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On the Stability of the Telegraph Equation with Time Delay

Allaberen Ashyralyev[a,b,c], Deniz Agirseven[d], Koray Turk[d]

[a]*Department of Mathematics, Near East University, Nicosia, TRNC, Mersin 10, Turkey*
[b]*Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan*
[c]*Peoples Friendship University of Russia (RUDN University) 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation*
[d]*Department of Mathematics Trakya University, Edirne, Turkey*

Abstract. In this study, the initial value problem for telegraph equations with time delay in a Hilbert space is considered. The main theorem on stability estimates for the solution of this problem is established. As a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary condition is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes.

1. Introduction

In many fields of the contemporary science and technology, systems with delaying terms appear. The dynamical processes are described by systems of delay ordinary and partial differential equations. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The second order delay differential equation with damping term is of interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots (see, [1]). These are illustrations of inverted pendulum problems. The theory of linear delay differential equations has been studied extensively by many researchers (see, [2]-[12] and the references given therein).

Telegraph equation is mostly interested in physical systems. Many physicists and engineers use telegraph equation without time delay (see, [13]-[17]), and also telegraph equation is studied by using classical energy methods and operator theory(see, [18]). For example, Ashyralyev and Modanli (see, [19]-[21]) studied the stability of Cauchy problem for telegraph differential and difference equations in a Hilbert space.

For several reasons, in problems encountered in real life, time delay should be considered in modeling. However, the stability theory of problems for a delay telegraph equation is not well-investigated. A few researchers are interested in these kinds of problems. Feireisl (see, [22]) proved the existence of small global (in time) solutions to an abstract evolution equation containing a damping term and applied the result to fully nonlinear telegraph equations and to nonlinear equations involving operators with time delay.

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Email addresses: allaberen.ashyralyev@neu.edu.tr (Allaberen Ashyralyev), denizagirseven@trakya.edu.tr (Deniz Agirseven), korayturk@trakya.edu.tr (Koray Turk)

In the present paper, we study the initial value problem for the telegraph differential equation with time delay

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = aAv([t]), \quad t > 0, \\ v(0) = \varphi, \quad v'(0) = \psi \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A, $A \geq \delta I$, φ and ψ are elements of $D(A)$ and $[t]$ denotes the greatest-integer function. Here $\delta > \frac{\alpha^2}{4}$.

A function $v(t)$ is called a solution of problem (1), if the following conditions are satisfied:

i. $v(t)$ is twice continuously differentiable on the interval $[0, \infty)$.

ii. The element $v(t)$ belongs to $D(A)$ for all $t \in [0, \infty)$, and the function $Av(t)$ is continuous on the interval $[0, \infty)$.

iii. $v(t)$ satisfies the equation and initial conditions in (1).

The main theorem on stability estimates for the solution of problem (1) is established. In applications, as a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary condition is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes.

2. The Main Theorem on the Stability

It is easy to show that for $\delta > \frac{\alpha^2}{4}$, the operator $B = A - \frac{\alpha^2}{4}I$ be a self-adjoint positive definite operator in a Hilbert space H with $B \geq (\delta - \frac{\alpha^2}{4})I$. Throughout this paper, $c(t)$ and $s(t)$ are operator-functions defined by formulas

$$c(t)u = \frac{e^{iB^{1/2}t} + e^{-iB^{1/2}t}}{2}u, \quad s(t)u = \int_0^t c(s)uds.$$

Let us give lemma that will be needed below.

Lemma 2.1. *For $t \geq 0$, the following estimates hold:*

$$||B^{-1/2}||_{H\to H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}}, \tag{2}$$

$$||c(t)||_{H\to H} \leq 1, \quad ||B^{1/2}s(t)||_{H\to H} \leq 1. \tag{3}$$

The proof of Lemma 2.1 is based on the spectral representation of unit self-adjoint positive definite operator B in a Hilbert space H (see, [12]).

Theorem 2.2. *For the solution of problem (1), the following estimates hold:*

$$\max_{0\leq t\leq 1} ||v(t)||_H \leq b\left\|\varphi\right\|_H + \left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\psi\right\|_H, \tag{4}$$

$$\max_{0\leq t\leq 1} \left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}v'(t)\right\|_H \leq c\left\|\varphi\right\|_H + d\left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\psi\right\|_H, \tag{5}$$

$$\max_{n\le t\le n+1}\|v(t)\|_H \le b\max_{n-1\le t\le n}\|v(t)\|_H + \max_{n-1\le t\le n}\left\|\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}v'(t)\right\|_H,\ n=1,2,..., \tag{6}$$

$$\max_{n\le t\le n+1}\left\|\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}v'(t)\right\|_H \le c\max_{n-1\le t\le n}\|v(t)\|_H + d\max_{n-1\le t\le n}\left\|\left(A-\frac{\alpha^2}{4}I\right)^{-1/2}v'(t)\right\|_H,\ n=1,2,..., \tag{7}$$

where

$$b=|a|+|1-a|d,\ \ c=|1-a|\frac{\delta}{\delta-\frac{\alpha^2}{4}},\ \ d=1+\frac{\frac{\alpha}{2}}{\sqrt{\delta-\frac{\alpha^2}{4}}}.$$

Proof. Problem (1) can be rewritten as the equivalent initial value problem for the system of first order linear differential equations

$$\begin{cases} v'(t)+\frac{\alpha}{2}v(t)+iB^{1/2}v(t)=z(t),\ t>0,\ v(0)=\varphi,\quad v'(0)=\psi \\ z'(t)+\frac{\alpha}{2}z(t)-iB^{1/2}z(t)=aAv([t]),\ t>0,\ z(0)=\psi+(\frac{\alpha}{2}+iB^{1/2})\varphi. \end{cases} \tag{8}$$

Integrating these equations, we can write

$$\begin{cases} v(t)=e^{-(t-n+1)(\frac{\alpha}{2}I+iB^{1/2})}v(n-1)+\int\limits_{n-1}^{t}e^{-(t-s)(\frac{\alpha}{2}I+iB^{1/2})}z(s)ds, \\ z(t)=e^{-(t-n+1)(\frac{\alpha}{2}I-iB^{1/2})}z(n-1)+\int\limits_{n-1}^{t}e^{-(t-s)(\frac{\alpha}{2}I-iB^{1/2})}aAv([s])ds \end{cases}$$

for any $n-1\le t\le n,\ n=1,2,....$ Therefore,

$$z(t)=e^{-(t-n+1)(\frac{\alpha}{2}I-iB^{1/2})}\left[v'(n-1)+\left(\frac{\alpha}{2}I+iB^{1/2}\right)v(n-1)\right]+\left(\frac{\alpha}{2}I+iB^{1/2}\right)\left(I-e^{-(t-n+1)(\frac{\alpha}{2}I-iB^{1/2})}\right)av(n-1)$$

$$=e^{-(t-n+1)(\frac{\alpha}{2}I-iB^{1/2})}v'(n-1)+\left(\frac{\alpha}{2}I+iB^{1/2}\right)\left[aI+(1-a)e^{-(t-n+1)(\frac{\alpha}{2}I-iB^{1/2})}\right]v(n-1).$$

Applying this formula, we get

$$v(t)=\left\{aI+(1-a)e^{-(t-n+1)(\frac{\alpha}{2}I+iB^{1/2})}+(1-a)e^{-(t-n+1)\frac{\alpha}{2}}\left(\frac{\alpha}{2}I+iB^{1/2}\right)s(t-n+1)\right\}v(n-1)$$

$$+e^{-(t-n+1)\frac{\alpha}{2}}s(t-n+1)v'(n-1). \tag{9}$$

Let $0\le t\le 1$. Then, using formula (9), we get

$$v(t)=\left\{aI+(1-a)e^{-\frac{\alpha t}{2}}\left[c(t)+\frac{\alpha}{2}s(t)\right]\right\}\varphi+e^{-\frac{\alpha t}{2}}s(t)\psi.$$

Applying the triangle inequality and estimates (2) and (3), we get

$$\|v(t)\|_H\le b\|\varphi\|_H+\|B^{-1/2}\psi\|_H,$$

$$\|B^{-1/2}v'(t)\|_H\le c\|\varphi\|_H+d\|B^{-1/2}\psi\|_H$$

for any $t \in [0,1]$. From these estimates, they follow estimates (4) and (5).
Let $n \leq t \leq n+1,\ n = 1,2,....$ Then using formula (9), we get

$$v(t) = \left\{aI + (1-a)e^{-(t-n+1)\frac{\alpha}{2}}\left[c(t-n+1) + \frac{\alpha}{2}s(t-n+1)\right]\right\}v(n-1) + e^{-(t-n+1)\frac{\alpha}{2}}s(t-n+1)v'(n-1).$$

Applying the triangle inequality and estimates (2) and (3), we get

$$\|v(t)\|_H \leq b \max_{n-1\leq t\leq n} \|v(t)\|_H + \max_{n-1\leq t\leq n} \left\|B^{-1/2}v'(t)\right\|_H,$$

$$\|B^{-1/2}v'(t)\| \leq c \max_{n-1\leq t\leq n} \|v(t)\|_H + d \max_{n-1\leq t\leq n} \left\|B^{-1/2}v'(t)\right\|_H$$

for any $t \in [n, n+1],\ n = 1,2,....$ From these estimates, they follow estimates (6) and (7).
The proof of Theorem 2.2 is completed. □

3. Numerical Results

We consider the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = 0.001\frac{\partial^2 u([t],x)}{\partial x^2},\ \ t>0,\ \ 0<x<\pi, \\ u(0,x) = \sin x,\ \ u_t(0,x) = -\sin x,\ \ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0,\ \ t \geq 0 \end{cases} \tag{10}$$

for the delay telegraph differential equation. The exact solution of problem (10) is

$$u(t,x) = \sin x \begin{cases} 1.001e^{-t} + 0.001te^{-t} - 0.001, & 0 \leq t \leq 1, \\ 1.001e^{-t} + [0.002002 - 0.001001e]\, te^{-t} - 0.001\left[1.002e^{-1} - 0.001\right], & 1 \leq t \leq 2, \\ \left[0.999994996 + 0.001005004e + 1.001(0.001)^2e^2\right]e^{-t} \\ +\left[0.003007004 - 0.002006004e - 1.001(0.001)^2e^2\right]te^{-t} \\ -0.001\left[1.005004e^{-2} - 0.003004e^{-1} - (0.001)^2\right], & 2 \leq t \leq 3, \\ ... \end{cases} \tag{11}$$

Using the first order of accuracy difference scheme for the approximate solution of problem (10), we get the following system of equations

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1} - u_n^k}{\tau} - \frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} = 0.001\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2}, \\ t_k = k\tau,\ \ x_n = nh,\ \ k \geq 1,\ \ 1 \leq n \leq M-1,\ N\tau = 1,\ Mh = \pi, \\ u_n^k = \sin x_n,\ \ \frac{u_n^{k+1} - u_n^k}{\tau} = -\sin x_n,\ \ x_n = nh,\ \ 0 \leq n \leq M,\ \ -N \leq k \leq 0, \\ u_0^k = u_M^k = 0,\ \ k \geq 0. \end{cases} \tag{12}$$

We can rewrite system (12) in the matrix form

$$\begin{cases} AU^{k+1} + BU^{k} + CU^{k-1} = 0.001R\varphi(U^{k-N}), \ \ 1 \le n \le N-1, \ \ k = 1, 2, ..., \\ U^{k} = \begin{bmatrix} \sin x_0 \\ \sin x_1 \\ \vdots \\ \sin x_{M-1} \\ \sin x_M \end{bmatrix}_{(M+1)\times 1}, U^{k+1} = (1-\tau)U^{k}, \ \ -N \le k \le 0, \end{cases} \tag{13}$$

where A, B, C are $(M+1)\times(M+1)$ matrices defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 1 \\ a & b & a & 0 & . & 0 & 0 & 0 & 0 \\ 0 & a & b & a & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & . & a & b & a & 0 \\ 0 & 0 & 0 & 0 & . & 0 & a & b & a \end{bmatrix}_{(M+1)\times(M+1)}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & c & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & d & 0 \end{bmatrix}_{(M+1)\times(M+1)}.$$

Here $a = -\frac{1}{h^2}$, $b = \frac{1}{\tau^2} + \frac{2}{\tau} + \frac{2}{h^2}$, $c = -\frac{2}{\tau^2} - \frac{2}{\tau}$, $d = \frac{1}{\tau^2}$ and R is the $(M+1)\times(M+1)$ identity matrix, and $\varphi(U^{k-N})$,U^s are $(M+1)\times 1$ column vectors as

$$\varphi(U^{k-N}) = \begin{bmatrix} 0 \\ \varphi_1^{k-N} \\ \vdots \\ \varphi_{M-1}^{k-N} \\ 0 \end{bmatrix}_{(M+1)\times 1}, \quad U^s = \begin{bmatrix} u_0^s \\ u_1^s \\ \vdots \\ u_{M-1}^s \\ u_M^s \end{bmatrix}_{(M+1)\times 1} \quad \text{for } \ s = k, \ k \pm 1,$$

where $\varphi_n^{k-N} = \frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2}$ for $1 \le n \le M-1$.

Hence, we have a second order difference equation with respect to k matrix coefficients. From (13) it follows that

$$\begin{cases} U^{k+1} = A^{-1}\left(0.001R\varphi(U^{k-N}) - BU^k - CU^{k-1}\right), \ \ k = 1, 2, ..., \\ U^k = \begin{bmatrix} \sin x_0 \\ \sin x_1 \\ \vdots \\ \sin x_{M-1} \\ \sin x_M \end{bmatrix}_{(M+1)\times 1}, U^{k+1} = (1-\tau)U^k, \ \ -N \le k \le 0. \end{cases} \tag{14}$$

Second, using the second order of accuracy difference scheme for the approximate solution of problem (10), we obtain the following system of equations

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + 2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{2h^2} - \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2} \\ \quad = 0.001\left[\frac{u_{n+1}^{k+1-N}-2u_n^{k+1-N}+u_{n-1}^{k+1-N}}{2h^2} + \frac{u_{n+1}^{k-1-N}-2u_n^{k-1-N}+u_{n-1}^{k-1-N}}{2h^2}\right], \\ t_k = k\tau, \ \ x_n = nh, \ \ k \ge 1, \ \ 1 \le n \le M-1, \ \ N\tau = 1, \ \ Mh = \pi, \\ u_n^k = \sin x_n, \ \ \frac{u_n^{k+1}-u_n^k}{\tau} = -\sin x_n + \frac{\tau}{2}\sin x_n, \ \ x_n = nh, \ \ 0 \le n \le M, \ \ -N \le k \le 0, \\ u_0^k = u_M^k = 0, \quad k \ge 0. \end{cases} \tag{15}$$

We can rewrite system (15) in the matrix form

$$\begin{cases} AU^{k+1} + BU^k + CU^{k-1} = 0.001R\varphi(U^{k-N}), \ \ 1 \le n \le N-1, \quad k = 1, 2, ..., \\ U^k = \begin{bmatrix} \sin x_0 \\ \sin x_1 \\ \vdots \\ \sin x_{M-1} \\ \sin x_M \end{bmatrix}_{(M+1)\times 1}, U^{k+1} = (1-\tau+\frac{\tau^2}{2})U^k, \ \ -N \le k \le 0, \end{cases} \tag{16}$$

where A, B, C are $(M+1)\times(M+1)$ matrices defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 1 \\ x & y & x & 0 & . & 0 & 0 & 0 & 0 \\ 0 & x & y & x & . & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & y & x & 0 & 0 \\ 0 & 0 & 0 & 0 & . & x & y & x & 0 \\ 0 & 0 & 0 & 0 & . & 0 & x & y & x \end{bmatrix}_{(M+1)\times(M+1)}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & z & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 1 \\ x & t & x & 0 & . & 0 & 0 & 0 & 0 \\ 0 & x & t & x & . & 0 & 0 & 0 & 0 \\ 0 & 0 & x & t & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & t & x & 0 & 0 \\ 0 & 0 & 0 & 0 & . & x & t & x & 0 \\ 0 & 0 & 0 & 0 & . & 0 & x & t & x \end{bmatrix}_{(M+1)\times(M+1)},$$

where $x=-\frac{1}{2h^2},\ y=\frac{1}{\tau^2}+\frac{1}{\tau}+\frac{1}{h^2},\ z=-\frac{2}{h^2},\ t=\frac{1}{\tau^2}-\frac{1}{\tau}+\frac{1}{h^2}$ and R is the $(M+1)\times(M+1)$ identity matrix, $\varphi(U^{k-N})$, U^s are $(M+1)\times 1$ column vectors as

$$\varphi(U^{k-N})=\begin{bmatrix} 0 \\ \varphi_1^{k-N} \\ \vdots \\ \varphi_{M-1}^{k-N} \\ 0 \end{bmatrix}_{(M+1)\times 1},\quad U^s=\begin{bmatrix} u_0^s \\ u_1^s \\ \vdots \\ u_{M-1}^s \\ u_M^s \end{bmatrix}_{(M+1)\times 1} \quad \text{for } s=k,\ k\pm 1,$$

where $\varphi_n^{k-N}=\frac{u_{n+1}^{k+1-N}-2u_n^{k+1-N}+u_{n-1}^{k+1-N}}{2h^2}+\frac{u_{n+1}^{k-1-N}-2u_n^{k-1-N}+u_{n-1}^{k-1-N}}{2h^2}$ for $1\leq n\leq M-1$.

Hence, we have a second order difference equation with respect to k matrix coefficients. Applying (16), we can obtain the solution of this difference scheme. We find the numerical solutions for different values of N and M. For $N=M=40$, 80, 160 in $t\in[0,1]$, $t\in[1,2]$, $t\in[2,3]$, the errors are given respectively in Table 1, Table 2 and Table 3. Thus, by using the second order of accuracy difference scheme, the accuracy of solution increases faster than the first order of accuracy difference scheme. While the error decreases in half by using the first order of accuracy difference scheme, it decreases quarter by using the second order of accuracy difference scheme. The error is computed by the following formula

$$E_M^N=\max_{\substack{0\leq k<\infty \\ 0\leq n\leq M}}\left|u\left(t_k,x_n\right)-u_n^k\right|.$$

Here, u_n^k represents the numerical solutions of these difference schemes at (t_k,x_n).

Table 1. Comparison of errors of difference schemes in $t\in[0,1]$

Method	N=M=40	N=M=80	N=M=160
Difference scheme (12) in $t\in[0,1]$	0.0046624	0.0023123	0.0011517
Difference scheme (15) in $t\in[0,1]$	0.0001080	0.0000282	0.0000076

Table 2. Comparison of errors of difference schemes in $t \in [1, 2]$

Method	N=M=40	N=M=80	N=M=160
Difference scheme (12) in $t \in [1, 2]$	0.0017215	0.0008538	0.0004252
Difference scheme (15) in $t \in [1, 2]$	0.0000367	0.0000089	0.0000020

Table 3. Comparison of errors of difference schemes in $t \in [2, 3]$

Method	N=M=40	N=M=80	N=M=160
Difference scheme (12) in $t \in [2, 3]$	0.0006350	0.0003149	0.0001568
Difference scheme (15) in $t \in [2, 3]$	0.0000123	0.0000027	0.0000004

4. Conclusion

In this paper, we studied the initial value problem for telegraph equations with time delay in a Hilbert space. Theorem on stability estimates for the solution of this problem is established. As a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary conditions is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes. Some of these statements were formulated in [23] without proof. Applying this approach and method of [9], we can study the initial value problem for the telegraph differential equation with time delay

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = Bv(t-w) + C\frac{dv(t-w)}{dt} + f(t), \quad t > 0, \\ v(t) = \varphi(t), \quad -w \leq t \leq 0 \end{cases} \tag{17}$$

in a Hilbert space H with a self-adjoint positive definite operator A, where B and C are closed operators. Here $\varphi(t)$ is a continuously differentiable abstract-function defined on the interval $[-w, 0]$ with values in H; $f(t)$ is continuous abstract-function defined on the interval $[0, \infty)$ with values in H.

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