Filomat 34:4 (2020), 1197–1207
https://doi.org/10.2298/FIL2004197K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quaternionic Fock Space on Slice Hyperholomorphic Functions

Sanjay Kumar[a], S. D. Sharma[a], Khalid Manzoor[b]

[a] *Department of Mathematics, Central University of Jammu, Rahya Suchani(Bagla), Samba-1811 43, INDIA*
[b] *Department of Mathematics, Cluster University of Jammu, Jammu-1800 01, INDIA*

Abstract. In this paper, we define the quaternionic Fock spaces $\mathfrak{F}^p_\alpha$ of entire slice hyperholomorphic functions in a quaternionic unit ball $\mathbb{B}$ in $\mathbb{H}$. We also study growth estimates and various results of entire slice regular functions in these spaces. The work of this paper is motivated by the recent work of [5] and [26].

1. Introduction

The notation of slice hyperholomorphicity is introduced in [15] in 2006 and till then a lot of works have been done in this direction. Several function spaces like Hardy spaces, Bergman spaces, Bloch spaces, Besov spaces, Dirichlet spaces, Pontryagin De Branges Rovnyak spaces, etc are studied in the slice hyperholomorphic settings, see [5–12, 23, 26]. We refer to survey [14] and the book [13] for details information and references for the systematic development of slice hyperholomorphic functions and their applications. The Fock spaces in the slice hyperholomorphic settings were studied by D. Alpay, F. Colombo and I. Sabadini, [7]. The Fock spaces are fundamental for their role in quantum mechanics, see [3, 9, 28] and references therein. By symbol

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : x_l \in \mathbb{R} \text{ for } 0 \leq l \leq 3\},$$

we denote the set of 4-dimensional non-commutative real algebra of quaternions generated by imaginary units i, j, k such that

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The Euclidean norm of a quaternion q is given by

$$|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{\sum_{l=0}^{3} x_l^2}, \text{ for } x_l \in \mathbb{R},$$

2010 *Mathematics Subject Classification*. Primary 30G35, 47B38
Keywords. Fock space, slice regular Fock space, slice hyperholomorphic functions
Received: 01 October 2016; Accepted: 23 March 2020
Communicated by Dragan S. Djordjević
The work of the first author is supported by UGC-BSR Research Start-Up Grant No.F.30-412/2018(BSR)
Email addresses: sanjaykmath@gmail.com (Sanjay Kumar), somdatt_jammu@yahoo.co.in (S. D. Sharma), khalidcuj14@gmail.com (Khalid Manzoor)

where $\bar{q} = Rel(q) - Im(q) = x_0 - (x_1 i + x_2 j + x_3 k)$, denote the congugate of q. The multiplicative inverse of non-zero quarternion q is given by $\frac{\bar{q}}{|q|^2}$.
The set

$$\mathbb{S} = \{q \in \mathbb{H} : q = x_1 i + x_2 j + x_3 k \text{ and } x_1^2 + x_2^2 + x_3^2 = 1\}$$

represents the two-dimensional unit sphere of purely imaginary quaternions. Any element $I \in \mathbb{S}$ is such that $I^2 = -1$. This implies that the elements of S are imaginary units. The quaternion is considered as the union of complex plane $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ (also called slices) each one is identified by an imaginary unit $I \in \mathbb{S}$. Let $\Omega_I = \Omega \cap \mathbb{C}_I$, for some domain Ω of $\mathbb{H}$. For any quaternion q we can write

$$q = x_0 + x_1 i + x_2 j + x_3 k = x_0 + Im(q) = x_0 + |Im(q)| I_q = x + y I_q$$

with $I_q = \frac{Im(q)}{|Im(q)|}$ if $|Im(q)| \neq 0$, otherwise we take arbitrary I in $\mathbb{S}$.

Here, we begin with some basic results in the quaternionic-valued slice regular functions.

Definition 1.1. *[14, Definition 2.1.1] Let Ω be a domain in $\mathbb{H}$. A real differentiable function $f : \Omega \to \mathbb{H}$ is said to be the (left) slice regular or slice hyperholomorphic if for any $I \in \mathbb{S}$, f_I is holomorphic in Ω_I, i.e:,*

$$\left(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial y}\right) f_I(x_0 + yI) = 0,$$

where f_I denote the restriction of f to Ω_I. The class of slice regular function on Ω is denoted by $SR(\Omega)$.

For slice regular functions, we have the following useful result.

Theorem 1.2. *[16, Theorem 2.7] A function $f : \mathbb{B} \to \mathbb{H}$ is said to be slice regular if and only if it has a power series of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \text{ where } a_n = \frac{1}{n!}\frac{\partial^n f(0)}{\partial x^n} \tag{1}$$

converging uniformly on $\mathbb{B}$.

Splitting Lemma gives the relation between classical holomorphy and slice regularity.

Lemma 1.3. *[14, Definition 2.1.4] (Splitting Lemma) If f is a slice regular function on the domain Ω, then for any $i, j \in \mathbb{S}$, with $i \perp j$ there exists two holomorphic functions $F, L : \Omega_I \to \mathbb{C}_I$ such that*

$$f_I(z) = F(z) + L(z)J \text{ for any } z = x + yI. \tag{2}$$

Definition 1.4. *[14, Definition 2.2.1] Let Ω be an open set in $\mathbb{H}$. We say Ω is axially symmetric if for any $q = x + yI_q \in \Omega$ all the elements $x + yI$ are contained in Ω, for all $I \in \mathbb{S}$ and Ω is said to be slice domain if $\Omega \cap \mathbb{R}$ is non empty and $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.*

Theorem 1.5. *[14, Theorem 2.2.4] (Representation Formula) Let f be a slice regular function in the domain $\Omega \subset \mathbb{H}$. Then for any $j \in \mathbb{S}$ and for all $z = x + yI \in \Omega$,*

$$f(x + yI) = \frac{1}{2}\{(1 + IJ)f(x - yI) + (1 - IJ)f(x + yI)\}.$$

Remark 1.6. *Let I, J be orthogonal imaginary units in $\mathbb{S}$ and Ω be an axially symmetric slice domain. Then the Splitting Lemma and the Representation formula generate a class of operators on the slice regular functions as follows:*

$$Q_I : SR(\Omega) \to hol(\Omega_I) + hol(\Omega_I)J$$

$$Q_I : f \mapsto f_1 + f_2 J$$

$$P_I : hol(\Omega_I) + hol(\Omega_I)J \to SR(\Omega)$$

$$PI[f](q) = P_I[f](x + yI_q) = \frac{1}{2}[(1 - I_q I)f(x + yI) + (1 + I_q I)f(x - yI)].$$

Also,

$$P_I \circ Q_I = \mathrm{I}_{SR(\Omega)} \text{ and } Q_I \circ P_I = \mathrm{I}_{SR(hol(\Omega_I)+hol(\Omega_I))},$$

where I *is an identity operator.*

Since pointwise product of functions does not preserve slice regularity (see [13]) a new multiplication operation for regular functions is defined. In the special case of power series, the regular product (or $\star-$product) of $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ is

$$f \star g(q) = \sum_{n\geq 0} q^n \sum_{k=0}^{n} a_k b_{n-k}.$$

The $\star-$product is related to the standard pointwise product by the following formula.

Theorem 1.7. *[8, Proposition 2.4] Let f, g be regular functions on $\mathbb{B}$. Then $f \star g(q) = 0$ if $f(q) = 0$ and $f(q)g(f(q)^{-1}qf(q))$ if $f(q) \neq 0$. The reciprocal $f^{-\star}$ of a regular function $f(q) = \sum_{n=0}^{\infty} q^n a_n$ with respect to the $\star-$product is*

$$f^{\star}(q) = \frac{1}{f \star f^c(q)} f^c(q),$$

where $f^c(q) = \sum_{n=0}^{\infty} q^n \overline{a_n}$ is the regular conjugate of f. The function $f^{-\star}$ is regular on $\mathbb{B} \setminus (q \in \mathbb{B} | f \star f^c(q) = 0)$ and $f \star f^{-\star} = 1$ there.

2. Fock spaces

In this section, we study some basic properties of Fock spaces in the slice hyperholomorphic settings. Fock spaces of holomorphic functions are discussed in details in the book [28]. The slice hyperholomorphic quaternionic Fock spaces are studied in [7]. Let dA be the normalized area measure on $\mathbb{C}$. For $0 < p < \infty$, the Fock space $\mathfrak{F}_{p,\mathbb{C}}$ is defined as the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\frac{\alpha p}{2\pi} \int_{\mathbb{C}} \left| f(z) e^{\frac{-\alpha}{2}|z|^2} \right|^p dA(z) < \infty,$$

where $z \in \mathbb{C}$ and $dA(z) = \frac{1}{\pi} dxdy,\ z = x + jy,\ x, y \in \mathbb{R}$. Let $\mathbb{B}(0, 1) = \mathbb{B} = \{q = x + yI_q : |q| < 1\}$ be the quaternionic unit ball centered at origin in $\mathbb{H}$ and $\mathbb{B} \cap \mathbb{C}_I = \mathbb{B}_I$ denote unit disk in the complex plane $\mathbb{C}_I$ for $I \in \mathbb{S}$. A function slice regular on the quaternionic space $\mathbb{H}$ is called all slice regular and have power series representation of the form (1) converging everywhere in $\mathbb{H}$ and uniformily on the compact subsets of $\mathbb{H}$. Let $SR(\mathbb{H})$ denote the space of entire slice regular functions on the unit ball $\mathbb{B}$. Here we begin with the following definition.

Definition 2.1. *For $0 < p < \infty$ and $I \in \mathbb{S}$, the quaternionic right linear space of entire slice regular functions f is said to be the quaternionic slice regular Fock space on the unit ball $\mathbb{B}$, if for any $q \in \mathbb{B}$*

$$\frac{\alpha p}{2\pi} \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left| f(q) e^{\frac{-\alpha}{2}|q|^2} \right|^p dA_I(q) < \infty,$$

that is,

$$\mathfrak{F}^p_\alpha = \{f \in SR(\mathbb{H}) : \frac{\alpha p}{2\pi} \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left| f(q) e^{\frac{-\alpha}{2}|q|^2} \right|^p dA_I(q) < \infty\},$$

where $dA_I(q)$ denote the normalized differential area in the complex plane $\mathbb{C}_I$ such that area of $\mathbb{B}_I$ is equal to one and is Möbius invariant measure on $\mathbb{B}$ with norm given by

$$\|f\|_{\mathfrak{F}^p_\alpha} = \left(\frac{\alpha p}{2\pi}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|f(q)e^{\frac{-\alpha}{2}|q|^2}\right|^p dA_I(q):\ q = x + yI_q \in \mathbb{B}\right)^{\frac{1}{p}}.$$

By $\mathfrak{F}^p_{\alpha,I}$, we denote the quaternionic right linear space of entire slice regular functions on $\mathbb{B}$ such that

$$\frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p dA_I(z) < \infty.$$

Furthermore, for each function $f \in \mathfrak{F}^p_{\alpha,I}$, we define

$$\|f\|_{\mathfrak{F}^p_{\alpha,I}} = \left(\frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p dA_I(z)\ : z = x + yI \in \mathbb{B}\cap\mathbb{C}_I\right)^{\frac{1}{p}}.$$

Remark 2.2. *[26] Let $I \in \mathbb{S}$ be such that $J\perp I$. Then there exist holomorphic functions $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}_I$ such that $f_I = Q_I[f] = f_1 + f_2 J$ for some holomorphic map $Q_I[f]$ in complex variable $z \in \mathbb{B}_I$. Then*

$$\left|f_I(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p \le \left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p \le 2^{max\{0,p-1\}}\left|f_1(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p + 2^{max\{0,p-1\}}\left|f_2(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p.$$

The condition $f \in \mathfrak{F}^p_{\alpha,I}$ is equivalent to f_1 and f_2 belonging to one dimensional complex Fock space.

Proposition 2.3. *Suppose $I \in \mathbb{S}$ and $\alpha > 0$. Then $f \in \mathfrak{F}^p_{\alpha,I}$, $p > 1$ if and only if $f \in \mathfrak{F}^p_\alpha$. Moreover, the spaces $(\mathfrak{F}^p_{\alpha,I}, \|.\|_{\mathfrak{F}^p_{\alpha,I}})$ and $(\mathfrak{F}^p_\alpha, \|.\|_{\mathfrak{F}^p_\alpha})$ have equivalent norms. More precisely, one has*

$$\|f\|^p_{\mathfrak{F}^p_{\alpha,I}} \le \|f\|^p_{\mathfrak{F}^p_\alpha} \le 2^p\|f\|^p_{\mathfrak{F}^p_{\alpha,I}}.$$

Proof. Let $f \in \mathfrak{F}^p_\alpha$. Since $\mathbb{B}_I \subset \mathbb{B}$. Then by definition, $\|f\|^p_{\mathfrak{F}^p_{\alpha,I}} \le \|f\|^p_{\mathfrak{F}^p_\alpha}$ which implies $\mathfrak{F}^p_\alpha \subset \mathfrak{F}^p_{\alpha,I}$. Now, let $f \in \mathfrak{F}^p_{\alpha,I}$. For $q = x + yI_q \in \mathbb{B}$ with $I_q = \frac{Im(q)}{|Im(q)|}$ and $z = x + yI \in \mathbb{B}_I$ and as $|q| = |z|$. Then by Representation Formula for slice regular functions, we have

$$\begin{aligned}\frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\left|f(q)e^{\frac{-\alpha}{2}|q|^2}\right|^p dA_I(q) &= \frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\frac{1}{2}|(1 - I_q I)(f(z)e^{\frac{-\alpha}{2}|z|^2})\\ &+ (1 + I_q I)(f(\bar{z})e^{\frac{-\alpha}{2}|\bar{z}|^2})|^p dA_I(z)\\ &\le 2^{max\{0,p-1\}}\frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p dA_I(z)\\ &+ 2^{max\{0,p-1\}}\frac{\alpha p}{2\pi}\int_{\mathbb{B}_I}\left|f(\bar{z})e^{\frac{-\alpha}{2}|\bar{z}|^2}\right|^p dA_I(\bar{z}).\end{aligned}$$

Hence on taking supremum over all $I \in \mathbb{S}$, we have

$$\begin{aligned}\|f\|^p_{\mathfrak{F}^p_\alpha} &\le 2^{max\{0,p-1\}}\frac{\alpha p}{2\pi}\left(\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^p dA_I(z) + \int_{\mathbb{B}_I}\left|f(\bar{z})e^{\frac{-\alpha}{2}|\bar{z}|^2}\right|^p dA_I(\bar{z})\right)\\ &\le 2^{p-1}2\|f\|_{\mathfrak{F}^p_{\alpha,I}}.\end{aligned}$$

This completes the proof. □

We can easily prove the following results.

Proposition 2.4. *Suppose $p > 1, \alpha > 0$. If $f \in SR(\mathbb{H})$, then following statements are equivalent:*
(a) $f \in \mathfrak{F}^p_\alpha$;
(b) $f \in \mathfrak{F}^p_{\alpha,I}$ for some $I \in \mathbb{S}$.

Proposition 2.5. *Let $I, J \in \mathbb{S}, p > 1$ and $\alpha > 0$. If $f \in SR(\mathbb{H})$, then $f \in \mathfrak{F}^p_{\alpha,I}$ if and only if $f \in \mathfrak{F}^p_{\alpha,J}$.*

Proposition 2.6. *The space $\mathfrak{F}^p_\alpha$, $p > 1$ and $\alpha > 0$ is complete.*

Proof. Let $\{f_m\}_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathfrak{F}^p_\alpha$. Then, for $I \in \mathbb{S}$, $\{f_m\}$ is Cauchy sequence in $\mathfrak{F}^p_{\alpha,I}$. Let $J \in \mathbb{S}$ be such that $J \perp I$ and let $f_{m,1}, f_{m,2}$ be holomorphic functions such that $f_I = Q_I[f] = f_{m,1} + f_{m,2}J$. Since $\{f_{m,1}\}_{m\geq 0}$ and $\{f_{m,2}\}_{m\geq 0}$ are Cauchy sequences in the complex Fock space $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ and the fact that $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ is complete, so we conclude that, there exist functions $f_l \in \mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ such that each $f_{m,l} \to f_l$ as $m \to \infty$ for $l = 1, 2$. Now set $f = P_I(f_1 + f_2 J)$. Therefore,

$$\|f_m - f\|^p_{\mathfrak{F}^p_{\alpha,I}} \leq \|f_{m,1} - f_1\|^p_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} + \|f_{m,2} - f_2\|^p_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} \to 0 \text{ as } m \to \infty.$$

This implies that $f_m \to f$ in $\mathfrak{F}^p_{\alpha,I}$. Hence $f \in \mathfrak{F}^p_{\alpha,I}$ and so $f \in \mathfrak{F}^p_\alpha$. Thus, the slice regular Fock space $\mathfrak{F}^p_\alpha$ is complete. □

Remark 2.7. *If we write*

$$d\lambda_{\alpha,I}(q) = \frac{\alpha}{\pi} e^{-\alpha|q|^2} dA_I(q); q = x + yI_q \in \mathbb{B},$$

then the slice regular Fock space has the structure of quaternionic Hilbert space with their inner product $\langle ., . \rangle_\alpha$ defined by

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}_I} f(q)\overline{g(q)} d\lambda_{\alpha,I}(q)$$

for $f, g \in \mathfrak{F}^p_\alpha$.

Proposition 2.8. *On the slice regular Fock $\mathfrak{F}^p_\alpha$, the function $\langle ., . \rangle_\alpha$ is a quaternionic right linear inner product, i.e., for all $f, g, h \in \mathfrak{F}^p_\alpha$ and $a \in \mathbb{H}$, we have*

(*i*) *positivity: $\langle f, f \rangle_\alpha \geq 0$ and $\langle f, f \rangle_\alpha = 0$ if and only if $f = 0$;*

(*ii*) *quaternionic hermiticity: $\langle f, g \rangle_\alpha = \overline{\langle g, f \rangle_\alpha}$;*

(*iii*) *right linearity: $\langle f, ga + h \rangle_\alpha = \langle f, g \rangle_\alpha a + \langle f, h \rangle_\alpha$.*

Proposition 2.9. *For $p > 1$ and $\alpha > 0$, the space $(\mathfrak{F}^p_\alpha, \langle ., . \rangle_\alpha)$ is quaternionic Hilbert space.*

Proof. From Proposition 2.8, it follows that the function $\langle ., . \rangle_\alpha$ is a quaternionic right linear inner product and Proposition 2.6 shows that the slice regular Fock space is complete. □

Remark 2.10. *By $L^p(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H})$, we define the set of functions $g : \mathbb{B}_I \to \mathbb{H}$ such that*

$$\int_{\mathbb{B}_I} |g(w)|^p d\lambda_{\alpha,I}(w) < \infty,$$

where $d\lambda_{\alpha,I}(w) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA_I(w)$ for $\alpha > 0$ is called the Gaussian probability measure. Note that for $J \in \mathbb{S}$ with $J \perp I$ and $g = g_1 + g_2 J$ with $g_1, g_2 : \mathbb{B}_I \to \mathbb{C}_I$, then $g \in L^p(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H})$ if and only if $g_1, g_2 \in L^p(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{C}_I)$.
Clearly, $\mathfrak{F}^p_\alpha$ is a closed subspace of $L^p(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H})$. In complex analysis, the reproducing kernel of complex Fock space for $p = 2$ is given by

$$K^{\mathbb{C}_I}_\alpha(z, w) = e^{\alpha\langle z, w\rangle}; \ z, w \in \mathbb{C}_I.$$

This gives the motivation for the following definition.

Definition 2.11. *For any $q \in \mathbb{B}$, the slice regular exponential function is given by*

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}.$$

Let $e^{zw} = \sum_{n=0}^{\infty} \frac{z^n w^n}{n}$ be a holomorphic function in variable z in the complex plane $\mathbb{C}_I$. Clearly, e^{zw} is not slice regular in both variable. Setting $e_\star^{qw} = \sum_{n=0}^{\infty} \frac{q^n w^n}{n!}$, then we see that the function is left slice regular in q and right slice regular in w, where $\star$ denote the slice regular product. By Representation Formula, we can obtain the extension of function e^{zw} to $\mathbb{H}$, as

$$ext(e^{zw}) = \frac{1}{2}\{(1-IJ)e^{zw} + (1+IJ)e^{\bar{z}w}\} = e^{qw},$$

where $q \in \mathbb{B}$ and for some arbitrary w. For $I \in \mathbb{S}$ and $\alpha > 0$, we define

$$B_\alpha(q,w) = e_\star^{\alpha q \bar{w}} \text{ for each } q \in \mathbb{B}$$

and is called slice regular reproducing kernel of quaternionic Fock space.

Proposition 2.12. *For any positive integer m, the set of the form $e_m(q) = q^m \sqrt{\frac{\alpha}{m}}$ is orthonormal in the quaternionic Fock space $\mathfrak{F}_\alpha^2(\mathbb{B})$.*

Proof. By Lemma 1.3, we can write $f_I = f_1 + f_2 J$ for some $\mathbb{C}_I$-valued holomorphic functions f_1, f_2. Now for any $m > 0$, we have

$$\begin{aligned}\langle f, e_m\rangle_\alpha &= \langle f_1 + f_2 J, e_m\rangle_\alpha = \langle f_1, e_m\rangle_\alpha + \langle f_2 J, e_m\rangle_\alpha \\ &= \int_{\mathbb{B}_I} f_1(z) e_m d\lambda_{\alpha,I}(z) + \int_{\mathbb{B}_I} f_2(z) e_m d\lambda_{\alpha,I}(z) J.\end{aligned}$$

In complex plane every power series of the form $f_l(z) = \sum_{k=0}^{\infty} z^k a_{l,k},\ l = 1,2$ converges uniformly on $|z| < R$, for each $z \in \mathbb{B}_I$.
Therefore, we obtain

$$\begin{aligned}< f, e_m >_\alpha &= \int_{|z|<R} \sum_{k=0}^{\infty} z^k a_{1,k} e_m(z) d\lambda_{\alpha,I}(z) + \int_{|z|<R} \sum_{k=0}^{\infty} z^k a_{2,k} e_m(z) d\lambda_{\alpha,I}(z) J \\ &= \sum_{k=0}^{\infty} a_{1,k} \int_{|z|<R} z^k e_m(z) d\lambda_{\alpha,I}(z) + \sum_{k=0}^{\infty} a_{2,k} \int_{|z|<R} z^k e_m(z) d\lambda_{\alpha,I}(z) J \\ &= \lim_{R\to\infty} (a_{1,m} + a_{2,m} J) \int_{|z|<R} z^k e_m d\lambda_{\alpha,I}(z) \\ &= \lim_{R\to\infty} d_m \int_{\mathbb{B}_I} q^k e_m(q) d\lambda_{\alpha,I}(q),\end{aligned}$$

where $d_m = a_{1,m} + a_{2,m} J$. But in complex Fock space $\mathfrak{F}_\alpha^2(\mathbb{B}_I)$, each $a_{l,k} = 0$ for $l = 1,2$ implies $d_m = 0$ and so $f = 0$. Thus the sequence $\{e_m\}_{m>0}$ is complete in $\mathfrak{F}_\alpha^2(\mathbb{B})$. □

Proposition 2.13. *For some $I \in \mathbb{S}$, the slice regular orthogonal projection on $\mathbb{B}$ is defined by $T_{\alpha,I} : L^2(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H}) \to \mathfrak{F}_\alpha^2$. Then for all $q, w \in \mathbb{B}$, the integral representaion for $T_{\alpha,I}$ is given by*

$$T_{\alpha,I} f(q) = \frac{\alpha p}{2\pi} \int_{\mathbb{B}_I} f(w) B_\alpha(q,w) e^{-\alpha|w|^2} dA_I(w)$$

for all $f \in L_2(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H})$, where $B_\alpha(q,w) = e_\star^{\alpha\langle q,w\rangle} = e_\star^{\alpha q \bar{w}}$ is reproducing kernel for $\mathfrak{F}_\alpha^2$.

Proof. Given $f \in L^2(\mathbb{B}_I, d\lambda_{\alpha,I}, \mathbb{H})$, let $Q_I[f] = f_I$ be its restriction. Then we write $Q_I[f] = f_1 + f_2 J$, where J is an element of $\mathbb{S}$ such that $J \perp I$ and f_1, f_2 are complex valued holomorphic functions. Further, if for all $z, w \in \mathbb{B}_I$, then the two functions $K_\alpha^{\mathbb{C}_I}(z,w)$ and $B_\alpha(z,w)$ coincide and from the fact that $f = \langle f, B_\alpha \rangle$ (see [7, Theorem 3.10]), one conclude

$$\begin{aligned} T_{\alpha,I} f &= \langle T_{\alpha,I} f, B_\alpha(.,.)\rangle_\alpha \\ &= \langle T_{\alpha,I}(f_1 + f_2 J), K_\alpha^{\mathbb{C}_I}\rangle_\alpha \\ &= \langle T_{\alpha,I} f_1, K_\alpha^{\mathbb{C}_I}\rangle_\alpha + \langle T_{\alpha,I} f_2 J, K_\alpha^{\mathbb{C}_I}\rangle_\alpha \\ &= \langle f_1, K_\alpha^{\mathbb{C}_I}\rangle_\alpha + \langle f_2 J, K_\alpha^{\mathbb{C}_I}\rangle_\alpha \\ &= \langle f_1 + f_2 J, K_\alpha^{\mathbb{C}_I}\rangle_\alpha \\ &= \langle f, B_\alpha\rangle_\alpha \\ &= \int_{\mathbb{B}_I} f(w) B_\alpha(q,w) d\lambda_{\alpha,I}(w) \\ &= \frac{\alpha p}{2\pi} \int_{\mathbb{B}_I} f(w) B_\alpha(q,w) e^{-\alpha|w|^2} dA_I(w). \end{aligned}$$

This completes the proof. □

In the next result, we give the growth rate estimation for entire slice regular functions in quaternionic Fock space.

Theorem 2.14. *Let* $1 < p \le \infty$ *and* $\alpha > 0$. *Then for every* $f \in \mathfrak{F}_\alpha^p$,

$$\sup_{q \in \mathbb{B}} \left\{ |f(q)| : \|f\|_{\mathfrak{F}_\alpha^p} \le 1 \right\} \le 2e^{\frac{\alpha}{2}|q|^2}, \ \textit{where } q = x + yI_q \textit{ and } I_q = \frac{Im(q)}{|Im(q)|}.$$

Proof. Let I, J be the orthogonal imaginary units in two dimensional sphere $\mathbb{S}$. If $f \in \mathfrak{F}_\alpha^p$, then by Proposition 2.4, $f \in \mathfrak{F}_{\alpha,I}^p$ and $\|f\|_{\mathfrak{F}_{\alpha,I}^p} \le 1$. Now, we can find two holomorphic functions f_1, f_2 in $\mathfrak{F}_{\alpha,\mathbb{C}_I}^p$ such that $Q_I[f] = f_1 + f_2 J$. By using [28, Theorem 2.7], each f_l satisfies $\sup_{z \in \mathbb{B}_I} \left\{ |f_l(z)| : \ \|f_l\|_{\mathfrak{F}_{\alpha,\mathbb{C}_I}^p} \le 1 \right\} = e^{\frac{\alpha}{2}|z|^2}$. Furthermore,

$$\sup_{z \in \mathbb{B}_I} \left\{ |f(z)| : \|f\|_{\mathfrak{F}_{\alpha,I}^p} \le 1 \right\} \le \sup_{z \in \mathbb{B}_I} \left\{ |f_1(z)| : \|f_1\|_{\mathfrak{F}_{\alpha,\mathbb{C}_I}^p} \le 1 \right\} + \sup_{z \in \mathbb{B}_I} \left\{ |f_2(z)| : \|f_2\|_{\mathfrak{F}_{\alpha,\mathbb{C}_I}^p} \le 1 \right\}.$$

Let $q = x + yI_q$ and $z = x + yI$. By using triangle inequality and Theorem 1.5, we have

$$|f(q)| \le |f(z)| + |f(\bar{z})|.$$

On taking supremum over all $q \in \mathbb{B}$, we conclude that

$$\begin{aligned} \sup_{q \in \mathbb{B}} \left\{ |f(q)| : \|f\|_{\mathfrak{F}_\alpha^p} \le 1 \right\} &\le \sup_{z \in \mathbb{B}_I} \left\{ |f(z)| : \|f\|_{\mathfrak{F}_{\alpha,I}^p} \le 1 \right\} + \sup_{z \in \mathbb{B}_I} \left\{ |f(\bar{z})| : \|f\|_{\mathfrak{F}_{\alpha,I}^p} \le 1 \right\} \\ &= 2e^{\frac{\alpha}{2}|z|^2} \\ &= 2e^{\frac{\alpha}{2}|q|^2}. \end{aligned}$$

Hence the result. □

Corollary 2.15. *Suppose* $p > 1$ *and* $\alpha > 0$. *If* f *is in* $\mathfrak{F}_\alpha^p(\mathbb{B})$, *then*

$$|f(q)| \le 2^{p+1} e^{\frac{\alpha}{2}|q|^2} \|f\|_{\mathfrak{F}_\alpha^p}, \textit{ for all } q = x + yI_q \in \mathbb{B}.$$

Proof. Let $f \in \mathfrak{F}_{\alpha,I}^p$. Then by Remark 2.2 and [28, Corollary 2.8], we have

$$\begin{aligned} |f(z)|^p &\le 2^{p-1}(|f_1(z)|^p + |f_2(z)|^p) \\ &\le 2^{p-1}(e^{\frac{\alpha p}{2}|z|^2} \|f_1\|^p_{\mathfrak{F}_{\alpha,\mathbb{C}_I}^p} + e^{\frac{\alpha p}{2}|z|^2} \|f_2\|^p_{\mathfrak{F}_{\alpha,\mathbb{C}_I}^p}) \\ &\le 2^p e^{\frac{\alpha p}{2}|z|^2} \|f\|^p_{\mathfrak{F}_{\alpha,I}^p}. \end{aligned} \tag{3}$$

Now, on applying Representation Formula, condition (3) and Remark 2.3, we obtain

$$|f(q)|^p \leq 2|f(z)|^p \leq 2^{p+1} e^{\frac{\alpha p}{2}|z|^2} \|f\|^p_{\mathfrak{F}^p_{\alpha,I}} \leq 2^{p+1} e^{\frac{\alpha p}{2}|z|^2} \|f\|^p_{\mathfrak{F}^p_{\alpha}}.$$

□

We can easily prove the following result.

Proposition 2.16. *Let $1 < p < \infty$ and $r \in (0,1)$. Then for any $f \in \mathfrak{F}^p_\alpha$*

$$\lim_{r\to 1} \|f_r - f\|^p_{\mathfrak{F}^p_\alpha} = 0,$$

where $f_r(q) = f(rq) = \sum_{k=0}^{\infty} r^k q^k a_k$, for all $q \in \mathbb{B}$.

Proof. Let $f \in \mathfrak{F}^p_\alpha$. Then $f \in \mathfrak{F}^p_{\alpha,I}$. Let $I, J \in \mathbb{S}$ be such that $I \perp J$. Let f_1, f_2 be holomorphic functions in $\mathbb{B}_I$. By Remark 2.2, it follows that f_1, f_2 lie in the complex Fock space $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$. By applying corresponding results [28, Proposition 2.9 (a)] to f_1, f_2 in $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$, we obatin $\lim_{r\to 1}\|f_{l,r} - f_l\|^p_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} = 0, l = 1, 2$. Since $\|f\|^p_{\mathfrak{F}^p_\alpha} \leq 2^p \|f\|^p_{\mathfrak{F}^p_{\alpha,I}}$, we have

$$\begin{aligned} \lim_{r\to 1} \|f_r - f\|^p_{\mathfrak{F}^p_\alpha} &\leq 2^p \lim_{r\to 1} \|f_r - f\|^p_{\mathfrak{F}^p_{\alpha,I}} \\ &\leq 2^p \left(\lim_{r\to 1} \|f_{1,r} - f_1\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} - \lim_{r\to 1} \|f_{2,r} - f_2\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} \right) \\ &= 0. \end{aligned}$$

Hence $\lim_{r\to 1} \|f_r - f\|^p_{\mathfrak{F}^p_\alpha} = 0$. □

Proposition 2.17. *For $1 < p < \infty$, the slice regular Fock space is the closure of the sequence $\{p_m\}$ of quaternionic polynomials of the form $p_m(q) = \sum_{k=0}^{m} q^k \beta_{m,k}$, where $\beta_{m,k} \in \mathbb{H}$ with norm $\|.\|_{\mathfrak{F}^p_\alpha}$. In particular, the slice regular Fock space $\mathfrak{F}^p_\alpha$ is separable.*

Proof. Suppose $f \in \mathfrak{F}^p_\alpha$. Then $f \in \mathfrak{F}^p_{\alpha,I}$ so that $f_1, f_2 \in hol(\mathbb{B}_I)$, where $f_l, l = 1, 2$, is given by Splitting Lemma 1.3. Let $\beta_{m,k} = \zeta_{m,k} + \gamma_{m,k} J$, where $\zeta_{m,k}, \gamma_{m,k} \in \mathbb{C}_I$. By denseness property of polynomials in complex Fock space, we can choose polynomials of the form $p_{1,m}(z) = \sum_{k=0}^{m} z^k \zeta_{m,k}$ and $p_{2,m}(z) = \sum_{k=0}^{m} z^k \gamma_{m,k}$. Applying [28, Proposition 2.9 (b)] to each $f_l, p_{l,m}, l = 1, 2$, we see $\|p_{l,m} - f_l\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} \to 0$ as $m \to \infty$. Thus, we have

$$\begin{aligned} \|f - p_m\|_{\mathfrak{F}^p_\alpha} &\leq 2^p \|f - p_m\|_{\mathfrak{F}^p_{\alpha,I}} \\ &= 2^p \|(f_1 + f_2 J) - (p_{1,m} + p_{2,m} J)\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} \\ &\leq 2^p \|f_1 - p_{1,m}\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} - 2^p \|f_2 - p_{2,m}\|_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} \to 0 \text{ as } m \to \infty. \end{aligned}$$

Hence, $\mathfrak{F}^p_\alpha$ is separable. □

Proposition 2.18. *For $1 < p < u < \infty$ with $\frac{1}{p} + \frac{1}{u} = 1$, $\mathfrak{F}^p_\alpha \subset \mathfrak{F}^u_\alpha$. Moreover $\|f\|^u_{\mathfrak{F}^u_\alpha} \leq 2^{u+1} \frac{u}{p} \|f\|^u_{\mathfrak{F}^p_\alpha}$.*

Proof. Let $f \in \mathfrak{F}^p_\alpha$. For any $I, J \in \mathbb{S}$ with $I \perp J$. Then Lemma 1.3, guarantees the existence of holomorphic functions $f_1, f_2 : \mathbb{B} \cap \mathbb{C}_I \to \mathbb{C}_I$ such that $Q_I[f](z) = f_1(z) + f_2(z)J$, for all $z = x + yI \in \mathbb{B}_I$. From Remark

2.2, it follows that f_1, f_2 lie in the complex Fock space $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$. Therefore, from [28, Theorem 2.10], we have $\|f_l\|^u_{\mathfrak{F}^u_{\alpha,\mathbb{C}_I}} \leq \frac{u}{p}\|f_l\|^u_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}}$ for $l = 1, 2$. Furthermore,

$$\begin{aligned}
\frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(z) &\leq 2^{u-1}\frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f_1(z)e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(z)\\
&+ 2^{u-1}\frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f_2(z)e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(z)\\
&= 2^{u-1}(\|f_1\|^u_{\mathfrak{F}^u_{\alpha,\mathbb{C}_I}} + \|f_2\|^u_{\mathfrak{F}^u_{\alpha,\mathbb{C}_I}})\\
&\leq 2^{u-1}\frac{u}{p}(\|f_1\|^u_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}} + \|f_2\|^u_{\mathfrak{F}^p_{\alpha,\mathbb{C}_I}})\\
&\leq 2^{u}\frac{u}{p}\|f\|^u_{\mathfrak{F}^p_{\alpha,I}}. \qquad (4)
\end{aligned}$$

Now, Theorem 1.5 follows $|f(q)| \leq |f(z)| + |f(\bar{z})|$, where $q = x + yI_q \in \mathbb{B}$ with $I_q = \frac{Im(q)}{|Im(q)|}$ and $z = x + yI \in \mathbb{B}_I$ for all $x, y \in \mathbb{R}$ and by equation (4), we conclude that

$$\begin{aligned}
\frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f(q)e^{\frac{-\alpha}{2}|q|^2}\right|^u dA_I(q) &\leq \frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(z)\\
&+ \frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f(\bar{z})e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(\bar{z})\\
&\leq 2\frac{\alpha u}{2\pi}\int_{\mathbb{B}_I}\left|f(z)e^{\frac{-\alpha}{2}|z|^2}\right|^u dA_I(z)\\
&\leq 2^{u+1}\frac{u}{p}\|f\|^u_{\mathfrak{F}^p_{\alpha,I}}\\
&\leq 2^{u+1}\frac{u}{p}\|f\|^u_{\mathfrak{F}^p_{\alpha}}.
\end{aligned}$$

Hence the result. □

Proposition 2.19. *Let $1 < p < \infty$. Then for all $q, w \in \mathbb{B}$, the function $f(q) = P_I\sum_{m=0}^{n} e^{\beta q\bar{w}_m}a_m$ is dense in $\mathfrak{F}^p_\alpha$ for some positive parameters α and β.*

Proof. If $f \in \mathfrak{F}^p_\alpha$, then $f \in \mathfrak{F}^p_{\alpha,I}$. Let $f_1, f_2 \in hol(\mathbb{B}_I)$ given as in Lemma 1.3 such that $Q_I[f] = f_1 + f_2J$. Therefore from [28, Lemma 2.11], each functions of the form $f_1(z) = \sum_{m=0}^{n} e^{\beta q\bar{w}_m}c_m$ and $f_2(z) = \sum_{m=0}^{n} e^{\beta q\bar{w}_m}d_m$ is dense on $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ on $\mathbb{B}_I$. Consequently,

$$Q_I[f](z) = f_1(z) + f_2(z)J = \sum_{m=0}^{n} e^{\beta z\bar{w}_m}c_m + \sum_{m=0}^{n} e^{\beta z\bar{w}_m}d_mJ.$$

This implies that for each $q, w \in \mathbb{B}$, we have

$$f = P_I \circ Q_I[f] = P_I\left[\sum_{m=0}^{n} e^{\beta q\bar{w}_m}(c_m + d_mJ)\right] = P_I\left[\sum_{m=0}^{n} e^{\beta q\bar{w}_m}a_m\right],$$

where the sequence $a_m = c_m + d_mJ$ lie in $l^p(\mathbb{H})$. Thus, the density of f_1 and f_2 in $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ implies f is dense in $\mathfrak{F}^p_{\alpha,I}$. Therefore, from Proposition 2.4, we conclude that the set of functions f is dense $\mathfrak{F}^p_\alpha$. □

Proposition 2.20. *Let $0 < p \leq \infty$ and for some $I \in \mathbb{S}$. Then $f \in \mathfrak{F}^p_\alpha(\mathbb{B})$ if and only if there exists $\mathbb{H}$-valued Borel measure μ such that*

$$f(q) = \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}q - \frac{\alpha}{2}|\zeta|^2} d\mu(\zeta) \text{ for each } \zeta, q \in \mathbb{B} \text{ and } \{|\mu|(G_1 + w) : w \in \mathbb{H}\} \in l^p(\mathbb{H}). \tag{5}$$

Proof. Suppose $f \in \mathfrak{F}^p_\alpha(\mathbb{B})$ implies $f \in \mathfrak{F}^p_{\alpha,I}(\mathbb{B}_I)$. Let $J \in \mathbb{S}$ be such that $J \perp I$. Then f decomposes as $f_I = f_1 + f_2 J$, where $f_1, f_2 : \mathbb{B} \cap \mathbb{C}_I \to \mathbb{C}_I$ with $J \perp I$. Clearly the holomorphic functions f_1, f_2 lie in the complex Fock space $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}$ on $\mathbb{B}_I$. Further, for each $f_l \in \mathfrak{F}^p_{\alpha,\mathbb{C}_I}(\mathbb{B}_I)$, $l = 1, 2$ (see [28, p 91]), there exist complex positive Borel measure μ_1 and μ_2 on $\mathbb{B}_I$ such that $f_l(z) = \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}z - \frac{\alpha}{2}|\zeta|^2} d\mu_l(\zeta)$, for each $z = x + yI \in \mathbb{B}_I$ and $\{|\mu_l|(G_r + w) : w \in r\mathbb{R}^2\} \in l^p(\mathbb{C}_I)$. Now if we decompose $\mu = \mu_1 + \mu_2 J$, then, we can write

$$\begin{aligned} f(q) = Q_I[f_1 + f_2](q) &= \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}q - \frac{\alpha}{2}|\zeta|^2} d\mu_1(\zeta) + \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}q - \frac{\alpha}{2}|\zeta|^2} d\mu_2(\zeta) J \\ &= \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}q - \frac{\alpha}{2}|\zeta|^2} d\mu(\zeta). \end{aligned}$$

Conversely, assume the condition (5) holds. So we can find complex valued Borel measure μ_1 and μ_2 in $\mathbb{C}_I$ such that $\mu = \mu_1 + \mu_2 J$. Therefore, for each $z \in \mathbb{B}_I$

$$f_1(z) + f_2(z)J = Q_I[f](z) = \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}z - \frac{\alpha}{2}|\zeta|^2} d\mu_1(\zeta) + \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}z - \frac{\alpha}{2}|\zeta|^2} d\mu_2(\zeta) J.$$

Therefore, $f_l(z) = \int_{\mathbb{B}_I} e^{\alpha\bar{\zeta}z - \frac{\alpha}{2}|\zeta|^2} d\mu_l(\zeta)$, $l = 1, 2$ and as $\{|\mu|(G_r + w) : w \in \mathbb{H}\} \in l^p(\mathbb{H})$, it follows that $\{|\mu_l|(G_r + w) : w \in r\mathbb{R}^2\} \in l^p(\mathbb{C}_I)$. This implies f_1 and f_2 belong to complex Fock space $\mathfrak{F}^p_{\alpha,\mathbb{C}_I}(\mathbb{B}_I)$ which is equivalent to $f \in \mathfrak{F}^p_{\alpha,I}$ and hence $f \in \mathfrak{F}^p_\alpha$ in $\mathbb{B}$. □

References

[1] K. Abu-Ghanem, D. Alpay, F. Colombo, D. P. Kimsey and I. Sabadini, Boundary interpolation for slice hyperholomorphic Schur functions, Integr. Equ. Oper. Theory 82 (2015), 223-248.

[2] K. Abu-Ghanem, D. Alpay, F. Colombo and I. Sabadini, Gleason's problem and Schur multipliers in the multivariable quaternionic setting, J. Math. Anal. Appl. 425 (2015), 1083-1096.

[3] S. Adler, Quaterninoic Quantum Field Theory, Oxford University Press, 1995.

[4] D. Alpay, V. Bolotnikov, F. Colombo and I. Sabadini, Interpolation problems for certain classes of slice hyperholomorphic functions, Integr. Equ. Oper. Theory 86 (2016), 165-183.

[5] D. Alpay, F. Colombo and I. Sabadini, Schur functions and their realizations in the slice hyperholomorphic setting, Integr. Equ. Oper. Theory 72 (2012), 253-289.

[6] D. Alpay, F. Colombo and I. Sabadini, Pontryagin-de Branges-Rovnyak spaces of slice hyperholomorphic functions, J. Anal. Math. 121 (2013), 87-125.

[7] D. Alpay, F. Colombo, I. Sabadini and G. Salomon, Fock spaces in the slice hyperholomorphic setting, In hypercomplex analysis: New Perspectives and Applications (2014), 43-59.

[8] N. Arcozzi and G. Sarfatti, From Hankel operators to Carleson measures in a quaternionic variable, Proc. Edimburg Math. Soc. 60 (3) (2017), 565-585.

[9] N. Arcozzi and G. Sarfatti, Invariant metrics for the quaternionic Hardy space, J. Geom. Anal. 25 (2015), 2028-2059.

[10] F. Colombo, J. O. Gonzalez-Cervantes and I. Sabadini, On slice biregular functions and isomorphisms of Bergman spaces, Complex Var. Elliptic Equ. 57 (2012), 825-839.

[11] F. Colombo, J. O. Gonzalez-Cervantes, M.E. Luna-Elizarraras, I. Sabadini and M.V. Shapiro, On two approaches to the Bergman theory for slice regular functions, Advances in Hypercomplex Analysis, Springer INdAM Series 1, (2013), 39-54.

[12] F. Colombo, J. O. Gonzalez-Cervantes and I. Sabadini, The C-property for slice regular functions and applications to the Bergman space, Complex Var. Elliptic Equ. 58 (2013), 1355-1372.

[13] F. Colombo, I. Sabadini and D. C. Struppa, Noncommutative Functional Calculus, Theory and Applications of Slice Regular Hyperholomorphic Functions, Progress in Mathematics V. 289, Birkhäuser Basel, 2011.

[14] F. Colombo, I. Sabadini and D. C. Struppa, Entire slice regular functions, SpringerBriefs in Mathematics, 2016.

[15] G. Gentili and D. C. Struppa, A new approach to Cullen-regular functions of a quaternionic variable, C. R. Math. Acad. Sci. Paris 342 (2006), 741-744.

[16] G. Gentili and D. C. Struppa, A new theory of regular functions of a quaternionic variable, Adv. Math. 216 (2007), 279-301
[17] S. Janson, J. Peetre and R. Rochberg, Hankel forms and the Fock spaces, Revista Mat. Iberoamericana 3 (1987), 61-138.
[18] X. Hu, S. Kumar, K. Manzoor and Z. H. Zhou, Composition operators on BMO-Spaces of slice hyperholomorphic functions, preprint 2018.
[19] S. Kumar, K. Manzoor and P. Singh, Composition operators on slice regular Bloch type spaces of hyperholomorphic functions, Advances in Applied Clifford Algebras 27 (2) (2017), 1459-1477.
[20] R. Kumar, K. Singh, H. Saini and S. Kumar, Bicomplex weighted Hardy spaces and bicomplex C*-algebras, Adv. Appl. Clifford Algebras 26 (1) (2016), 217-235.
[21] K. R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Springer Science & Business Media, 1992.
[22] A. M. Rodriguez, The essential norm of a composition operator on Bloch spaces, Pacific. J. Math. 188 (2) (1999), 339-351.
[23] G. Sarfatti, Elements of function theory in the unit ball of quaternions, Ph. D thesis, Universitá di Firenze, 2013.
[24] S. Ueki, Characterization for Fock-type spaces via higher order derivatives and its applications, Complex Anal. Oper. Theory 8 (2014), 1475-1486.
[25] S. Ueki, Higher order derivative characterization for Fock-type spaces, Integr. Equ. Oper. Theory 84 (2016), 89-104.
[26] C. M. P. C. Villalba, F. Colombo, J. Gantner and J. O. G. Cervantes, Bloch, Besov asnd Dirichlet spaces of slice hyerhlomorphic functions, Complex Anal. Oper. Theory 9 (2015), 479-517.
[27] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.
[28] K. Zhu, Analysis on Fock Spaces, Graduate Texts in Mathematics 263, Springer, New York, 2012.