Filomat 34:4 (2020), 1223–1239
https://doi.org/10.2298/FIL2004223R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Galois Connection of Stabilizers in Residuated Lattices

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Abstract. The paper is devoted to introduce the notions of some types of stabilizers in non-commutative residuated lattices and to investigate their properties. We establish a connection between (contravariant) Galois connection and stabilizers of a residuated lattices. If $\mathfrak{A}$ is a residuated lattice and F be a filter of $\mathfrak{A}$, we show that the set of all stabilizers relative to F of a same type forms a complete lattice. Furthermore, we prove that $ST-F_l^{\square}$, $ST-F_l$ and $ST-F_s$ are pseudocomplemented lattices.

1. Introduction

Various logical algebras have been proposed as the semantical systems of non-classical logical systems, for example, residuated lattices, divisible residuated lattices, MTL-algebras, Girard monoids, BL-algebras, Gödel algebras, etc. Among these algebras, residuated lattices are very basic and important algebraic structures because the other logical algebras are all particular cases of residuated lattices.

In Gentzen-style systems, a structural rule is an inference rule that does not refer to any logical connective. Substructural logics were introduced as logics which, when formulated as Gentzen-style systems, lack some of the three basic structural rules as follows:

Weakening rule:

$$\frac{\Gamma, \Delta \Rightarrow \phi}{\Gamma, \alpha, \Delta \Rightarrow \phi}.$$

Contraction rule:

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \phi}{\Gamma, \alpha, \Delta \Rightarrow \phi}.$$

Exchange rule:

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \phi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \phi}.$$

2010 *Mathematics Subject Classification*. Primary 06F99; Secondary 06D20
Keywords. residuated lattice, filter; stabilizer, Galois connection, complete lattice, pseudocomplemented lattice
Received: 16 November 2016; Accepted: 23 March 2020
Communicated by Dragan S. Djordjević
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Commutative residuated lattices are the algebraic counterpart of logics without contraction rule. The concept of commutative residuated lattice firstly introduced by W. Krull in [21] who discussed decomposition into isolated component ideals. After him, they were investigated by M. Ward and R. P. Dilworth in [37], as the main tool in the abstract study of ideal lattices in ring theory. The properties of residuated lattices were presented in [13, 20, 26–35]. For a survey of residuated lattices we refer to [19].

Non-commutative residuated lattices, sometimes called pseudo-residuated lattices, biresiduated lattices or generalized residuated lattices, are the algebraic counterparts of substructural logics; i.e. logics which lack at least one of the three structural rules, namely contraction, weakening and exchange. Complete studies on non-commutative residuated lattices were developed in [1, 7, 8] and [19]. In this paper, a residuated lattice will be a FL_w-algebra. We denote by $\mathcal{RL}$ the class of residuated lattices. Following the results of Blount and Tsinakis [2], we deduce that the class $\mathcal{RL}$ of residuated lattices is equational, hence it forms a variety. A subclass $\mathcal{V}$ of the variety $\mathcal{RL}$ which is also a variety is called a subvariety of $\mathcal{RL}$.

The deductive system theory of the logical algebras plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various deductive systems correspond to various sets of provable formulas. Since deductive systems correspond to subsets closed with respect to Modus Ponens so they are sometimes called (implicative) filters.

Di Nola, Georgescu and Iorgulescu in [9] introduced the notion of left stabilizers in pseudo-BL algebras. After that Haveshki and Mohamadhasani in [14] generalized the notion of stabilizers to the stabilizers with respect to a subset and introduced the notion of left stabilizer with respect to a subset in BL-algebras. Borzooei and Paad in [5] introduced some new types of stabilizers in BL-algebras. Borumand and Mohtashamnia in [3] introduced the notion of right and left stabilizer in (commutative) residuated lattices. Haveshki in [15] improved some results in [3]. Motamed and Torkzadeh in [22] introduced the notion of right stabilizers in BL-algebras and define a class of BL-algebras, called RS-BL-algebra. In this paper we introduced the notion of some type of stabilizers in (non-commutative) residuated lattices and we establish a connection between them and Galois connection.

This paper is organized in four sections. In Section 2 we recall some definitions, properties and results relative to (non-commutative) residuated lattices and Galois connection. In this section we give some examples of residuated lattices which will be used in the following sections of the paper. In Section 3, we introduce the notions of ll, lr, rl, rr, left, right stabilizer relative to a filter in (non-commutative) residuated lattices and we investigate their properties and give some examples of them. In section 4 we establish a connection between stabilizers and Galois connection. We show that the set of all stabilizers relative to a filter of a same type forms a complete lattice.

2. A brief excursion into residuated Lattices and Galois connections

In this section we recall some definitions, properties and results relative to residuated lattices and Galois connection which will be used in the following sections of this paper.

2.1. residuated Lattices

Definition 2.1. *[19] A residuated lattice is an algebra* $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow_l, \rightarrow_r, 1)$ *of type* $(2, 2, 2, 2, 2, 0)$ *satisfying the following conditions.*

RL_1 $(A; \vee, \wedge, 0, 1)$ *is a bounded lattice.*

RL_2 $(A, \odot, 1)$ *is a monoid.*

RL_3 $x \odot y \leq z$ *iff* $x \leq y \rightarrow_l z$ *iff* $y \leq x \rightarrow_r z$ *for* $x, y, z \in A$.

The operations $\rightarrow_l$ and $\rightarrow_r$ are referred to as the left and right residual of $\odot$, respectively. Note that, in general, 1 is not the top element of the lattice reduct of $\mathfrak{A}$, $\ell(\mathfrak{A})$. A residuated lattice with a constant 0 (which can denote any element) is called a pointed residuated lattice or a *full Lambek algebra* (*FL*-algebra). If 1 is a top a element of $\ell(\mathfrak{A})$, then $\mathfrak{A}$ is called an integral residuated lattice. A *FL*-algebra $\mathfrak{A}$ in which $(A; \vee, \wedge, 0, 1)$ is a bounded lattice is called a FL_w-algebra. A FL_w-algebra is sometimes called a bounded integral residuated lattice. A residuated lattice $\mathfrak{A}$ is called commutative if $\rightarrow_l = \rightarrow_r$. It is obvious that $\mathfrak{A}$ is a commutative residuated lattice if and only if $\odot$ is a commutative binary operation. A residuated lattice $\mathfrak{A}$ in which $x \odot y = x \wedge y$ (or equivalently, $x^2 = x$)for all $x, y \in A$ is called a *Heyting algebra* or *pseudo-Boolean algebra* [36]. Clearly, a Heyting algebra is a commutative residuated lattice.

In this paper, a residuated lattice will be a FL_w-algebra. A residuated lattice $\mathfrak{A}$ is nontrivial if and only if $0 \neq 1$. In a residuated lattice $\mathfrak{A}$, for any $a \in A$, we put $\neg_l a := a \rightarrow_l 0$ and $\neg_r a := a \rightarrow_r 0$. Also, $\neg_l\neg_l a$, $\neg_l\neg_r a$, $\neg_r\neg_l a$ and $\neg_r\neg_r a$ are denoted by $\neg_{ll} a$, $\neg_{rl} a$, $\neg_{lr} a$ and $\neg_{rr} a$, respectively.

A residuated lattice $\mathfrak{A}$ is called a *pseudo-MTL algebra* [12] if it satisfies the pseudo-pre-linearity condition (denoted by pprel) $(x \rightarrow_l y) \vee (y \rightarrow_l x) = (x \rightarrow_r y) \vee (y \rightarrow_r x) = 1$. It is easy to see that each linearly-ordered residuated lattice is a pseudo-MTL algebra. We denote by $\mathcal{PMTL}$ the class of pseudo-MTL algebras. Obviously, the class $\mathcal{PMTL}$ of pseudo-MTL algebras is equational, hence it forms a subvariety of the variety $\mathcal{RL}$. $\mathfrak{A}$ is called a *pseudo divisible residuated lattice* [11] (or a *bounded Rℓ monoid* in [25]) if it satisfies the pseudo-divisibility condition (denoted by pdiv) $x \odot (x \rightarrow_r y) = (x \rightarrow_l y) \odot x = x \wedge y$. We denote by $\mathcal{R}\ell$ the class of bounded Rℓ monoids. Obviously, the class $\mathcal{R}\ell$ of bounded Rℓ monoids is equational, hence it forms a subvariety of the variety $\mathcal{RL}$. A residuated lattice is called proper if it is not a pseudo-MTL algebra or a bounded Rℓ monoid, i.e. if (pprel) and (pdiv) do not hold. $\mathfrak{A}$ is called a *pseudo-BL algebra* if it satisfies both (pprel) and (pdiv). Denote by $\mathcal{PBL}$ the class of pseudo-BL algebras. A pseudo-MTL algebra is called proper if it is not a pseudo-BL algebra, i.e. if (pdiv) does not hold. A bounded Rℓ monoid is called proper if it is not a pseudo-BL algebra, i.e. if (pprel) does not hold. A pseudo BL-algebra $\mathfrak{A}$ is called a pseudo MV-algebra (GMV-algebra) [16] if it is an involutive (or regular) i.e. $\neg_{lr} x = \neg_{rl} x = x$. Denote by $\mathcal{PMV}$ the class of pseudo-MV algebras. It is well-known that a residuated lattice $\mathfrak{A}$ is a pseudo MV-algebra if and only if it satisfies the following assertions:

mv_l $(x \rightarrow_l y) \rightarrow_r y = (y \rightarrow_l x) \rightarrow_r x$.

mv_r $(x \rightarrow_r y) \rightarrow_l y = (y \rightarrow_r x) \rightarrow_l x$.

A pseudo-BL algebra is called proper if it is not a pseudo-MV algebra, i.e. if mv_l or mv_r do not hold. Note that $\mathcal{PMTL}, \mathcal{R}\ell, \mathcal{PBL}$ are all subvarieties of $\mathcal{RL}$, connected as Figure 2.

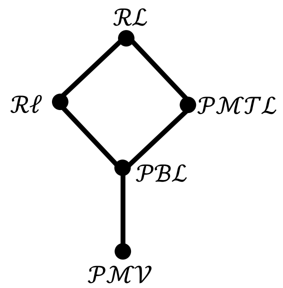

Figure 1: The interrelation between some subvarieties of $\mathcal{RL}$

Example 2.2. *[7] Let $A_5 = \{0, a, b, c, 1\}$ be a lattice whose Hasse diagram is below (see Figure 2). Define $\odot$, $\rightarrow_l$ and $\rightarrow_r$ on A_5 by the following tables.*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\rightarrow_l$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightarrow_r$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

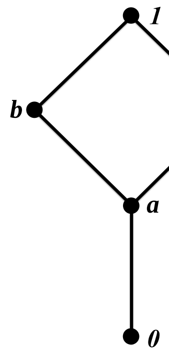

Figure 2: The Hasse diagram of $\mathfrak{A}_5$.

Routine calculation shows that $\mathfrak{A}_5 = (A_5; \vee, \wedge, \odot, \to_l, \to_r, 0, 1)$ *is a residuated lattice. One can check that* $\mathfrak{A}_5$ *is a proper pseudo-MTL algebra, because the property (pdiv) does not hold:*

$$c \odot (c \to_r a) = c \odot b = 0 \neq a = c \wedge a.$$

Example 2.3. *Let* $A_7 = \{0, a, b, c, d, e, 1\}$ *be a lattice whose Hasse diagram is below (see Figure 3). Define* $\odot$, $\to_l$ *and* $\to_r$ *on* A_7 *as follows:*

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	b	c	d	e	e
1	0	a	b	c	d	e	1

$\to_l$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$\to_r$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	e	1	1	1
e	0	a	b	c	d	1	1
1	0	a	b	c	d	e	1

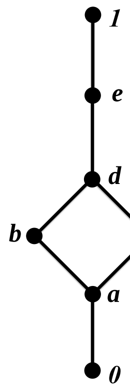

Figure 3: The Hasse diagram of $\mathfrak{A}_7$.

Routine calculation shows that $\mathfrak{A}_7 = (A_7; \vee, \wedge, \odot, \to_l, \to_r, 0, 1)$ *is a proper residuated lattice, because the property (pprel) does not hold:* $(b \to_l c) \vee (c \to_l b) = d \vee b = d \neq 1$ *and the property (pdiv) also does not hold:* $d \odot (d \to_r b) = d \odot b = a \neq d \wedge b$.

Example 2.4. *Let* $B_5 = \{0, a, b, c, 1\}$ *be a lattice whose Hasse diagram is below (see Figure 4). Define* $\odot$, $\to_l$ *and* $\to_r$ *on* B_5 *by the following tables.*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow_l$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightarrow_r$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

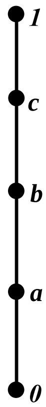

Figure 4: The Hasse diagram of $\mathfrak{B}_5$.

Routine calculation shows that $\mathfrak{B}_5 = (B_5; \vee, \wedge, \odot, \rightarrow_l, \rightarrow_r, 0, 1)$ *is a proper linearly-ordered pseudo-MTL algebra, because (pdiv) does not hold:*

$$a = b \wedge a \neq b \odot (b \rightarrow_r a) = b \odot b = 0.$$

Example 2.5. *[17, 18] Let* $A_6 = \{0, a, b, c, d, 1\}$ *be a lattice whose Hasse diagram is below (see Figure 5). Define* $\odot = \wedge$ *and* $\rightarrow$ *on* A_6 *as follows.*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Routine calculation shows that $\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 1)$ *is a proper commutative bounded Rℓ monoid, since the property (pprel) (here (prel)) is not verified:* $(a \rightarrow b) \vee (b \rightarrow a) = b \vee a = c \neq 1$*; more precisely* $\mathfrak{A}_6$ *is a Heyting algebra.*

The following proposition provides some rules of calculus in a residuated lattice.

Proposition 2.6. *[1, 4, 9] Let* $\mathfrak{A}$ *be a residuated lattice. Then the following assertions are satisfied for any* $x, y, z \in A$, $\square \in \{l, r\}$.

r_1 $x \leq y \Leftrightarrow x \rightarrow_\square y = 1$.
r_2 $x \rightarrow_\square x = 0 \rightarrow_\square x = x \rightarrow_\square 1 = 1$ *and* $1 \rightarrow_\square x = x$.
r_3 $x \rightarrow_l (y \rightarrow_l z) = (x \odot y) \rightarrow_l z$ *and* $x \rightarrow_r (y \rightarrow_r z) = (y \odot x) \rightarrow_r z$.
r_4 $x \odot y \leq (x \odot (x \rightarrow_r y)) \wedge ((x \rightarrow_l y) \odot y) \leq x \wedge y$. *In particular,* $x \leq y \rightarrow_\square x$ *and* $x \leq (x \rightarrow_{l(r)} y) \rightarrow_{r(l)} y$.
r_5 $x \leq y \Rightarrow x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$.

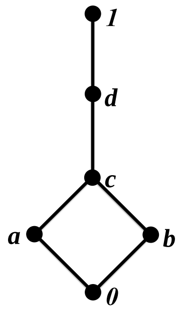

Figure 5: The Hasse diagram of $\mathfrak{A}_6$.

r_6 $x \leq y \Rightarrow z \rightarrow_\square x \leq z \rightarrow_\square y$ *and* $y \rightarrow_\square z \leq x \rightarrow_\square z$.

r_7 $x \rightarrow_{l(r)} y \leq (y \rightarrow_{l(r)} z) \rightarrow_{r(l)} (x \rightarrow_{l(r)} z)$.

r_8 $x \rightarrow_{l(r)} y \leq (z \rightarrow_{l(r)} x) \rightarrow_{r(l)} (z \rightarrow_{l(r)} y)$.

r_9 $x \rightarrow_l (y \rightarrow_r z) = y \rightarrow_r (x \rightarrow_l z)$

r_{10} $x \rightarrow_\square (y \wedge z) = (x \rightarrow_\square y) \wedge (x \rightarrow_\square z)$. *In particular,* $x \rightarrow_\square y = x \rightarrow_\square (x \wedge y)$.

r_{11} $((x \rightarrow_{l(r)} y) \rightarrow_{r(l)} y) \rightarrow_{l(r)} y = x \rightarrow_{l(r)} y$.

Let $\mathfrak{A}$ be a residuated lattice and F be a subset of A. For convenience, we enumerate some conditions which will be used in this paper.

$c_\emptyset$ $F \neq \emptyset$.

c_1 $1 \in F$.

$c_\odot$ $x, y \in F \Rightarrow x \odot y \in F$.

$c_\leq$ $x \leq y, x \in F \Rightarrow y \in F$.

$c_\vee$ $x \in F$ and $y \in A \Rightarrow x \vee y \in F$.

c_l $x, x \rightarrow_l y \in F \Rightarrow y \in F$.

c_r $x, x \rightarrow_r y \in F \Rightarrow y \in F$.

Definition 2.7. *Let $\mathfrak{A}$ be a residuated lattice and F be a subset of A.*

- *F is called an ordered-filter of $\mathfrak{A}$ if it satisfies $c_\emptyset$ and $c_\leq$.*
- *F is called a filter of $\mathfrak{A}$ if it satisfies $c_\emptyset$, $c_\odot$ and $c_\leq$.*
- *F is called a 1-ideal of $\mathfrak{A}$ if it satisfies $c_\emptyset$, $c_\odot$ and $c_\vee$.*
- *F is called a left deductive system of $\mathfrak{A}$ if it satisfies c_1 and c_l.*
- *F is called a right deductive system of $\mathfrak{A}$ if it satisfies c_1 and c_r.*

Proposition 2.8. *Let $\mathfrak{A}$ be a residuated lattice and F be a subset of A containing 1. Then the following assertions are equivalent for any $x, y, z \in A$.*

F_1 *F is a filter.*

F_2 *F is a 1-ideal.*

F_3 *F is a left deductive system.*

F_4 *F is a right deductive system*

F_5 $x \rightarrow_l y, y \rightarrow_l z \in F \Rightarrow x \rightarrow_l z \in F$.

F_6 $x \rightarrow_r y, y \rightarrow_r z \in F \Rightarrow x \rightarrow_r z \in F$.

F_7 $x \rightarrow_l y, x \odot z \in F \Rightarrow y \odot z \in F$.

F_8 $x \rightarrow_r y, z \odot x \in F \Rightarrow z \odot y \in F$.

F_9 $x \to_l y, \neg_l y \in F \Rightarrow \neg_l x \in F$.

F_{10} $x \to_r y, \neg_r y \in F \Rightarrow \neg_r x \in F$.

F_{11} $x, y \in F$ *and* $x \leq y \to_l z \Rightarrow z \in F$.

F_{12} $x, y \in F$ *and* $x \leq y \to_r z \Rightarrow z \in F$.

Proof. It is straightforward by Proposition 2.6. □

The set of ordered-filters and filters of a residuated lattice $\mathfrak{A}$ will be denoted by $OF(\mathfrak{A})$ and $F(\mathfrak{A})$, respectively. It is clear that $F(\mathfrak{A}) \subseteq OF(\mathfrak{A})$. Trivial examples of filters are $\mathbf{1} = \{1\}$ and A. A filter F of $\mathfrak{A}$ is proper if $F \neq A$. Clearly, F is a proper filter if and only if $0 \notin F$.

Example 2.9. *Consider the proper pseudo-MTL algebra $\mathfrak{A}_5$ from Example 2.2. Then $F(\mathfrak{A}_5) = \{F_1 = \mathbf{1}, F_2 = \{b, 1\}, F_3 = \{c, 1\}, F_4 = A_5\}$.*

Example 2.10. *Consider the proper residuated lattice $\mathfrak{A}_7$ from Example 2.3. Then $F(\mathfrak{A}_7) = \{F_1 = \mathbf{1}, F_2 = \{e, 1\}, F_3 = \{c, d, e, 1\}, F_4 = \{a, b, c, d, e, 1\}, F_5 = A_7\}$.*

Example 2.11. *Consider the proper linearly-ordered pseudo-MTL algebra $\mathfrak{B}_5$ from Example 2.4. Then $F(\mathfrak{B}_5) = \{F_1 = \mathbf{1}, F_2 = \{c, 1\}, F_3 = B_5\}$.*

Example 2.12. *Consider the proper commutative bounded $R\ell$ monoid $\mathfrak{A}_6$ from Example 2.5. Then $F(\mathfrak{A}_6) = \{F_1 = \mathbf{1}, F_2 = \{d, 1\}, F_3 = \{c, d, 1\}, F_4 = \{b, c, d, 1\}, F_5 = \{a, c, d, 1\}, F_6 = A_6\}$.*

Let $\mathfrak{A}$ be a residuated lattice. We define the *distance functions* as for pseudo-BL algebras [9] $d^l(a, b) = (a \to_l b) \odot (b \to_l a)$ and $d^r(a, b) = (a \to_r b) \odot (b \to_r a)$, for any $a, b \in A$. With any filter of a residuated lattice $\mathfrak{A}$ we associate two binary relations $\equiv^l_F$ and $\equiv^r_F$ on A by defining as for pseudo-BL algebras [9];

$$(a, b) \in \equiv^l_F \text{ if and only if } d^l(a, b) \in F,$$
$$(a, b) \in \equiv^r_F \text{ if and only if } d^r(a, b) \in F,$$

As for pseudo-BL algebras [9], the binary relations $\equiv^l_F$ and $\equiv^r_F$ are equivalence relations on A. $\equiv^l_F$ and $\equiv^r_F$ are called the left equivalence relation and the right equivalence relation induced by F, respectively. In the following, for any $a \in A$ the equivalence classes $a/\equiv^l_F$ and $a/\equiv^r_F$ are denoted by $[a]^l_F$ and $[a]^l_F$, respectively.

Definition 2.13. *[10] Let $\mathfrak{A}$ be a residuated lattice. A filter F of $\mathfrak{A}$ is called normal if the following condition holds, for any $x, y \in A$:*

$$x \to_l y \in F \textit{ if and only if } x \to_r y \in F.$$

We shall denote by $F_n(\mathfrak{A})$ the set of normal filters of $\mathfrak{A}$.

Example 2.14. *Consider the proper residuated lattice $\mathfrak{A}_7$ from Example 2.3. Then we have $F_n(\mathfrak{A}_7) = \{F_1 = \mathbf{1}, F_3 = \{c, d, e, 1\}, F_4 = \{a, b, c, d, e, 1\}, F_5 = A_7\}$*

It is obvious that if F is a normal filter of the residuated lattice $\mathfrak{A}$ then the right and the left equivalence relations induced by F are equal and both of them are denoted by $\equiv_F$. So $(x, y) \in \equiv_F$ if and only if $d^l(x, y) \in F$ if and only if $d^r(x, y) \in F$. As for pseudo-BL algebras [10, Proposition 1.7], if F is a normal filter of a residuated lattice $\mathfrak{A}$ then $\equiv_F$ is a congruence relation on $\mathfrak{A}$. In this case, For any $a \in A$, let a/F be the equivalence class $a/\equiv_F$ and $A/F = \{a/F | a \in A\}$. A/F becomes a residuated lattice with the natural operations induced from those of $\mathfrak{A}$ and it is denoted by $\mathfrak{A}/F$.

Let $\mathfrak{A}$ be a residuated lattice. It is obvious that $(A; F(\mathfrak{A}))$ is an algebraic closed set system. The closure operator associated with the closed set system $(A; F(\mathfrak{A}))$ is denoted by $Fi^{\mathfrak{A}} : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$. Thus for any subset X of A, $Fi^{\mathfrak{A}}(X) = \cap\{F \in F(\mathfrak{A}) | X \subseteq F\}$ is the smallest filter of $\mathfrak{A}$ contains X. $Fi^{\mathfrak{A}}(X)$ is called the filter generated by X. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $Fi^{\mathfrak{A}}(x)$ and it is called the principle filter of $\mathfrak{A}$. When there is no ambiguity we will drop the superscript $\mathfrak{A}$.

If $\{F_i\}_{i \in I}$ is a family of all filters of $\mathfrak{A}$, we define $\wedge_{i \in I} F_i = \cap_{i \in I} F_i$ and $\veebar_{i \in I} F_i = Fi(\cup_{i \in I} F_i)$. According to [7], $(F(\mathfrak{A}), \wedge, \veebar)$ is a complete Browerian algebraic lattice which its compact elements are exactly the principal filter of $\mathfrak{A}$.

Proposition 2.15. *[7] Let $\mathfrak{A}$ be a residuated lattice and X be a subset of A. Then we have*

$$Fi(X) = \{a \in A | x_1 \odot \cdots \odot x_n \leq a, for\ some\ integer\ n,\ x_1, \cdots, x_n \in X\}.$$

Let $\mathfrak{A}$ be a residuated lattice. The set of all complemented elements in the lattice reduct $\mathfrak{A}$ is denoted by $B(\mathfrak{A})$ and it is called the Boolean center of $\mathfrak{A}$. Complements are generally not unique unless the lattice is distributive. In residuated lattices however, although the underlying lattices need not be distributive, according to [8], the complements are unique.

Proposition 2.16. *[8] Let $\mathfrak{A}$ be a residuated lattice, $e \in B(\mathfrak{A})$ and $a \in A$. Then the following assertions hold for any $\square_1, \square_2 \in \{l, r\}$.*

(1) $e^c = \neg_l e = \neg_r e$, $\neg_{lr} e = \neg_{rl} e = e$ *and* $e^2 = e$.
(2) $e \rightarrow_{\square_1} \neg_{\square_2} e = \neg_{\square_2} e$, $\neg_{\square_1} e \rightarrow_{\square_2} e = e$.
(3) $e \odot a = e \wedge a$.

Let $\mathfrak{A}$ be a residuated lattice. An element a of A is said to be the left (right) dense element of $\mathfrak{A}$ if and only if $\neg_l a = 0 (\neg_r a = 0)$. We denote by $D_s^l(\mathfrak{A})$ and $D_s^r(\mathfrak{A})$ the sets of the left and the right dense elements of $\mathfrak{A}$, respectively. Also, the intersection of the left and the right dense elements of $\mathfrak{A}$ is said to be the dense elements of $\mathfrak{A}$ and denoted by $D_s(\mathfrak{A})$. One can check that $D_s(\mathfrak{A}) = \{a \in A | \neg_{lr} a = 1 = \neg_{rl} a\}$. Obviously, if $\mathfrak{A}$ is an involutive residuated lattice then $D_s^l(\mathfrak{A}) = D_s^r(\mathfrak{A}) = D_s(\mathfrak{A}) = \{1\}$.

Proposition 2.17. *[7] Let $\mathfrak{A}$ be a residuated lattice. Then $D_s^l(\mathfrak{A})$ and $D_s^r(\mathfrak{A})$ are proper filters of $\mathfrak{A}$. In particular, $D_s(\mathfrak{A})$ is a proper filter of $\mathfrak{A}$.*

Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$ and X be a subset of A. The generalized co-annihilator of X (relative to F) is denoted by $(F : X)$ and defined as follow.

$$(F : X) = \{a \in A | x \vee a \in F, \forall x \in X\}.$$

In the following proposition, we collect the properties of generalized co-annihilators.

Proposition 2.18. *[28] Let $\mathfrak{A}$ be a residuated lattice, F, G be filters of $\mathfrak{A}$ and X, Y be subsets of A. Then the following conditions satisfy.*

(1) $(F : X)$ *is a filter of* $\mathfrak{A}$.
(2) $F \subseteq (F : X)$.
(3) $(F : X) = A$ *if and only if* $X \subseteq F$.
(4) $X \subseteq (F : (F : X))$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be residuated lattices. A mapping $h : A \longrightarrow B$ is called a homomorphism, in symbols $h : \mathfrak{A} \longrightarrow \mathfrak{B}$, if it preserves the fundamental operations. If $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a homomorphism we put $coker(h) = h^{\leftarrow}(1)$. It is easy to check that $coker(h)$ is a normal filter of $\mathfrak{A}$. Also, it is obvious that $h(a_1) = h(a_2)$ if and only if $a_1 \rightarrow_{\square} a_2, a_2 \rightarrow_{\square} a_1 \in coker(h)$ and it implies that h is an injective homomorphism if and only if $coker(h) = \{1\}$.

Proposition 2.19. *Let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism.*

(1) *If h is surjective and $F \in F(\mathfrak{A}) (F \in F_n(\mathfrak{A}))$ such that $coker(h) \subseteq F$ then $h(F) \in F(\mathfrak{B}) (h(F) \in F_n(\mathfrak{B}))$.*
(2) *If $F \in F(\mathfrak{B}) (F \in F_n(\mathfrak{B}))$ then $h^{\leftarrow}(F) \in F(\mathfrak{A}) (h^{\leftarrow}(F) \in F_n(\mathfrak{A}))$ and $coker(h) \subseteq h^{\leftarrow}(F)$.*

Proof. It is straightforward. □

2.2. Galois connection

This section is devoted to recall some definitions, properties and results relative to Galois connection.

Definition 2.20. *Let $\mathscr{A} = (A; \leq)$ and $\mathscr{B} = (B; \preccurlyeq)$ be posets and $f : A \longrightarrow B$ be a map between posets.*

1. *f is antitone if $a_1 \leq a_2$ implies $f(a_2) \preccurlyeq f(a_1)$, for all $a_1, a_2 \in A$.*

In particular case which $\mathscr{A} = \mathscr{B}$,

1. *f is inflationary (also called extensive) if $a \leq f(a)$ for all $a \in A$.*
2. *f is idempotent if $f^2 = f$.*
3. *f is a closure operator on $\mathscr{A}$ if it is inflationary, isotone and idempotent. A fixpoint of the closure operator f, i.e. an element a of A that satisfies $f(a) = a$, is called a closed element of f. The set of closed elements of the closure operator f will be denoted by $\mathscr{C}_f$.*

Definition 2.21. *Let $\mathscr{A} = (A; \leq)$ and $\mathscr{B} = (B; \preccurlyeq)$ be posets. Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are functions such that for all $a \in A$ and $b \in B$ we have*

$$a \leq g(b) \text{ if and only if } b \preccurlyeq f(a).$$

Then the pair (f, g) is called a (contravariant or antitone) Galois connection between $\mathscr{A}$ and $\mathscr{B}$

Proposition 2.22. *Let $\mathscr{A}$ and $\mathscr{B}$ be posets and $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be two functions. Then the pair (f, g) forms a Galois connection between $\mathscr{A}$ and $\mathscr{B}$ if and only if the following assertions hold.*

(1) *$I_A \leq gf$ and $I_B \leq fg$.*
(2) *f and g are antitone functions.*

Proof. Let (f, g) forms a Galois connection between $\mathscr{A}$ and $\mathscr{B}$. Consider $a \in A$. We have $f(a) \preccurlyeq f(a)$ and it implies that $a \leq g(f(a))$. So $I_A \leq gf$. Analogously, we can show that $I_B \leq fg$. If we have $a_1 \leq a_2$ then we have $a_1 \leq g(f(a_2))$ and it states that $f(a_2) \preccurlyeq f(a_1)$. In a similar way, we can obtain that g is an antitone function.

Now, let (1) and (2) holds. Assume that $a \leq g(b)$ for some $a \in A$ and $b \in B$. So we have $b \preccurlyeq f(g(b)) \preccurlyeq f(a)$. Analogously, we can show that $b \preccurlyeq f(a)$ implies $a \leq g(b)$. Therefore, (f, g) forms a Galois connection between $\mathscr{A}$ and $\mathscr{B}$. □

Proposition 2.23. *Let $\mathscr{A}$ and $\mathscr{B}$ be posets and (f, g) forms a Galois connection between $\mathscr{A}$ and $\mathscr{B}$. Then the following assertions hold.*

(1) *$fgf = f$ and $gfg = g$.*
(2) *If $\vee X$ exists for some $X \subseteq A$ then $\wedge f(X)$ exists and $\wedge f(X) = f(\vee X)$.*
(3) *If $\vee Y$ exists for some $Y \subseteq B$ then $\wedge g(Y)$ exists and $\wedge g(Y) = g(\vee Y)$.*
(4) *$f(a) = \max\{b \in B | a \leq g(b)\}$, $g(b) = \max\{a \in A | b \preccurlyeq f(a)\}$*
(5) *gf is a closure operator on $\mathscr{A}$ and $\mathscr{C}_{gf} = g(B)$.*
(6) *fg is a closure operator on $\mathscr{B}$ and $\mathscr{C}_{fg} = f(A)$.*

Proof.

1. Let $a \in A$. By Proposition 2.22(1) we have $a \leq g(f(a))$ and $f(a) \preccurlyeq f(g(f(a)))$ and by 2.22(2) we get that $f(g(f(a))) \preccurlyeq f(a)$. It shows that $f = fgf$. Analogously, we can show that $g = gfg$.
2. Let $x \in X$. Then $x \leq \vee X$ and it implies that $f(\vee X) \preccurlyeq f(x)$ and this means that $f(\vee X)$ is a lower bound of the set $f(X)$. Assume that $b \preccurlyeq f(x)$ for any $x \in X$. So we obtain that $x \leq g(b)$ for any $x \in X$. So we have $\vee X \leq g(b)$ and this states that $b \leq f(\vee X)$. Therefore, $f(\vee X) = \wedge f(X)$.
3. Let $a \in A$. By Proposition 2.22(1) we obtain that $f(a) \in \{b \in B | a \leq g(b)\}$. Assume that $b \in \{b \in B | a \leq g(b)\}$. Then $a \leq g(b)$ and it implies that $b \leq f(a)$. Analogously, we can show that $g(b) = \max\{a \in A | b \preccurlyeq f(a)\}$.
4. By Proposition 2.22(1), gf is inflationary and by Proposition 2.22(1), gf is isotone. Also, by (1) we can conclude that gf is idempotent. It states that gf is a closure operator on $\mathscr{A}$.
   Let $b \in B$. By (1) we have $g(B) \subseteq \mathscr{C}_{gf}$. Also, for each $a \in A$, $a \in \mathscr{C}_{gf}$ implies $a = g(f(a)) \in g(B)$ and this shows that $\mathscr{C}_{gf} \subseteq g(B)$. Hence, we have $\mathscr{C}_{gf} = g(B)$. Analogously, we can show that fg is a closure operator on $\mathscr{B}$ and $\mathscr{C}_{fg} = f(A)$.
   □

Theorem 2.24. *[6] Let A be set and $f : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ be a closure operator. Then the set of closed elements of f, $\mathscr{C}_f$, is a complete lattice with respect to the following operations.*

$$\begin{array}{cccc} \wedge^f: & \mathscr{C}_f \times \mathscr{C}_f & \longrightarrow & \mathscr{C}_f \\ & (X,Y) & \longmapsto & X \cap Y, \end{array} \qquad \begin{array}{cccc} \vee^f: & \mathscr{C}_f \times \mathscr{C}_f & \longrightarrow & \mathscr{C}_f \\ & (X,Y) & \longmapsto & f(X \cup Y). \end{array}$$

Corollary 2.25. *Let A be a set and (f,g) forms a Galois connection between $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Then the following assertions hold.*

(1) *$\mathscr{L}_g = (g(\mathcal{P}(B)); \wedge^g, \vee^g, 0 = g(B), 1 = g(\emptyset))$ is a complete lattice where $\wedge^g_{i\in I} g(Y_i) = g(\cup_{i\in I} Y_i)$ and $\vee^g_{i\in I} g(Y_i) = g(\cap_{i\in I} fg(Y_i))$ for any family $\{Y_i\}_{i\in I} \in \mathcal{P}(B)$.*

(2) *$\mathscr{L}_f = (f(\mathcal{P}(A)); \wedge^f, \vee^f, 0 = f(A), 1 = f(\emptyset))$ is a complete lattice where $\wedge^f_{i\in I} f(X_i) = f(\cup_{i\in I} X_i)$ and $\vee^f_{i\in I} f(X_i) = f(\cap_{i\in I} gf(X_i))$ for any family $\{X_i\}_{i\in I} \in \mathcal{P}(B)$.*

Proof. By Proposition 2.23(5), gf is a closure operator on $\mathcal{P}(A)$ and $\mathscr{C}_{gf} = g(\mathcal{P}(B))$. So by Theorem 2.24, $(g(\mathcal{P}(B)); \wedge^{gf}, \vee^{gf})$ is a complete lattice where $\wedge^{gf}_{i\in I} g(Y_i) = \cap_{i\in I} g(Y_i)$ and $\vee^{gf}_{i\in I} g(Y_i) = gf(\cup_{i\in I} g(Y_i))$ for any family $\{Y_i\}_{i\in I} \in \mathcal{P}(B)$. Now, let $\{Y_i\}_{i\in I}$ be a family of subset of the set B. By Proposition 2.23(3) we have $\cap_{i\in I} g(Y_i) = g(\cup_{i\in I} Y_i)$ and this shows that $\wedge^{gf}_{i\in I} g(Y_i) = \wedge^g_{i\in I} g(Y_i)$. Also, we have $gf(\cup_{i\in I} g(Y_i)) = g(\cap_{i\in I} fg(Y_i))$ and it implies that $\vee^{gf}_{i\in I} g(Y_i) = \vee^g_{i\in I} g(Y_i)$. Since g is an antitone function so we have $g(B) \subseteq g(Y) \subseteq g(\emptyset)$ for any $Y \subseteq B$. Therefore, $(g(\mathcal{P}(B)); \wedge, \vee, 0 = g(B), 1 = g(\emptyset))$ is a complete lattice. Analogously, we can show that (2) holds. □

3. Stabilizer in residuated lattice

In this section we introduce and investigate the notion of stabilizer relative to a filter in residuated lattices.

Definition 3.1. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$ and X be a subset of A. The ll, lr, rl and rr-stabilizer of X relative to F is denoted by $(F : X)^l_l$, $(F : X)^r_l$, $(F : X)^l_r$ and $(F : X)^r_r$, respectively and defined as follows.*

1. $(F : X)^l_l = \{a \in A | (a \rightarrow_l x) \rightarrow_r x \in F,\ \forall x \in X\}$.
2. $(F : X)^l_l = \{a \in A | (a \rightarrow_r x) \rightarrow_l x \in F,\ \forall x \in X\}$.
3. $(F : X)^l_r = \{a \in A | (x \rightarrow_l a) \rightarrow_r a \in F,\ \forall x \in X\}$.
4. $(F : X)^l_r = \{a \in A | (x \rightarrow_r a) \rightarrow_l a \in F,\ \forall x \in X\}$.

$(F : X)_{l(r)} = (F : X)^l_{l(r)} \cap (F : X)^r_{l(r)}$ is called the left (right) stabilizer of X relative to F and $(F : X)_s = (F : X)_l \cap (F : X)_r$ is called the stabilizer of X relative to F. Let $\square_1, \square_2 \in \{l, r\}$. If $X = \{x\}$ then $(F : \{x\})^{\square_2}_{\square_1}$ is denoted by $(F : x)^{\square_2}_{\square_1}$. Also, $(\mathbf{1}, X)^{\square_2}_{\square_1}$ is called the $\square_1\square_2$-stabilizer of X and it is denoted by $(X)^{\square_2}_{\square_1}$.

Example 3.2. *Consider the proper pseudo-MTL algebra $\mathfrak{A}_5$ from Example 2.2 and its filters from Example 2.9.*

In the following proposition, we collect some properties of stabilizers.

Proposition 3.3. *Let $\mathfrak{A}$ be a residuated lattice. Then the following assertions hold for any family $\{X\} \cup \{Y\} \cup \{X_i\}_{i\in I} \in \mathcal{P}(A)$, $\{F\} \cup \{G\} \cup \{F_i\}_{i\in I} \in Fi(\mathfrak{A})$, $\square_1, \square_2 \in \{l, r\}$ and $\square \in \{l, r, s\}$.*

(1) *$X^{\square_1}_l = \{a \in A | a \rightarrow_{\square_1} x = x, \forall x \in X\}$ and $X^{\square_1}_r = \{a \in A | x \rightarrow_{\square_1} a = a, \forall x \in X\}$.*
(2) *$(F : X) \subseteq (F : X)_s$. In particular, $F \subseteq (F : X)_s$.*
(3) *$X \subseteq Y$ implies $(F : Y)^{\square_2}_{\square_1} \subseteq (F : X)^{\square_2}_{\square_1}$. In particular, if $X \subseteq Y$ then $(F : Y)_\square \subseteq (F : X)_\square$.*
(4) *$(F : Fi(X))^{\square_2}_{\square_1} \subseteq (F : X)^{\square_2}_{\square_1}$. In particular, $(F : Fi(X))_\square \subseteq (F : X)_\square$.*
(5) *$F \subseteq G$ implies $(F : X)^{\square_2}_{\square_1} \subseteq (G : X)^{\square_2}_{\square_1}$. In particular, if $F \subseteq G$ then $(F : X)_\square \subseteq (G : X)_\square$.*
(6) *$X \cap (F : X)^{\square_2}_{\square_1} \subseteq F$. In particular, $X \cap (F : X)_\square \subseteq F$.*
(7) *If X contains F then $X \cap (F : X)^{\square_2}_{\square_1} = F$. In particular, if X contains F then $X \cap (F : X)_\square = F$.*

		0	a	b	c	1
$F_1 = \{1\}$	ll	F_3	F_1	F_3	F_2	A_5
	lr	F_2	F_1	F_3	F_2	A_5
	rl	F_1	F_1	F_3	{0,b,1}	A_5
	rr	F_1	F_1	{0,c,1}	F_2	A_5
$F_2 = \{b, 1\}$	ll	{a,b,c,1}	F_2	A_5	F_2	A_5
	lr	F_2	F_2	A_5	F_2	A_5
	rl	F_2	{0,b,1}	A_5	{0,b,1}	A_5
	rr	F_2	F_2	A_5	F_2	A_5
$F_3 = \{c, 1\}$	ll	F_3	F_3	F_3	A_5	A_5
	lr	{a,b,c,1}	F_3	F_3	A_5	A_5
	rl	F_3	F_3	F_3	A_5	A_5
	rr	F_3	{0,c,1}	{0,c,1}	A_5	A_5
$F_4 = A_5$	ll	A_5	A_5	A_5	A_5	A_5
	lr	A_5	A_5	A_5	A_5	A_5
	rl	A_5	A_5	A_5	A_5	A_5
	rr	A_5	A_5	A_5	A_5	A_5

Table 1: Table of stabilizers of the residuated lattice $\mathfrak{A}_5$.

(8) $(F:X)^{\Box_2}_{\Box_1} = A$ *if and only if* $X \subseteq F$. *In particular,* $(F:X)_{\Box} = A$ *if and only if* $X \subseteq F$.

(9) $(F:\emptyset)_s = (F:1)_s = (F:F)_s = A$.

(10) $X \subseteq (F:(F:X)^{l(r)}_l)^{l(r)}_r$, $(F:(F:X)^{l(r)}_r)^{l(r)}_l$, $(F:(F:X)_{l(r)})_{r(l)}$, $(F:(F:X)_s)_s$.

(11) $\cap_{i\in I}(F_i:X)^{\Box_2}_{\Box_1} = (\cap_{i\in I}F_i:X)^{\Box_2}_{\Box_1}$. *In particular,* $\cap_{i\in I}(F_i:X)_{\Box} = (\cap_{i\in I}F_i:X)_{\Box}$.

(12) $(F:0)^{l(r)}_r = (F:0)_r = (F:0)_s = (F:A)^{\Box_2}_{\Box_1} = F$.

(13) $(F:0)^{l(r)}_l = \{a \in A | \neg_{lr(rl)} a \in F\}$. *In particular,* $(0)^{\Box}_l = D^{\Box}_s(\mathfrak{A})$.

Proof. 1. By r_1 we have $X^{l(r)}_l = \{a \in A | (a \to_{l(r)} x) \to_{r(l)} x = 1,\ \forall x \in X\} = \{a \in A | a \to_{l(r)} x \leq x,\ \forall x \in X\}$. On the other hand, by r_4 we have $x \leq a \to_{\Box} x$, for any $a, x \in A$. It implies that $X^{l(r)}_l = \{a \in A | a \to_{l(r)} x = x,\ \forall x \in X\}$. It shows that $X^{\Box}_l = \{a \in A | a \to_{\Box} x = x, \forall x \in X\}$. Analogously, we can show that $X^{\Box}_r = \{a \in A | x \to_{\Box} a = a, \forall x \in X\}$.

2. Let $a \in (F:X)$. Then for any $x \in X$ we have $a \vee x \in F$. By r_4 we have $a \vee x \leq ((a \to_{l(r)} x) \to_{r(l)} x) \wedge ((x \to_{l(r)} a) \to_{r(l)} a)$. Since F is a filter so we have $((a \to_{l(r)} x) \to_{r(l)} x) \wedge ((x \to_{l(r)} a) \to_{r(l)} a) \in F$. It shows that $a \in (F:X)_l \cap (F:X)_r = (F:X)_s$. By Proposition 2.18(2) we can conclude that $F \subseteq (F:X)_s$.

3. Let $X \subseteq Y$ and $a \in (F:Y)^{l(r)}_l$. Then for any $x \in X$, since $X \subseteq Y$, we have $(a \to_{l(r)} x) \to_{r(l)} x \in F$ and it shows that $a \in (F:X)^{l(r)}_l$. So $(F:Y)^{l(r)}_l \subseteq (F:X)^{l(r)}_l$. Analogously, we can obtain the other cases.

4. It is an immediate consequence of (3).

5. Let $F \subseteq G$ and $a \in (F:X)^{l(r)}_l$. For any $x \in X$ we have $(a \to_{l(r)} x) \to_{r(l)} x \in F$. It implies that $(a \to_{l(r)} x) \to_{r(l)} x \in G$ and it shows that $a \in (G:X)^{l(r)}_l$. Analogously, we can show that the other cases.

6. Let $a \in X \cap (F:X)^{l(r)}_l$. Then $(a \to_{l(r)} x) \to_{r(l)} x \in F$ for any $x \in X$. Let $x = a$. By r_2 we have $a \in F$ and it implies that $X \cap (F:X)^{l(r)}_l \subseteq F$. Similarly, we can obtain the other cases.

7. It is an immediate consequence of (2) and (6)

8. Let $(F:X)^{l(r)}_l = A$ and $x \in X$. We have $x = 1 \to_{r(l)} x = (x \to_{l(r)} x) \to_{r(l)} x \in F$ and it shows that $X \subseteq F$. Conversely, if $X \subseteq F$, then by Proposition 2.18(3) we have $(F:X) = A$ and (2) implies that $(F:X)^{l(r)}_l = A$. Analogously, we can obtain the other cases.

9. It is an immediate consequence of (8).

10. Let $x \in X$. Then for any $a \in (F:X)^{l(r)}_l$ we have $(a \to_{l(r)} x) \to_{r(l)} x \in F$ and it implies that $x \in (F:(F:X)^{l(r)}_l)^{l(r)}_r$. Analogously, we can show that $X \subseteq (F:(F:X)^{l(r)}_r)^{l(r)}_l$, $(F:(F:X)_{l(r)})_{r(l)}$, $(F:(F:X)_s)_s$.

11. By (5), for each $i \in I$ we have $(\cap_{i\in I} F_i : X)^{\Box_2}_{\Box_1} \subseteq (F_i : X)^{\Box_2}_{\Box_1}$ and it shows that $(\cap_{i\in I} F_i : X)^{\Box_2}_{\Box_1} \subseteq \cap_{i\in I}(F_i : X)^{\Box_2}_{\Box_1}$. Conversely, let $a \in \cap_{i\in I}(F_i : X)^{l(r)}_l$. Thus, for any $i \in I$ and $x \in X$ we have $(a \to_{l(r)} x) \to_{r(l)} x \in F_i$ and it implies that we have $(a \to_{l(r)} x) \to_{r(l)} x \in \cap_{i\in I} F_i$ for any $x \in X$. Hence we obtain that $a \in (\cap_{i\in I} F_i : X)^{l(r)}_l$. Analogously, we can obtain the other cases.
12. By (2) we know that $F \subseteq (F : 0)^{l(r)}_r$, $(F : A)^{\Box_2}_{\Box_1}$. Let $a \in (F : 0)^{l(r)}_r$. Thus we have $a = 1 \to_{r(l)} a = (0 \to_{l(r)} a) \to_{r(l)} a \in F$. It means that $(F : 0)^{l(r)}_r = F$. If $a \in (F : A)^{l(r)}_l$, then for each $x \in A$ we have $(a \to_{l(r)} x) \to_{r(l)} x \in F$. Consider $x = a$. So we have $a = 1 \to_{l(r)} a = (a \to_{l(r)} a) \to_{r(l)} a \in F$. It shows that $(F : A)^{l(r)}_l = F$. Analogously, we can obtain the other cases.
13. It is straightforward.

□

Proposition 3.4. *Let $\mathfrak{A}$ be a residuated lattice and F be filters of $\mathfrak{A}$. Then the following assertions hold for any $X \subseteq A$.*

(1) $(X)^l_l$ *and* $(X)^r_l$ *are filters of* $\mathfrak{A}$.
(2) $(X)^\Box_r = (Fi(X))^\Box_r$ *for* $\Box \in \{l, r\}$.
(3) $Fi(X) \cap X_r = \{1\}$.

Proof. 1. See [9, Proposition 4.38].
2. By (4) we have $(Fi(X))^l_r \subseteq (X)^l_r$. Let $a \in (X)^l_r$ and $x \in Fi(X)$. By Proposition 2.15, there exist $x_1, x_2, \cdots, x_n \in X$, such that $x_1 \odot \cdots \odot x_n \leq x$. By Proposition 2.6($r_3$ and r_6) we have $x \to_l a \leq (x_1 \odot \cdots \odot x_n) \to_l a = x_1 \to_l (x_2 \to_l \cdots (x_n \to_l a) \cdots)$. On the other hand, we have $x_i \to_l a = a$ for $i = 1, \cdots, n$. This states that $x \to_l a = a$ and it implies $a \in (Fi(X))^l_r$. Analogously, we can show that $(X)^r_r = (Fi(X))^r_r$.
3. It is an immediate consequence of Proposition 3.3(6) and (2).

□

In the following example we show that assertions of Proposition 3.4 may not be true for the any stabilizer of any subset.

Example 3.5. *Consider Example 3.2. Then we have $(F_2 : 0)^l_l$, $(F_3 : 0)^r_l$, $(F_2 : a)^l_r$ and $(F_3 : a)^r_r$ are not filters of $\mathfrak{A}$. Also, we have $(F_2 : 0)^l_l \neq (F_2 : Fi(0))^l_l$, $(F_3 : 0)^r_l \neq (F_3 : Fi(0))^r_l$, $(F_2 : a)^l_r \neq (F_2 : Fi(a))^l_r$ and $(F_3 : a)^r_r \neq (F_3 : Fi(a))^r_r$.*

Proposition 3.6. *Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$, $x, y \in A$ and $e \in B(\mathfrak{A})$. Then the following assertions hold.*

(1) $x \leq y$ *implies* $(F : x)^\Box_r \subseteq (F : y)^\Box_r$. *In particular,* $x \leq y$ *implies* $(F : x)_r \subseteq (F : y)_r$.
(2) $(F : x \odot y)^\Box_r \subseteq (F : x \wedge y)^\Box_r \subseteq (F : \{x, y\})^\Box_r$.
(3) *If F is a normal filter of* $\mathfrak{A}$ *then* $(F : x/F)^{\Box_2}_{\Box_1} = (F : x)^{\Box_2}_{\Box_1}$.
(4) $e \in B(\mathfrak{A})$ *if and only if* $\neg_l e \in (e)_s$ *if and only if* $\neg_r e \in (e)_s$.

Proof. 1. Let $x \leq y$ and $a \in (F : x)^l_r$. By r_6 we have $(y \to_l a) \to_r a \geq (x \to_l a) \to_r a \in F$ and it implies that $a \in (F : y)^l_r$.
2. It follows by (1).
3. By Proposition 3.3(3), it is obvious that $(F : x/F)^{\Box_2}_{\Box_1} \subseteq (F : x)^{\Box_2}_{\Box_1}$. Now, let $a \in (F : X)^{l(r)}_l$ and $y \in x/F$. Therefore, $d^l(x, y) \in F$ and this means $d^l((a \to_{l(r)} x) \to_{r(l)} x, (a \to_{l(r)} y) \to_{r(l)} y) \in F$. On the other hand, we have $(a \to_{l(r)} x) \to_{r(l)} x \in F$ and this implies that $(a \to_{l(r)} y) \to_{r(l)} y \in F$. Thus $a \in (F : x/F)^{l(r)}_l$ and this shows the equality.
4. It is obvious by Proposition 2.16(2) and Proposition 3.3(1).

□

In the following example we show that assertions of Proposition 3.6 may not be true for the any stabilizer of any subset.

Example 3.7. *Consider Example 3.2. Then we have* $0 \leq c$ *but* $(F_2 : 0)^l_l \nsubseteq (F : c)^l_l$*. Also, we have* $(F_2 : a \odot a)^l_r \subsetneq (F_2 : a \wedge a)^l_r$*.*

Proposition 3.8. *Let* $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ *be a surjective homomorphism and* $\Box_1, \Box_2 \in \{l, r\}$*.*

(1) *If* F *is a filter of* $\mathfrak{A}$ *containing* $coker(h)$ *and* $X \subseteq A$ *then* $h((F : X)^{\Box_2}_{\Box_1}) = (h(F) : h(X))^{\Box_2}_{\Box_1}$*.*

(2) *If* F *is a filter of* $\mathfrak{B}$ *and* $Y \subseteq B$ *then* $h^{\leftarrow}((F : Y)^{\Box_2}_{\Box_1}) = (h^{\leftarrow}(F) : h^{\leftarrow}(Y))^{\Box_2}_{\Box_1}$*.*

Proof. 1. Let F be a filter of $\mathfrak{A}$ and $X \subseteq A$. By Proposition 2.19(1), $h(F)$ is a filter of $\mathfrak{B}$. If $X = \emptyset$ then by Proposition 3.3(8) we have $(F : X) = A$ and $(h(F) : h(X)) = B$. Since h is surjective so the equality holds. So let X be a nonempty subset of A. Assume that $b \in (h(F) : h(X))^{l(r)}_l$. So for each $y \in h(X)$ we have $(b \rightarrow_{l(r)} y) \rightarrow_{r(l)} y \in h(F)$. Hence, there are $x \in X$, $a \in A$ and $f \in F$ such that $h(x) = y$, $h(a) = b$ and $(b \rightarrow_{l(r)} y) \rightarrow_{r(l)} y = h(f)$. It means that $(h(a) \rightarrow_{l(r)} h(x)) \rightarrow_{r(l)} h(x) = h(f)$ and it implies that $f \rightarrow_l ((a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x) \in coker(h) \subseteq F$. Since F is a filter so we can conclude that $(a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x \in F$. Thus $a \in (F : X)^{l(r)}_l$ and it states that $b \in h((F : X)^{l(r)}_l)$.

Now, let $b \in h((F : X)^{l(r)}_l)$ and $y \in h(X)$. So there are $a \in (F : X)^{l(r)}_l$ and $x \in X$ such that $h(a) = b$ and $h(x) = y$ and it results that $(a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x \in F$. Therefore, $(h(a) \rightarrow_{l(r)} h(x)) \rightarrow_{r(l)} h(x) \in h(F)$ and it implies that $b \in (h(F) : h(X))^{l(r)}_l$. Analogously, we can show that $h((F : X)^{l(r)}_r) = (h(F) : h(X))^{l(r)}_r$

2. Let F be a filter of $\mathfrak{B}$ and $Y \subseteq A$. By Proposition 2.19(2), $h(F)$ is a filter of $\mathfrak{A}$. If $Y = \emptyset$ then we have $(F : Y)^{l(r)}_l = B$ and $(h^{\leftarrow}(F) : h^{\leftarrow}(Y))^{l(r)}_l = A$. Since h is surjective so the equality holds. Suppose that $a \in (h^{\leftarrow}(F) : h^{\leftarrow}(Y))^{l(r)}_l$. Consider $y \in Y$. So there is $x \in A$ such that $h(x) = y$. We have $(h(a) \rightarrow_{l(r)} y) \rightarrow_{r(l)} y = (h(a) \rightarrow_{l(r)} h(x)) \rightarrow_{r(l)} h(x) = h((a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x)$. On the other hand, we have $x \in h^{\leftarrow}(Y)$ and it implies that $(a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x \in h^{\leftarrow}(F)$. Therefore, $(h(a) \rightarrow_{l(r)} y) \rightarrow_{r(l)} y \in F$ for each $y \in Y$ and it states that $h(a) \in (F : Y)^{l(r)}_l$. It shows that $a \in h^{\leftarrow}((F : Y)^{l(r)}_l)$.

Conversely, assume that $a \in h^{\leftarrow}((F : Y)^{l(r)}_l)$ and $x \in h^{\leftarrow}(Y)$. Hence $h(a) \in (F : Y)^{l(r)}_l$ and $h(x) \in Y$. It implies that $h((a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x) = (h(a) \rightarrow_{l(r)} y) \rightarrow_{r(l)} y \in F$. Therefore, $(a \rightarrow_{l(r)} x) \rightarrow_{r(l)} x \in h^{\leftarrow}(F)$ and it concludes that $a \in (h^{\leftarrow}(F) : h^{\leftarrow}(Y))^{l(r)}_l$. Analogously, we can show that $h^{\leftarrow}((F : Y)^{l(r)}_r) = (h^{\leftarrow}(F) : h^{\leftarrow}(Y))^{l(r)}_r$. □

Let $\mathfrak{A}$ be a residuated lattice and F be a normal filter of $\mathfrak{A}$. The mapping $\pi^{\mathfrak{A}}_F : \mathfrak{A} \longrightarrow \mathfrak{A}/F$ defined by $\pi^{\mathfrak{A}}_F(a) = a/F$ is called the natural homomorphism. It is obvious that the natural homomorphism $\pi^{\mathfrak{A}}_F$ is surjective and $coker(\pi^{\mathfrak{A}}_F) = F$. Therefore, by Proposition 2.19 we have

$$\mathcal{F}(\mathfrak{A}/F) = \{H/F | F \subseteq H \in \mathcal{F}(\mathfrak{A})\}.$$

Lemma 3.9. *Let* $\mathfrak{A}$ *be a residuated lattice and* F *be a normal filter of* $\mathfrak{A}$*. Then for any* $x \in A - F$ *there is a subset* $X \neq A$ *of* A *such that* $x \in X$*,* $(F : X)^{\Box}_l \subseteq (F : x)^{\Box}_l$ *and* $(F : X)^{\Box}_l$ *is a filter of* $\mathfrak{A}$*.*

Proof. Let $x \in A$. By Proposition 3.8(1) we have $\pi_F(F : x)^{\Box}_l = (x/F)^{\Box}_l$. By Proposition 3.4(2), $(x/F)^{\Box}_l$ is a filter of $\mathfrak{A}/F$ and by 2.19(1) and Proposition 3.4(1) it implies that $(\pi^{\leftarrow}_F(F/F), \pi^{\leftarrow}_F(x/F))^{\Box}_l = (F : \pi^{\leftarrow}_F(x/F))^{\Box}_l$ is a filter of $\mathfrak{A}$. On the other hand we have $x \in \pi^{\leftarrow}_F(x/F)$ and $\pi^{\leftarrow}_F(x/F) \neq A$. Since if $\pi^{\leftarrow}_F(x/F) = A$ then $1 \in \pi^{\leftarrow}_F(x/F)$ which it implies that $x \in F$. Now, by Proposition 3.3(3) we get that $(F : \pi^{\leftarrow}_F(x/F))^{\Box}_l \subseteq (F : x)^{\Box}_l$. □

4. Galois connection of stabilizers in residuated lattice

Let $\mathfrak{A}$ be a residuated lattice, F be a filter of $\mathfrak{A}$, $\Box_1, \Box_2 \in \{l, r\}$ and $\Box \in \{l, r, s\}$. We define the following functions.

$$F^{\Box_2}_{\Box_1} : \begin{array}{ccc} \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A) \\ X & \longmapsto & (F : X)^{\Box_2}_{\Box_1}, \end{array} \qquad F_{\Box} : \begin{array}{ccc} \mathcal{P}(A) & \longrightarrow & \mathcal{P}(A) \\ X & \longmapsto & (F : X)_{\Box}. \end{array}$$

Proposition 4.1. *Let* $\mathfrak{A}$ *be a residuated lattice and* F *be a filter of* $\mathfrak{A}$*. Then the following pairs* (F^l_l, F^l_r)*,* (F^r_l, F^r_r)*,* (F_l, F_r) *and* (F_s, F_s) *are Galois connections on* $\mathcal{P}(A)$*.*

Proof. By Proposition 3.3(3), functions F_l^l and F_r^l are antitone and by 3.3(10), $F_l^lF_r^l$ and $F_r^lF_l^l$ are inflationary functions. So by Proposition 2.22 we obtain that (F_l^l, F_r^l) is a Galois connection on $\mathcal{P}(A)$. Analogously, we can show that (F_l^r, F_r^r), (F_l, F_r) and (F_s, F_s) are Galois connections on $\mathcal{P}(A)$. □

Corollary 4.2. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold for any $X, Y \subseteq A$.*

(1) $X \subseteq (F : Y)_l^\square$ *if and only if* $Y \subseteq (F : X)_r^\square$.
(2) $X \subseteq (F : Y)_l$ *if and only if* $Y \subseteq (F : X)_r$.
(3) $X \subseteq (F : Y)_s$ *if and only if* $Y \subseteq (F : X)_s$.

Proof. It follows by Proposition 4.1 and Definition 2.21. □

Corollary 4.3. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold for any $X \subseteq A$ and $\square \in \{l, r\}$.*

(1) $(F : X)_{l(r)}^\square = (F : (F : (F : X)_{l(r)}^\square)_{r(l)}^\square)_{l(r)}^\square$.
(2) $(F : X)_{l(r)} = (F : (F : (F : X)_{l(r)})_{r(l)})_{l(r)}$.
(3) $(F : X)_s = (F : (F : (F : X)_s)_s)_s$.

Proof. It follows by Proposition 4.1 and Proposition 2.23(1). □

Corollary 4.4. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold for any family $\{X\} \cup \{X_i\}_{i\in I} \in \mathcal{P}(A)$, $\square_1, \square_2 \in \{l, r\}$ and $\square \in \{l, r, s\}$.*

(1) $(F : \cup_{i\in I} X_i)_{\square_1}^{\square_2} = \cap_{i\in I}(F : X_i)_{\square_1}^{\square_2}$. *In particular,* $(F : \cup_{i\in I} X_i)_\square = \cap_{i\in I}(F : X_i)_\square$.
(2) $(F : X)_{\square_1}^{\square_2} = \cap_{x\in X}(F : x)_{\square_1}^{\square_2}$. *In particular,* $(F : X)_\square = \cap_{x\in X}(F : x)_\square$.
(3) $(F : X)_{\square_1}^{\square_2} = (F : X - F)_{\square_1}^{\square_2}$. *In particular,* $(F : X)_\square = (F : X - F)_\square$.
(4) *If* $0 \in X$, *then* $(F : X)_r^{l(r)} = (F : X)_r = (F : X)_s = F$.

Proof. 1. It is straightforward by Proposition 4.1 and Proposition 2.23(2) and Proposition 2.23(3).
2. By taking $X = \cup_{x\in X} x$ it follows by (1).
3. By (1) we have $(F : X)_{\square_1}^{\square_2} = (F : (X - F) \cap (X \cap F))_{\square_1}^{\square_2} = (F : X - F)_{\square_1}^{\square_2} \cap (F : X \cap F)_{\square_1}^{\square_2}$ and by Proposition 3.3(8) we have $(F : X \cap F)_{\square_1}^{\square_2} = A$. It states that $(F : X)_{\square_1}^{\square_2} = (F : X - F)_{\square_1}^{\square_2}$.
4. Let $0 \in X$. By (2) we have $(F : X)_r^{l(r)} = \cap_{x\in X}(F : x)_r^{l(r)}$ and by Proposition 3.3(12) we have $(F : 0)_r^{l(r)} = F$. Since $F \subseteq (F : x)_r^{l(r)}$ for any $x \in X$ it implies that $(F : X)_r^{l(r)} = F$. Analogously, we have $(F : X)_r = (F : X)_s = F$.
□

Corollary 4.5. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold for any $X \subseteq A$ and $\square \in \{l, r\}$.*

(1) $(F : X)_{l(r)}^\square = \cup\{Y \in \mathcal{P}(A) | X \subseteq (F : Y)_{r(l)}^\square\}$.
(2) $(F : X)_{l(r)} = \cup\{Y \in \mathcal{P}(A) | X \subseteq (F : Y)_{r(l)}\}$.
(3) $(F : X)_s = \cup\{Y \in \mathcal{P}(A) | X \subseteq (F : Y)_s\}$.

Proof. Let $X \subseteq A$. By Proposition 4.1 and Proposition 2.23(4) we have $(F : X)_{l(r)}^\square = \max\{Y \in \mathcal{P}(A) | X \subseteq (F : Y)_{r(l)}^\square\}$. let $\Gamma = \{Y \in \mathcal{P}(A) | X \subseteq (F : Y)_{r(l)}^\square\}$. We have $\max\Gamma \subseteq \cup\Gamma$. By considering $Y \in \Gamma$ we obtain that $X \subseteq (F : Y)_{r(l)}^\square$ and it implies $Y \subseteq (F : X)_{l(r)}^\square$ by Proposition 4.2(1). Therefore, $\cup\Gamma \subseteq (F : X)_{l(r)}^\square$ and again by Proposition 4.2(1) we obtain that $X \subseteq (F : \cup\Gamma)_{r(l)}^\square$. So $\cup\Gamma \in \Gamma$ and it means that $\cup\Gamma = \max\Gamma$. Similarly, (2) and (3) hold. □

Corollary 4.6. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold for any $\square \in \{l, r\}$.*

(1) $F_{l(r)}^\square F_{r(l)}^\square$ *is a closure operator on* $\mathcal{P}(A)$ *and* $\mathscr{C}_{F_{l(r)}^\square F_{r(l)}^\square} = \{(F : X)_{l(r)}^\square | X \subseteq A\}$.
(2) $F_{l(r)} F_{r(l)}$ *is a closure operator on* $\mathcal{P}(A)$ *and* $\mathscr{C}_{F_{l(r)} F_{r(l)}} = \{(F : X)_{l(r)} | X \subseteq A\}$.

(3) F_sF_s *is a closure operator on* $\mathcal{P}(A)$ *and* $\mathscr{C}_{F_sF_s} = \{(F : X)_s | X \subseteq A\}$.

Proof. By Proposition 4.1 and Proposition 2.23((5) and (6)) we obtain that $F^{\square}_{l(r)}F^{\square}_{r(l)}$ is a closure operator on $\mathcal{P}(A)$ and $\mathscr{C}_{F^{\square}_{l(r)}F^{\square}_{r(l)}} = \{F^{\square}_{l(r)}(X) | X \subseteq A\} = \{(F : X)^{\square}_{l(r)} | X \subseteq A\}$. Analogously, (2) and (3) hold. □

Corollary 4.7. *Let* $\mathfrak{A}$ *be a residuated lattice and F be a filter of* $\mathfrak{A}$. *Then the following assertions hold for any* $\square \in \{l, r\}$.

(1) $ST - F^{\square}_{l(r)} = (F^{\square}_{l(r)}(\mathcal{P}(A)); \wedge^{ST-F^{\square}_{l(r)}}, \vee^{ST-F^{\square}_{l(r)}}, F, A)$ *is a complete lattice where operations* $\wedge^{ST-F^{\square}_{l(r)}}$ *and* $\vee^{ST-F^{\square}_{l(r)}}$ *are defined as follows.*

$$\wedge_{i\in I}^{ST-F^{\square}_{l(r)}}(F : X_i)^{\square}_{l(r)} = (F : \cup_{i\in I}X_i)^{\square}_{l(r)},$$

and

$$\vee_{i\in I}^{ST-F^{\square}_{l(r)}}(F : X_i)^{\square}_{l(r)} = (F : \cap_{i\in I}(F : (F : X_i)^{\square}_{l(r)})^{\square}_{r(l)})^{\square}_{l(r)}.$$

(2) $ST - F_{l(r)} = (F_{l(r)}(\mathcal{P}(A)); \wedge^{ST-F_{l(r)}}, \vee^{ST-F_{l(r)}}, F, A)$ *is a complete lattice where operations* $\wedge^{ST-F_{l(r)}}$ *and* $\vee^{ST-F_{l(r)}}$ *are defined as follows.*

$$\wedge_{i\in I}^{ST-F_{l(r)}}(F : X_i)_{l(r)} = (F : \cup_{i\in I}X_i)_{l(r)},$$

and

$$\vee_{i\in I}^{ST-F_{l(r)}}(F : X_i)_{l(r)} = (F : \cap_{i\in I}(F : (F : X_i)_{l(r)})_{r(l)})_{l(r)}.$$

(3) $ST - F_s = (F_s(\mathcal{P}(A)); \wedge^{ST-F_s}, \vee^{ST-F_s}, F, A)$ *is a complete lattice where operations* $\wedge^{ST-F_s}$ *and* $\vee^{ST-F_s}$ *are defined as follows.*

$$\wedge_{i\in I}^{ST-F_s}(F : X_i)_s = (F : \cup_{i\in I}X_i)_s,$$

and

$$\vee_{i\in I}^{ST-F_s}(F : X_i)_s = (F : \cap_{i\in I}(F : (F : X_i)_s)_s)_s.$$

Proof. By Proposition 4.1, $(F^{\square}_l, F^{\square}_r)$ is a Galois connection and by Proposition 4.6(1), $F^{\square}_{l(r)}, F^{\square}_{r(l)}$ is a closure operator on $\mathcal{P}(A)$ and $\mathscr{C}_{F_{l(r)}F_{r(l)}} = F^{\square}_{l(r)}(\mathcal{P}(A))$. So by Proposition 2.25, $ST - F^{\square}_{l(r)} = (F^{\square}_{l(r)}(\mathcal{P}(A)); \wedge^{ST-F^{\square}_{l(r)}}, \vee^{ST-F^{\square}_{l(r)}}, F^{\square}_{l(r)}(A), F^{\square}_{l(r)}(\emptyset))$ is a complete lattice. Also, by Proposition 3.3(9) we have $F^{\square}_{l(r)}(\emptyset) = (F : \emptyset)^{\square}_{l(r)} = A$ and by Proposition 3.3(12) we have $F^{\square}_{l(r)}(A) = (F : A)^{\square}_{l(r)} = F$. Analogously, we can show that (2) and (3) hold. □

Proposition 4.8. *Let* $\mathfrak{A}$ *be a residuated lattice and F be a filter of* $\mathfrak{A}$. *Then the following assertions hold for any* $X \subseteq A$.

(1) *If* $a, b \in (F : X)^{l(r)}_r$ *then* $a \wedge b \in (F : X)^{l(r)}_r$. *In particular,* $a, b \in (F : X)_r$ *implies* $a \wedge b \in (F : X)_r$.

(2) $(F : X)^{\square}_l \in OF(\mathfrak{A})[F, A]$. *In particular,* $(F : X)_l \in OF(\mathfrak{A})[F, A]$.

Proof. 1. Let $a, b \in (F : X)^{l(r)}_r$. By r_6 we have $(x \to_{l(r)} a) \to_{r(l)} a \leq (x \to_{l(r)} (a \wedge b)) \to_{r(l)} a$ for any $x \in X$. Since F is a filter and $(x \to_{l(r)} a) \to_{r(l)} a \in F$ so we have $(x \to_{l(r)} (a \wedge b)) \to_{r(l)} a \in F$ for any $x \in X$. Analogously, we can obtain that $(x \to_{l(r)} (a \wedge b)) \to_{r(l)} b \in F$. By r_{10} we have $(x \to_{l(r)} (a \wedge b)) \to_{r(l)} (a \wedge b) = [(x \to_{l(r)} (a \wedge b)) \to_{r(l)} a] \wedge [(x \to_{l(r)} (a \wedge b)) \to_{r(l)} b]$. It means that $(x \to_{l(r)} (a \wedge b)) \to_{r(l)} (a \wedge b) \in F$ for any $x \in X$. Hence, we have $a \wedge b \in (F : X)^{l(r)}_r$.

2. Let $a \leq b$ and $a \in (F : X)^{l(r)}_l$. By r_6 we have $(a \to_{l(r)} x) \to_{r(l)} x \leq (b \to_{l(r)} x) \to_{r(l)} x$ for any $x \in X$. Since F is a filter and $(x \to_{l(r)} a) \to_{r(l)} a \in F$ so we have $(b \to_{l(r)} x) \to_{r(l)} x \in F$ for any $x \in X$. Hence, we have $b \in (F : X)^{l(r)}_l$.
□

Proposition 4.9. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold.*

(1) $\mathscr{C}_{F_r^\square F_l^\square} = \{(F:G)_r^\square | G \in OF(\mathfrak{A})[F,A]\}$. *In particular,* $\mathscr{C}_{\mathbf{1}_r^\square \mathbf{1}_l^\square} = \{(F)_r^\square | F \in F(\mathfrak{A})\}$.
(2) $\mathscr{C}_{F_r F_l} = \{(F:G)_r | G \in OF(\mathfrak{A})[F,A]\}$. *In particular,* $\mathscr{C}_{\mathbf{1}_r \mathbf{1}_l} = \{(F)_r | F \in F(\mathfrak{A})\}$.
(3) $\mathscr{C}_{F_s F_s} = \{(F:G)_s | G \in OF(\mathfrak{A})[F,A]\}$. *In particular,* $\mathscr{C}_{\mathbf{1}_s \mathbf{1}_s} = \{(F)_s | F \in F(\mathfrak{A})\}$.

Proof. It is obvious that $\{(F:G)_r^\square | G \in OF(\mathfrak{A})[F,A]\} \subseteq \mathscr{C}_{F_r^\square F_l^\square}$. Now, let $H = (F:X)_r^\square$ for some $X \subseteq A$. By Proposition 4.3(1) we have $(F:(F:H)_l^\square)_r^\square = H$ and by Proposition 4.8(2) we have $(F:H)_l^\square \in OF(\mathfrak{A})[F,A]$. It shows that $\mathscr{C}_{F_r^\square F_l^\square} \subseteq \{(F:G)_r^\square | G \in OF(\mathfrak{A})[F,A]\}$.

Let $H = (F)_r^\square$ for some $F \in F(\mathfrak{A})$. By Proposition 4.3(1) we have $((H)_l^\square)_r^\square = H$ and by Proposition 3.4(1) we have $(H)_l^\square \in F(\mathfrak{A})$. It shows that $\mathscr{C}_{\mathbf{1}_r^\square \mathbf{1}_l^\square} \subseteq \{(F)_r^\square | F \in F(\mathfrak{A})\}$. Analogously, we can show that (2) and (3) hold. □

Proposition 4.10. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Also let F_1 and F_2 be two ordered-filters of $\mathfrak{A}$ such that $F \subseteq F_1 \cap F_2$. Then the following assertions are equivalent.*

(1) $F_1 \cap F_2 = F$.
(2) $F_1 \subseteq (F:F_2)$.
(3) $F_1 \subseteq (F:F_2)_s$.
(4) $F_1 \subseteq (F:F_2)_l^l$.
(5) $F_1 \subseteq (F:F_2)_l^r$.
(6) $F_1 \subseteq (F:F_2)_r^l$.
(7) $F_1 \subseteq (F:F_2)_r^r$.
(8) $F_1 \subseteq (F:F_2)_l$.
(9) $F_1 \subseteq (F:F_2)_r$.

Proof. Let $f_1 \in F_1$ and $f_2 \in F_2$. We have $f_1, f_2 \leq f_1 \vee f_2$ and it states that $f_1 \in (F:F_2)$. Thus we have $F_1 \subseteq (F:F_2)$. Therefore (1) implies (2). By Proposition 3.3(2), (2) implies (3), (4), (5), (6), (7), (8) and (9). Now, let $f \in F_1 \cap F_2$. So we have $f = 1 \rightarrow_{r(l)} f = (f \rightarrow_{l(r)} f) \rightarrow_{r(l)} f \in F$ and it shows that (4), (5), (6), (7) and consequently (2), (3), (8) and (9) implies (1). □

Corollary 4.11. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Also let G be an ordered-filter of $\mathfrak{A}$ containing F. Then the following assertions hold.*

(1) $(F:G) = (F:G)_s = (F:G)_l = (F:G)_l^l = (F:G)_l^r \subseteq (F:G)_r$.
(2) $(F:G)_s$, $(F:G)_l$, $(F:G)_l^l$ *and* $(F:G)_l^r$ *are filters of* $\mathfrak{A}$.
(3) $G \subseteq (F:(F:G)_l^\square)_l^\square \cap (F:(F:G)_l^\square)_l \cap (F:(F:G)_l)_l \cap (F:(F:G)_l)_l^\square \cap (F:(F:G)_s)_l^\square \cap (F:(F:G)_l^\square)_s \cap (F:(F:G)_s)_s$.

Proof. 1. Let G be an ordered-filter of $\mathfrak{A}$ containing F. By Proposition 4.8(2) we know that $(F:G)_l^l$ is an ordered-filter of $\mathfrak{A}$. Also, by hypothesis and Proposition 3.3(8) we have $(F:G)_l^l \cap G = F$. So by Proposition 4.10 we obtain that $(F:G)_l^l \subseteq (F:G)_s$. It shows that $(F:G) = (F:G)_s = (F:G)_l = (F:G)_l^l \subseteq (F:G)_r$. Analogously, we can show that $(F:G) = (F:G)_s = (F:G)_l = (F:G)_l^r \subseteq (F:G)_r$.
2. It follows by (1) and Proposition 2.18(1).
3. It follows by (1) and Proposition 2.18(4).
□

Proposition 4.12. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. Then the following assertions hold.*

(1) *The complete lattice* $ST - F_l^\square = (F_l^\square(\mathcal{P}(A)); \wedge^{ST-F_l^\square}, \vee^{ST-F_l^\square}, (F:-), F, A)$ *is a pseudocomplemented lattice.*
(2) *The complete lattice* $ST - F_l = (F_l(\mathcal{P}(A)); \wedge^{ST-F_l}, \vee^{ST-F_l}, (F:-), F, A)$ *is a pseudocomplemented lattice.*
(3) *The complete lattice* $ST - F_s = (F_s(\mathcal{P}(A)); \wedge^{ST-F_s}, \vee^{ST-F_s}, (F:-), F, A)$ *is a pseudocomplemented lattice.*

Proof. Let $(F:X)_l^\square \in F_l^\square(\mathcal{P}(A))$. By Proposition 3.3((2) and (7)) we have $(F:X)_l^\square \cap (F:(F:X)_l^\square)_l^\square = F$. Also, Proposition 4.11(2) states that $(F:X)_l^\square$ is an ordered-filter of $\mathfrak{A}$. So by Corollary 4.11(2) we obtain that $(F:(F:X)_l^\square)_l^\square = (F:(F:X)_l^\square)$. It shows that $(F:X)_l^\square \cap (F:(F:X)_l^\square) = F$. Now, let $(F:X)_l^\square \cap (F:Y)_l^\square = F$. Since, $(F:X)_l^\square$ and $(F:Y)_l^\square$ are ordered-filters of $\mathfrak{A}$ containing F so by Proposition 4.10 we obtain that $(F:Y)_l^\square \subseteq (F:(F:X)_l^\square)$. It shows that the meet semilattice $(F_l^\square(\mathcal{P}(A)); \cap, (F:-), F)$ is a pseudocomplemented lattice. Analogously, we can show that (2) and (3) hold. □

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