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On Abundant Semigroups with Weak Normal Idempotents

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Abstract. A weak normal idempotent of an abundant semigroup was introduced by Guo [7]. In this paper, weak normal idempotents and normal idempotents of abundant semigroups are respectively characterized in many different ways. These results enable us to obtain an example which shows that the class of normal idempotents of abundant semigroups is a proper subclass of normal idempotents of abundant semigroups. Furthermore, this example tell us that there exists a non-regular abundant semigroup containing a weak normal idempotent. At last, we investigate the relationships between weak normal idempotents and normal idempotents and deduce that the main result of [2] can not be generalized into the class of abundant semigroups.

1. Introduction and Preliminaries

Blyth and McFadden [1] introduce the concept of *normal idempotent* of a regular semigroup. An idempotent u of a regular semigroup S is a *medial idempotent* if for any element x of the regular semigroup $\overline{E}$ generated by the set E of all idempotents of S, $xux = x$. A medial idempotent u is called *normal* if $u\overline{E}u$ is a semilattice. They describe all regular semigroups that contain a normal idempotent in terms of idempotent-generated regular semigroup with a normal idempotent, and an inverse semigroup with an identity. After that, much attention has been paid to this kind of idempotents. The normal idempotent of an *abundant semigroup* is introduced by Jing [11]. As analogous to it, Guo [7] introduces the *weak normal idempotent* of abundant semigroup. They all devoted themselves to explore construction theorems.

In this paper, we characterize weak normal idempotents and normal idempotents of an abundant semigroup in several different ways. By using the alternative description, we obtain a non-regular abundant semigroup with a weak normal idempotent, which is not a normal idempotent. It shows that the class of normal idempotents of abundant semigroups is a proper subclass of normal idempotents of abundant semigroups. Obviously, it also helps to improve that the weak normal idempotent of an abundant semigroup introduced by Guo [7] make sense. Furthermore, we investigate the relationships between weak normal idempotents and normal idempotents. Based on the results, we claim that the main result of [2] can not be generalized into the class of abundant semigroups.

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For our purposes here we mention only the following notions and some properties. The reader is referred to [4], [5], [6], [8], [9] and [12] for the notation and terminology not defined in this paper.

The class of all abundant semigroups is an important class of generalized regular semigroups. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S are generalizations of the familiar Green's relations $\mathcal{L}$ and $\mathcal{R}$. Two elements a and b in S are said to be $\mathcal{L}^*$-related if and only if they are $\mathcal{L}$-related in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. A semigroup S is called *abundant* if each $\mathcal{L}^*$-class and $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup S is called *quasi-adequate* [3] if its idempotents form a subsemigroup. An *adequate semigroup* [6] is a quasi-adequate semigroup in which the idempotents commute. Adequate semigroups are analogues of inverse semigroups in the range of abundant semigroups.

Lemma 1.1. *([5]) Let S be a semigroup and e be an idempotent of S. For any $a \in S$, the following conditions are equivalent:*

(1) $a\,\mathcal{L}^*\,e$ ($a\,\mathcal{R}^*\,e$);

(2) $a = ae$ ($ea = a$) and for all $x, y \in S^1$, $ax = ay$ ($xa = ya$) $\Rightarrow ex = ey$ ($xe = ye$).

Definition 1.2. *([11]) An idempotent u of an abundant semigroup S is a medial idempotent if for any element x of the semigroup $\overline{E}$ generated by the set E of all idempotents of S, $xux = x$. A medial idempotent u is called normal if $u\overline{E}u$ is a semilattice.*

Definition 1.3. *([7]) An idempotent u of an abundant semigroup S is a weak medial idempotent if for any idempotent x of S, $xux = x$. A weak medial idempotent is called a weak normal idempotent if uSu is an adequate semigroup.*

Lemma 1.4. *([7]) Let S be abundant semigroup and u be a weak medial idempotent for S. Then for any $a \in S, e \in E$ and for any $a^+, a^* \in E(a^+\mathcal{R}^*\,a\,\mathcal{L}^*\,a^*)$:*

(1) $ue, eu, ueu \in E$;

(2) $a^+u\,\mathcal{R}^*\,a\,\mathcal{L}^*\,ua^*$;

(3) $ua^+u\,\mathcal{R}^*\,uau\,\mathcal{L}^*\,ua^*u$.

Lemma 1.5. *([7]) Let S be an abundant semigroup and u be a weak medial idempotent for S. Then the following statements are equivalent:*

(1) u is weak normal;

(2) $(\forall e \in E(S))$ eSe is an adequate semigroup;

(3) $(\forall e \in E(S))$ $E(eS)$ is a right normal band;

(4) $(\forall e \in E(S))$ $E(Se)$ is a left normal band;

(5) $E(uS)$ is a right normal band;

(6) $E(Su)$ is a left normal band.

For any abundant semigroup S and for any $a \in S$, let a^+ (a^*) be the typical idempotent such that $a\,\mathcal{R}^*\,a^+$ ($a\,\mathcal{L}^*\,a^*$).

Let U be an abundant subsemigroup of S and E be the set of idempotents of S. U is called a left (right) $*$-subsemigroup if for all $a \in U$, there exists $e \in U \cap E$ such that $a\,\mathcal{L}^*(S)\,e$ ($a\,\mathcal{R}^*(S)\,e$). If U is both a left and a right $*$-subsemigroup, then we call it a $*$-subsemigroup. If for any element s of S and any element x of U, $sus \in U$ then U is called a quasi-ideal of S.

Suppose that S° is an adequate $*$-subsemigroup of an abundant semigroup S and that E° is the idempotent semilattice of S°. Let

$$C_{S^\circ}(x) = \{x^\circ \mid x = ex^\circ f, e, f \in E, e\,\mathcal{L}\,x^{\circ+}, f\,\mathcal{R}\,x^{\circ*}, x^{\circ+}, x^{\circ*} \in E^\circ\}.$$

S° is called an *adequate transversal* if for any $x \in S$, $|C_{S^\circ}(x)| = 1$. In this case, the unique element of $C_{S^\circ}(x)$ will be denoted by x°. Then e and f are uniquely determined by x and x°. For convenience, write e and f by e_x and f_x, respectively. Particularly, S° is *weakly multiplicative* if for any $e \in E(S)$, $e^\circ \in E^\circ$ and S° is *multiplicative* if for any $x, y \in S$, $f_x e_y \in E^\circ$. Furthermore, if an (weakly) multiplicative adequate transversal S° is a quasi-ideal then it also called a quasi-ideal (weakly) multiplicative adequate transversal.

For any adequate transversal S°, two subsets of E such as

$$I = \{\, e \in E \mid \exists\, g \in E^\circ,\ e\,\mathcal{L}\,g \,\} \text{ and } \Lambda = \{\, f \in E \mid \exists\, h \in E^\circ,\ f\,\mathcal{R}\,h\}$$

always play an important role in studying the construction of an abundant semigroup with an adequate transversal. Moreover, they can be described in an alternative way: $I = \{\, e_x \mid x \in S\}$ and $\Lambda = \{\, f_x \mid x \in S\}$.

2. On Weak Normal Idempotents

This section will be devoted to study the properties of weak normal idempotents of abundant semigroups and to describe them in many different terms. Meanwhile, an example is given to show that there exists an abundant semigroup with a weak normal idempotent. Let S be an abundant semigroup and E be the set of all idempotents of S.

Lemma 2.1. *If u is a weak normal idempotent of S then*
(1) $(\forall x \in S)\ xu\, \mathcal{R}\, x\, \mathcal{L}\, ux$;
(2) Eu is a left normal band;
(3) uE is a right normal band;
(4) uEu is a semilattice;
(5) $(\forall e, f \in E)\ e\, \mathcal{R}\, f \Rightarrow eu = fu$;
(6) $(\forall e, f \in E)\ e\, \mathcal{L}\, f \Rightarrow ue = uf$;
(7) uSu is a $*$-subsemigroup.

Proof. (1) Let $x^* \in E$ be such that $x\, \mathcal{R}^*\, x^*$. Then $x^*ux^* = x^*$ and $x = xx^*$. It follows that $x = xx^*ux^* = xux^*$. Hence, $x\, \mathcal{R}\, xu$. Similarly, we have $x\, \mathcal{L}\, ux$.

(2) To prove the desired result, we need to show that $Eu = E(Su)$. Since u is a weak normal idempotent, $(eu)^2 = eueu = eu$ for any $e \in E$, that is to say, $eu \in E(Su)$. The reverse inclusion, i.e., $E(Su) \subseteq Eu$, is obvious. Hence, by Lemma 1.5, Eu is a left normal band.

(3) Similar to the above argument, we establish that $uE = E(uS)$ is a right normal band.

(4) Since u is a weak normal idempotent, $ueu \in E$ for any $e \in E$. It means $uEu \subseteq E(uSu)$. Clearly, $E(uSu) \subseteq uEu$. Hence, $uEu = E(uSu)$. Notice that uSu is an adequate semigroup. We deduce that uEu is a semilattice.

(5) Suppose that $e\, \mathcal{R}\, f$. Then $eu\, \mathcal{R}\, fu$. Since Eu is a left normal band, $fu = eufu = eufueu = eu$.

(6) It is dual of (5).

(7) The result is a consequence of 1.4 (3) and 1.5 (2). □

Lemma 2.2. *If u is a weak medial idempotent of S then the following statements are equivalent:*
(1) uSu is an adequate semigroup;
(2) uEu is a semilattice.

Proof. (1) ⇒ (2) With the given information, u is weak normal idempotent. It follows that uEu is a semilattice.

(2) ⇒ (1) Suppose that uEu is a semilattice. Easily, $uEu = E(uSu)$ and so $E(uSu)$ is a semilattice. Notice that u is also a weak medial idempotent. We have for any $a \in uSu$ $ua^+u\ \mathcal{R}^*\ uau\ \mathcal{L}^*\ ua^*u$, where $ua^+u, ua^*u \in uEu$. Hence, uSu is an adequate semigroup. □

Every weak normal idempotent can be characterized in the following way, which should be viewed in comparison with Definition 1.3.

Theorem 2.3. *For any $u \in E$, u is a weak normal idempotent of S if and only if for any $e \in E$, $eue = e$ and uEu is a semilattice.*

Example 2.4. *Let T be a cancellative semigroup with an identity e. Suppose that $P = \begin{pmatrix} e & p \\ q & a \end{pmatrix}$, where $p, q \in Reg(T)$ and $a \in T$ but $p, q \neq e$ and $a \notin Reg(T)$. Let $G = \mathcal{M}[T; \{1,2\}, \{1,2\}; P]$. Then G is an abundant semigroup. It is easy to check that $E(G) = \{(1, e, 1), (2, p^{-1}, 1), (1, q^{-1}, 2)\}$ and that $(1, e, 1)E(G)(1, e, 1) = \{(1, e, 1)\}$. By computing, for any $x \in E(G)$, $x(1, e, 1)x = x$. Therefore, $(1, e, 1)$ is a weak normal idempotent of G.*

Guided by Lemma 2.1(7) and 1.4(2), we consider characterizing weak normal idempotents in terms of adequate transversals.

Proposition 2.5. *If u is a weak normal idempotent of S then uSu is a weakly multiplicative adequate transversal for S.*

Proof. As u is weak normal, $x^+u \,\mathcal{L}\, ux^+u \,\mathcal{R}^*\, uxu$, $ux^* \,\mathcal{R}\, ux^*u \,\mathcal{L}^*\, uxu$ and $x^+u, ux^* \in E$ for any $x \in S$. Then $x = x^+u(uxu)ux^*$ implies $uxu \in C_{uSu}(x)$. Next, we show $|C_{uSu}(x)| = 1$. For any $x^\circ \in C_{uSu}(x)$, there exist idempotents e and f such that $x = ex^\circ f$ and $e \,\mathcal{L}\, x^{\circ+}, f \,\mathcal{R}\, x^{\circ*}$ for some $x^{\circ+}, x^{\circ*} \in E(uSu)$. It is sufficient to verify $e \,\mathcal{R}^*\, x \,\mathcal{L}^*\, f$, which together with $x \,\mathcal{R}^*\, x^+u$ implies $x^{\circ+} \,\mathcal{L}\, e \,\mathcal{R}\, x^+u$. Since uEu is a semilattice, $x^{\circ+}x^+u = ux^{\circ+}ux^+u \in uEu$. Then $x^{\circ+} \,\mathcal{R}\, x^{\circ+}x^+u \,\mathcal{L}\, x^+u$ and so $ux^+u \,\mathcal{L}\, x^+u \,\mathcal{L}\, x^{\circ+}x^+u$. Hence, $x^{\circ+}x^+u = ux^+u$ as uSu is an adequate semigroup. Similarly, we have $ux^*x^{\circ*} = ux^*u$. Thus, $x^\circ = x^{\circ+}xx^{\circ*} = x^{\circ+}x^+uxux^*x^{\circ*} = uxu$. Particularly, for any $e \in E$, $e^\circ = ueu \in uEu \subseteq E$. In conclusion, uSu is a weakly multiplicative adequate transversal for S. □

To wonder whether the converse of Theorem 2.1 is true, some crucial properties of weak normal idempotents are required.

Lemma 2.6. *If u is a weak normal idempotent of S then*
(1) $I = Eu$ is a right normal band;
(2) $\Lambda = uE$ is a left normal band;
(3) $I \cap \Lambda = uEu$ is a semilattice.

Proof. (1) If u is a weak normal idempotent then uSu is a weakly multiplicative adequate transversal. Then $I = \{ e \in E \mid \exists\, g \in E^\circ, e \,\mathcal{L}\, g \}$. With the given notation, $e = eg = egu$ and so $e = eu \in Eu$. To obtain the reverse inclusion, let e be an arbitrary element of E. We have known that $eu \,\mathcal{L}\, ueu$ and $ueu, eu \in E$. Then $eu \in I$. Hence, $I = Eu$ is a right normal band.

(2) A similar argument using the condition $\Lambda = \{ f \in E \mid \exists\, h \in E^\circ, e \,\mathcal{R}\, h\}$ establishes that $\Lambda = uE$.

(3) The result for $I \cap \Lambda$ is a consequence of the results for I and Λ. □

Lemma 2.7. *If u is an arbitrary idempotent of S, then*
(1) $(\forall g \in E)\ C_{Su}(g) \neq \emptyset \Rightarrow g \,\mathcal{L}\, ug$;
(2) $(\forall g \in E)\ C_{uS}(g) \neq \emptyset \Rightarrow g \,\mathcal{R}\, gu$;
(3) $(\forall g \in E)\ C_{uSu}(g) \neq \emptyset \Rightarrow gu \,\mathcal{R}\, g \,\mathcal{L}\, ug$.

Proof. (1) Suppose $g^\circ \in C_{Su}(g)$. Then $g = e_g g^\circ f_g$. Since $e_g \,\mathcal{L}\, g^{\circ+}$ for some $g^{\circ+} \in E(Su)$, $e_g = e_g g^{\circ+} = e_g g^{\circ+}u$. It follows that $e_g = e_g u$ and so $g = e_g u e_g g^\circ f_g = e_g ug$. Hence, we deduce $g \,\mathcal{L}\, ug$.

(2) It is a dual result of (1).

(3) It follows from (1) and (2). □

Theorem 2.8. *For every idempotent u of S, the following statements are equivalent:*
(1) u is a weak normal idempotent;
(2) uEu is a semilattice and $C_{uSu}(g) \neq \emptyset$ for any $g \in E$;
(3) uE is a right normal band and $C_{Su}(g) \neq \emptyset$ for any $g \in E$;
(4) Eu is a left normal band and $C_{uS}(g) \neq \emptyset$ for any $g \in E$.

Proof. We just need to prove (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2) Suppose that u is a weak normal idempotent. Then by Proposition 2.5, uEu is a semilattice and $C_{uSu}(g) \neq \emptyset$ for any $g \in E$

(2) ⇒ (3) With the given notation, we have $g \,\mathcal{R}\, gu$. Then $ug \,\mathcal{R}\, ugu$ as $\mathcal{R}$ is a left congruence. Since $ugu \in E$, $(ug)^2 = (ugu)ug = ug$ and so $ug \in E$. For any $h, k \in E$, $(uguh)^2 = uguhuguh = uguh$ and $uguhuk = uhuguk$. Hence, uE is a right normal band.

(3) $\Rightarrow$ (1) $C_{Su}(g) \neq \emptyset$ implies $g \mathcal{L} ug$. Then $g = gug$ since $ug \in E$. That is to say, u is a weak medial idempotent. Hence, $ugu \in E$. For any $e \in E$, $(ueugu)^2 = ueugueugu = ueugu$ and $ueugu = ugueu$. Thus, uEu is a semilattice. From Lemma 2.2 that u is weak normal. $\square$

By Proposition 2.5 and Theorem 2.8, we now immediately deduce

Corollary 2.9. *Let $u \in E$. The following statements are equivalent:*
(1) u is a weak normal idempotent;
(2) uEu is a semilattice and $C_{uSu}(x) \neq \emptyset$ for any $x \in S$;
(3) uE is a right normal band and $C_{Su}(x) \neq \emptyset$ for any $x \in S$;
(4) Eu is a left normal band and $C_{uS}(x) \neq \emptyset$ for any $x \in S$.

The following result will be required in investigating relationships between weak normal idempotents and weakly multiplicative adequate transversals.

Corollary 2.10. *Let $u \in E$ and uSu be an adequate semigroup. If for any $g \in E$, $C_{uSu}(g) \cap E(uSu) \neq \emptyset$, then u is a weak normal idempotent.*

Proof. (1) To get the required result, we just verify $uEu = E(uSu)$. Suppose that $g^{\circ} \in C_{uSu}(g) \cap E(uSu)$. Then $g = e_g g^{\circ} f_g$ and $f_g \mathcal{R} g^{\circ *}$ for some $g^{\circ *} \in E(uSu)$. Clearly, $f_g = g^{\circ *} f_g = ug^{\circ *} f_g$ and so $f_g = uf_g$. Then $(f_g u)^2 = f_g u f_g u = f_g u$, i.e., $f_g u \in E(uSu)$. Similarly, we have $ue_g \in E(uSu)$. Notice that $g^{\circ} \in E(uSu)$. We have $ugu = ue_g e^{\circ} f_g u \in E(uSu)$. It means $uEu \subseteq E(uSu)$, which together with $E(uSu) \subseteq uEu$ implies $uEu = E(uSu)$. Since uSu is an adequate semigroup, uEu is a semilattice. Thus, by Theorem 2.8, u is a weak normal idempotent. $\square$

By reviewing Proposition 2.5 and Corollary 2.10, every weak normal idempotent can be characterized as follow.

Theorem 2.11. *For any $u \in E$, u is a weak normal idempotent of S if and only if uSu is a weakly multiplicative adequate transversal for S.*

3. On Normal Idempotents

As analogous to Section 2, we focus on characterizing normal idempotents in many different ways. Suppose that S is an abundant semigroup and that $\overline{E}$ be a subsemigroup generated by the set E of all idempotents of S.

The normal idempotent can be characterized in the following way, which should be viewed in comparison with Definition1.2.

Theorem 3.1. *For any $u \in E$, u is a normal idempotent of S if and only if for any $e \in \overline{E}$, $eue = e$ and uEu is a semilattice.*

Proof. Suppose first that u is a normal idempotent. Then for any $e \in \overline{E}$, $eue = e$ and $u\overline{E}u$ is a semilattice. Clearly, uEu is a subsemilattice of $u\overline{E}u$.

Conversely, suppose that for any $e \in \overline{E}$, $eue = e$ and uEu is a semilattice. Since $eue = e$ implies $ueu \in E$, $u\overline{E}u \subseteq uEu$. Then $u\overline{E}u$ is obviously a semilattice. We now conclude that u is a normal idempotent. $\square$

Note 1. *Generally, a weak normal idempotent of an abundant semigroup is not normal.*

By comparing the above Theorem with Theorem 2.3, a normal idempotent is clearly a weak normal idempotent. For the converse statement, recall from Example 1 that $G = \mathcal{M}[T; \{1, 2\}, \{1, 2\}; P]$ *is an abundant semigroup with an idempotent set*

$$E(G) = \{\ (1, e, 1),\ (2, p^{-1}, 1),\ (1, q^{-1}, 2)\ \}$$

and that $(1, e, 1)$ *is a weak normal idempotent of G. Since*

$$(1, q^{-1}, 2)(2, p^{-1}, 1) = (1, q^{-1}ap^{-1}, 1)$$

and

$$\begin{aligned} &(1, q^{-1}, 2)(2, p^{-1}, 1)(1, e, 1)(1, q^{-1}, 2)(2, p^{-1}, 1) \\ &= (1, q^{-1}ap^{-1}, 1)(1, e, 1)(1, q^{-1}ap^{-1}, 1) \\ &= (1, q^{-1}ap^{-1}q^{-1}ap^{-1}, 1) \neq (1, q^{-1}ap^{-1}, 1), \end{aligned}$$

$(1, e, 1)$ *is not a normal idempotent of G.*

As Note 1 shows, the class of normal idempotents is a proper subclass of weak normal idempotents. Taking Theorem 2.11 into account, it is interesting to discuss which kind of adequate transversals could be related to normal idempotents. First of all, let us describe normal idempotents in many other terms.

Lemma 3.2. *Let u be a weak normal idempotent of S. If for any* $g, h \in E$, $C_{uSu}(gh) \cap E \neq \emptyset$ *then*

(1) $\overline{E}u = Eu$ *is a left normal band;*

(2) $u\overline{E} = uE$ *is a right normal band;*

(3) $u\overline{E}u = uEu$ *is a semilattice.*

Proof. (1) To verify $\overline{E}u = Eu$, we just need to establish that $\overline{E}u \subseteq E$. Suppose $x \in \overline{E}$. Then there exist an integral n and some $g_1, g_2, \cdots, g_n \in E$ such that $x = g_1g_2\cdots g_n$. If $n = 1$ then $x = g_1$. Since u is weak normal, $xu \in E$. For a general case, we consider the mathematical induction.
(i) If $n = 2$ then $x = g_1g_2$. We have shown that uSu is a weakly multiplicative adequate transversal. Then for any $x \in S$, $|C_{uSu}(x)| = 1$. That $ug_1g_2u \in C_{uSu}(g_1g_2)$ is an easy consequence of the proof of Proposition 2.5. Hence, $\{\ ug_1g_2u\ \} = C_{uSu}(g_1g_2)$. Since $C_{uSu}(g_1g_2) \cap E \neq \emptyset$, $ug_1g_2u \in E$. Notice that $ug_1g_2u\ \mathcal{L}\ g_1g_2u$. So $g_1g_2u = g_1g_2u(ug_1g_2u) = (g_1g_2u)^2$, i.e., $g_1g_2u \in E$.
(ii) Suppose that $xu \in E$ as long as $n = k (k \geq 2)$.
(iii) If $n = k + 1$ then $x = g_1g_2\cdots g_ng_{n+1}$. Denote $g_2\cdots g_ng_{n+1}$ by y. Then by the hypothesis $yu \in E$ and so $xu = g_1yu = g_1(yu)u \in E$.

To sum up, $\overline{E}u = Eu$ is a left normal band by Lemma 2.1.

Part (2) is similar, and once again we can prove $u\overline{E} = uE$.

To prove part (3), notice that $u\overline{E}u = \overline{E}u \cap u\overline{E}$. The rest is clear. □

Theorem 3.3. *Let* $u \in E$. *The following statements are equivalent:*

(1) u is a normal idempotent;

(2) uEu is a semilattice and $C_{uSu}(gh) \cap E \neq \emptyset$ *for any* $g, h \in E$;

(3) uE is a right normal band and $C_{Su}(gh) \cap E \neq \emptyset$ *for any* $g, h \in E$;

(4) Eu is a left normal band and $C_{uS}(gh) \cap E \neq \emptyset$ *for any* $g, h \in E$.

Proof. As (4) is a dual result of (3), we only prove (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2) If u is a normal idempotent then uEu is a semilattice and $ughu \in u\overline{E}u \subseteq E$. We have in fact established that $ughu \in C_{uSu}(gh)$. Hence, $C_{uSu}(gh) \cap E \neq \emptyset$.

(2) ⇒ (3) By Theorem 2.8, u is a weak normal idempotent and so uE is a right normal band. Obviously, for any $g, h \in E$, $C_{uSu}(gh) \cap E \neq \emptyset$ implies $C_{Su}(gh) \cap E \neq \emptyset$.

(3) ⇒ (1) Suppose $(gh)^\circ \in C_{Su}(gh) \cap E$. Then there exist $e_{gh}, f_{gh} \in E$ such that $e_{gh} \,\mathcal{L}\, (gh)^{\circ+}, f_{gh} \,\mathcal{R}\, (gh)^{\circ*}$ for some $(gh)^{\circ+}, (gh)^{\circ*} \in E(Su)$ and $gh = e_{gh}(gh)^\circ f_{gh}$. To obtain the required result, we first verify $u(gh)^\circ \in C_{uSu}(gh) \cap E$. Indeed, $u(gh)^\circ \in E$ as uE is a band and $gh = e_{gh}(gh)^\circ(u(gh)^\circ)f_{gh}$. According to Theorem 2.8, u is a weak normal idempotent and so Eu is also a band. Hence, $e_{gh}(gh)^\circ = e_{gh}u(gh)^\circ u \in E$. Notice that $e_{gh} \,\mathcal{L}\, (gh)^{\circ+} \,\mathcal{R}\, (gh)^\circ$. Then $e_{gh}(gh)^\circ \,\mathcal{L}\, (gh)^\circ \,\mathcal{L}\, u(gh)^\circ$, which together with $f_{gh} \,\mathcal{R}\, (gh)^{\circ*} \,\mathcal{L}\, (gh)^\circ \,\mathcal{L}\, u(gh)^\circ$ implies $u(gh)^\circ \in C_{uSu}(gh)$.

Next we show that for any $x \in \overline{E}$, $xux = x$. In fact, the above observation tell us that u is a weak normal idempotent and $C_{uSu}(gh) \cap E \neq \emptyset$ for any $g, h \in E$. Hence, by Lemma 3.2, $\overline{E}u$ is a band and $u\overline{E}u$ is a semilattice. We have known that $xu \,\mathcal{R}\, x$. Thus, $x = xux$ as required. Therefore, we conclude that u is a normal idempotent. □

An alternative way to express the above result is sometimes useful.

Theorem 3.4. *Let $u \in E$. The following statements are equivalent:*
(1) u is a normal idempotent;
(2) $u\overline{E}u$ is a semilattice and $C_{uSu}(g) \neq \emptyset$ for any $g \in E$;
(3) $u\overline{E}$ is a right normal band and $C_{Su}(g) \neq \emptyset$ for any $g \in E$;
(4) $\overline{E}u$ is a left normal band and $C_{uS}(g) \neq \emptyset$ for any $g \in E$.

Proof. We just prove (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2) It is an easy consequence of Lemma 3.2 and the above Theorem.

(2) ⇒ (3) $u\overline{E}u \subseteq E$ implies $u\overline{E}u = uEu$. Then by Theorem 2.8, u is a weak normal idempotent. Hence, for any $x \in \overline{E}$, $x \,\mathcal{R}\, xu$ and so $ux \,\mathcal{R}\, uxu$. Since $uxu \in E$, $ux = (uxu)ux = (ux)^2$. It means $u\overline{E} \subseteq E$. As $u\overline{E}u$ is a semilattice, it is easy to see that $u\overline{E}$ is a right normal.

(3) ⇒ (1) $u\overline{E} \subseteq E$ implies $u\overline{E} = uE$. By Theorem 2.8, u is a weak normal idempotent. Then for any $g, h \in E$, $ughu \in C_{uSu}(gh) \cap E$ since $u\overline{E}$ is a band. As a result of the above theorem, u is a normal idempotent. □

To summarize, the close correspondence between weak normal idempotents and weakly multiplicative adequate transversals discussed in Section 2 enables us to consider the following alternative characterization of normal idempotents.

Theorem 3.5. *For any $u \in E$, u is a normal idempotent of S if and only if uSu is a multiplicative adequate transversal for S.*

Proof. As we know, if u is a normal idempotent then uSu is a weakly multiplicative adequate transversal. Easily, $\Lambda I \subseteq u\overline{E}u \subseteq E$, i.e, $f_y e_x \in E(uSu)$ for any $x, y \in S$. Thus, uSu is a multiplicative adequate transversal.

Conversely, if uSu is a multiplicative adequate transversal then u is a weak normal idempotent and so for any $g, h \in E$, $ughu \in C_{uSu}(gh) \cap E$. We deduce that u is a normal idempotent. □

4. Relationships between Weak Normal Idempotents and Normal Idempotents

In this sequel, we explore the conditions under which a weak normal idempotents is normal. For this purpose, we here add a basic property of regular elements that will be useful. Denote the set of regular elements of an arbitrary semigroup S by Reg(S).

Lemma 4.1. *([10]) Let S be an arbitrary semigroup. Then the following statements are equivalent:*
(1) For all idempotents e and f of S the element ef is regular;
(2) $< E(S) >$ is a regular subsemigroup;
(3) $Reg(S)$ is a regular subsemigroup.

A semigroup S is said to satisfy the regular condition if $Reg(S)$ is a regular semigroup. Throughout what follows, let S be an abundant subsemigroup with an idempotent set E and $\overline{E}$ be a semigroup generated by E.

Lemma 4.2. *Let u be a weak normal idempotent of S. For any $i \in Eu$ and $\lambda \in uE$, if $\lambda i \in Reg(S)$ then $\lambda i \in uEu$.*

Proof. Suppose that $x \in V(\lambda i)$. Then $uix\lambda u(\lambda i)ix\lambda = ix(\lambda i)x\lambda = ix\lambda$ and $\lambda i(ix\lambda)\lambda i = \lambda ix\lambda i = \lambda i$. Hence, $ix\lambda \in V(\lambda i)$. On the other hand, $(ix\lambda)^2 = ix\lambda ix\lambda = ix\lambda$, i.e, $ix\lambda \in E$. So $uix\lambda u \in E$. By a routine argument, we have $uix\lambda u \in V(\lambda i)$. Hence, $\lambda i \in uEu$ since $Reg(uSu)$ is an inverse semigroup. □

Theorem 4.3. *Let u be a weak normal idempotent of S. Then the following statements are equivalent:*
(1) u is a normal idempotent;
(2) $\Lambda I \subseteq uEu$;
(3) $\Lambda I \subseteq Reg(S)$;
(4) $uEu = u\overline{E}u$.

Proof. (1) ⇒ (2) Since u is a normal idempotent, $I = Eu$ and $\Lambda = uE$ and $u\overline{E}u = uEu$. Hence, $\Lambda I \subseteq uEu$.

(2) ⇒ (3) With the given information, uEu is a band. Then $\Lambda I \subseteq Reg(S)$.

(3) ⇒ (4) We have known that for any $g, h \in E$, $ughu \in C_{uSu}(gh)$. Clearly, $ughu \in uEu \subseteq E$. Hence, by Lemma 3.2, $uEu = u\overline{E}u$.

(4) ⇒ (1) By Theorem 2.8, uEu is a semilattice and for any $g \in E$ $C_{uSu}(g) \neq \emptyset$. Notice that $uEu = u\overline{E}u$. Then $u\overline{E}u$ is a semilattice. Hence, in view of Theorem 3.4, we hold that u is a normal idempotent. □

Proposition 4.4. *For any $u \in E$, if u is a weak normal idempotent of S then the following statements are equivalent:*
(1) $Reg(S)$ is a regular subsemigroup;
(2) u is a normal idempotent.

Proof. (1) ⇒ (2) It follows from Theorem 4.3 immediately.

(2) ⇒ (1) For any $g, h \in E$, $ghughugh = gh$ and $ughughughu = ughu$. It means $ughu \in V(gh)$. Hence, $Reg(S)$ is a regular subsemigroup. □

A main result of [2] states that a weakly multiplicative inverse transversal S° for a regular semigroup S is multiplicative if and only if S° is a quasi-ideal of S. Analogously, by Theorem 2.11, 3.5 and Proposition 4.4, we also have the following result for the class of abundant semigroups.

Corollary 4.5. *Suppose that S satisfies the regular condition. For any $u \in E$, uSu is a weakly multiplicative adequate transversal for S if and only if uSu is a multiplicative transversal for S.*

As Example 2.1 shows, $G = \mathcal{M}[T; \{1, 2\}, \{1, 2\}; P]$ is a non-regular abundant semigroup and $(1, e, 1)$ is a weak normal idempotent but not a normal idempotent. Moreover, by easily calculating, G does not satisfy the regular condition. According to Theorem 2.11 and 3.5, we deduce that

Note 2. *In general, a weakly multiplicative adequate transversal of an abundant semigroup is not multiplicative even if it is a quasi-ideal.*

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