr.* , 141-167 (1995).
[7] I. C. Gohberg, A. S. Markus and I. A. Feldman, *Normally solvable operators and ideals associated with them,* Amer. Math. Soc. Transl. ser. 2, 61, 63-84 (1967).
[8] N. Moalla, M. Dammak, A. Jeribi, *Essential spectra of some matrix operators and application to two-group transport operators with general boundary conditions.*, J. Math. Anal. Appl. 323 (2006) 1071-1090.
[9] V. Müller, *Spectral theory of linear operator and spectral system in Banach algebras.*, Operator theory advance and application vol. 139, (2003).
[10] R. Nagel, *Well-posedness and positivity for systems of linear evolution equations,* Confer. Sem. Univ. Math. Bari., 203, 29 (1985).
[11] R. Nagel, *Towrds a matrix theory for unbounded operator matrices*, Math. Z., 201 No. 1, 57-68 (1989).
[12] R. Nagel, *The spectrum of unbounded operator matrices with non-diagonal domain,* J. Func. Anal., 89 No. 2, 291-302 (1990).
[13] M. Schechter, *Riesz operators and Fredholm perturbations,* Bull. Amer. Math. Soc. 64 (1968), 1139-1144.
[14] A.Jeribi, N.Moalla, I.Walha, *Spectra of some block operator matrices and application to transport operators.*, J.Math. Anal. Appl. 351 (2009) 315-325.
[15] A.Jeribi, N.Moalla, S. Yengui, *Perturbation theory, M-essential spectra of* 2×2 *operator matrices and application to transport operators.*, (submitted)
[16] C. Tretter, *Spectral issues for block operator matrices in differential equations and mathematical physics,* (Birmingham, al, 1999), vol 16 of AMS-IP Stud. Adv. Math, pp 407-423
[17] C. Tretter, *Spectral inclusion for unbounded block operator matrices*, J. Funct. Anal., 256, 3806-3829 (2009).
[18] C. Tretter, *Spectral theory of block operator matrices and applications*, Imperial college. Press London (2008).
[19] A. A. Shkalikov, *On the essential spectrum of some matrix operators.*, Math. Notes, **58, 6** (1995), 1359-1362.