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Systems of k Boolean Inequations and a Boolean Equation

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Abstract. In this paper elementary generalized systems of Boolean equations are investigated. The formula for solving systems of k Boolean inequations and a Boolean equation is presented. This systems have many applications in computer science for solving logical problems. Presented formulas can accelerate application of elementary generalized systems of Boolean equations.

1. Introduction

The study of Boolean equations in arbitrary Boolean algebras began with Bool, Schröder and Löwenheim. The basic facts and various forms of solutions of Boolean equations can be found in Rudeanu's books [5],[6]. Let $(B, \cap, \cup, ', 0, 1)$ be a Boolean algebra and n be a natural number.

Definition 1.1. *Let $x \in B$. Then*

$$x^1 = x, \qquad x^0 = x'.$$

If $X = (x_1, \ldots, x_n) \in B^n$ and $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$ then

$$X^A = x_1^{a_1} \cap \cdots \cap x_n^{a_n}.$$

In the sequel $\cap$ will be omitted. For the following definitions and theorems, see e.g. Rudeanu [5].

Definition 1.2. *The Boolean functions of n variables (BF n) over the Boolean algebra $(B, \cup, \cdot, ', 0, 1)$ are determined by the following rules:*

0) For every $a \in B$, constant function $f_a : B^n \to B$ defined by

$$f_a(x_1, \ldots, x_n) = a \ \ (\forall x_1, \ldots, x_n \in B)$$

is a BF n.

1) For every $i = 1, 2, \ldots, n$, the projection function $\varepsilon_i : B^n \to B$ defined by

$$\varepsilon_i(x_1, \ldots, x_n) = x_i \ \ (\forall x_1, \ldots, x_n \in B)$$

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is a BF n.

2) If $f, g : B^n \to B$ are BF n, then the functions $f \cup g$, fg, $f' : B^n \to B$ defined by

$$(f \cup g)(x_1, \dots, x_n) = f(x_1, \dots, x_n) \cup g(x_1, \dots, x_n)\ (\forall x_1, \dots, x_n \in B),$$

$$(fg)(x_1, \dots, x_n) = f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)\ (\forall x_1, \dots, x_n \in B),$$

$$f'(x_1, \dots, x_n) = (f(x_1, \dots, x_n))'\ \ (\forall x_1, \dots, x_n \in B)$$

are BF n.

3) Any BF n is obtained by applying the rules 0), 1) and 2) a finite number of times.

Theorem 1.3. *(Corollary 1 in [5]) The function $f : B^n \to B$ is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_A f(A) X^A.$$

A Boolean equation in n unknown is an equation of the form

$$f(X) = g(X),$$

where $f, g : B^n \to B$ are Boolean function.

Theorem 1.4. *(Theorem 2.1 in [5]) Every Boolean equations is equivalent to a single Boolean equation of the form f(X)=0.*

Theorem 1.5. *(Theorem 1.5,(1.52) in [5]) Let $x_1, ...x_n, a_c, b_c (C \in \{0,1\}^n \subseteq B^n)$ be elements of a Boolean algebra $(B, \cup, \cdot, ', 0, 1)$; put $X = (x_1, x_2, ..., x_n)$. The following relation holds:*

$$(\bigcup_c a_c X^c)(\bigcup_c b_c X^c) = (\bigcup_c a_c b_c X^c).$$

2. Generalized systems of Boolean equations

Definition 2.1. *The generalized systems of Boolean equations (GSBE's for short) over a Boolean algebra are defined recursively as follows:*

(i) every Boolean equation $f(X) = 0$ is a GSBE;

(ii) the negation, logical conjunction and logical disjunction of any GSBE's is a GSBE;

(iii) every GSBE is obtained by applying rules (i) and (ii) finitely many times.

Definition 2.2. *Let $S(x_1, \dots, x_n)$ denote a GSBE whose (free!) variables belong to the set $\{x_1, \dots, x_n\}$. By a solution of $S(x_1, \dots, x_n)$ is meant any vector $(a_1, \dots, a_n) \in B^n$ such that the statement $S(a_1, \dots, a_n)$ obtained by replacing each x_i by a_i is true. A GSBE which has solutions is said to be consistent or satisfiable. Two GSBE's $S(x_1, \dots, x_n)$ and $T(x_1, \dots, x_n)$ are said to be equivalent provided they have the same set of solutions.*

Definition 2.3. *An elementary GSBE is either a Boolean equation $f(X) = 0$ or the system of the form*

$$f_1(X) \neq 0\ \wedge \cdots \wedge\ f_k(X) \neq 0 \tag{1}$$

or of the form

$$g(X) = 0 \wedge f_1(X) \neq 0\ \wedge \cdots \wedge\ f_k(X) \neq 0. \tag{2}$$

If $k = 1$ then the GSBE is atomic. An atomic GSBE of the form $f(X) \neq 0$ will be called a Boolean inequation. The problem of solving GSBE's reduces to a particular case of it.

The previous definitions and more on generalized systems of Boolean equations can be found in Rudeanu [6]. The problem of solving GSBE's is not completely solved. In the sequel we describe all solutions of elementary GSBE's.

3. Boolean equations

To solve a Boolean equation $f(X) = 0$ means to determine all $X \in B^n$ such that $f(X) = 0$ holds i.e. to determine the set $S = \{X | f(X) = 0 \wedge X \in B^n\}$.

Theorem 3.1. *(Theorem 2.3 in [5]) Let $f : B^n \to B$ be a Boolean function. The equation $f(X) = 0$ has a solution if and only if*

$$\prod_A f(A) = 0.$$

Let $T = (t_1, \ldots, t_n) \in B^n$.

Definition 3.2. *Let $f, F_1, \ldots, F_n : B^n \to B$ be Boolean functions and $F = (F_1, \ldots, F_n)$. The formula*

$$X = F(T),$$

or in scalar form

$$x_i = F_i(t_1, \ldots, t_n), \quad (i = 1, \ldots, n)$$

expresses a general solution of the Boolean equation $f(X) = 0$ if and only if, for every $X \in B^n$,

$$f(X) = 0 \Leftrightarrow (\exists T) X = F(T).$$

Definition 3.3. *Let $f, F_1, \ldots, F_n : B^n \times B^m \to B$ be Boolean functions and $F = (F_1, \ldots, F_n)$. The formula*

$$X = F(T, Y),$$

or in scalar form

$$x_i = F_i(t_1, \ldots, t_n, Y), \quad (i = 1, \ldots, n)$$

expresses a general solution of the Boolean equation $f(X, Y) = 0$ by X if and only if, for every $X \in B^n$ and every $Y \in B^m$,

$$f(X, Y) = 0 \Leftrightarrow (\exists S \in B^n) f(S, Y) = 0 \wedge (\exists T \in B^n) X = F(T, Y).$$

In accordance with Theorem 3.1. the previous formula can be written as

$$f(X, Y) = 0 \Leftrightarrow \prod_A f(A, Y) = 0 \wedge (\exists T \in B^n) X = F(T, Y).$$

Lemma 3.4. *(Lemma 2.2 in [5]). Suppose that the equation*

$$ax \cup bx' = 0$$

has a solution ($ab = 0$). Then

$$ax \cup bx' = 0 \Leftrightarrow (\exists t)(x = a't \cup bt') \tag{3}$$

$$ax \cup bx' = 0 \Leftrightarrow b \leq x \leq a' \tag{4}$$

for all $x \in B$.

Theorem 3.5. *(Theorem 3. in [1]) Let $f : B^n \to B$ be a Boolean function. If $f(X) = 0$ is consistent then , for every $X \in B^n$*

$$f(X) = 0 \Leftrightarrow (\exists T) X = \bigcup_{i=0}^{k} (f'(A_i)A_i \cup f(A_i)f'(A_{i_1})A_{i_1} \cup f(A_i)f(A_{i_1})f'(A_{i_2})A_{i_2}$$

$$\cup \cdots \cup f(A_i)f(A_{i_1})f(A_{i_2}) \cdots f(A_{i_{k-1}})f'(A_{i_k})A_{i_k})T^{A_i}$$

where, for every $i \in \{0, 1, \ldots, k\}$, $A_i, A_{i_1}, \ldots, A_{i_k}$ is a permutation of $\{0, 1\}^n$.

4. Systems of Boolean inequations

We shall use the following obvious equivalence

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge f(X) = p). \tag{5}$$

Let $f : B^n \to B$ be a Boolean function. The relation

$$f(X) \neq 0$$

is called a Boolean inequation. To solve a Boolean inequation $f(X) \neq 0$ means to determine all $X \in B^n$ such that $f(X) \neq 0$ holds.

Theorem 4.1. *(Remark 10.5 in [5]) Let $f : B^n \to B$ be a Boolean function. The inequation $f(X) \neq 0$ has a solution if and only if $\bigcup_A f(A) \neq 0$.*

Theorem 4.2. *(Theorem 5 in [2]) Let $f : B^n \to B$ be a Boolean function. Then*

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge \bigcup_A ((f(A) + p)X^A) = 0).$$

Theorem 4.3. *(Theorem 6 in [2]) Let $f : B^n \to B$ be a Boolean function. Suppose that the inequation $f(X) \neq 0$ has a solution. Let $X = \Phi(T, p)$ expresses the general solution of the equation*

$$\bigcup_A ((f(A) + p)X^A) = 0.$$

Then, for every $X \in B^n$,

$$f(X) \neq 0 \Leftrightarrow (\exists p)(\exists T)(p \neq 0 \wedge \prod_A f(A) \leq p \leq \bigcup_A f(A) \wedge X = \Phi(T, p)).$$

Lemma 4.4. *(Lemma 4 in [7]) Let $f_1, \dots, f_k : B^n \to B$ be Boolean functions. Then the equation*

$$\prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0 \tag{6}$$

in $p_1, \dots, p_k$ has a solution.

Theorem 4.5. *(Theorem 11 in [7]) Let $f_1, \dots, f_k : B^n \to B$ be Boolean function. Then*

$$f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0 \Leftrightarrow$$
$$(\exists p_1) \cdots (\exists p_k)(\exists T)(p_1 \neq 0 \wedge \cdots \wedge p_k \neq 0 \wedge X = \Phi(p_1, \dots, p_k, T)$$
$$\wedge \ \textstyle\prod_A f_1(A) \leq p_1 \leq \bigcup_A f_1(A)$$
$$\wedge \ p_1 \textstyle\prod_A (f_1'(A) \cup f_2(A)) \cup p_1' \prod_A (f_1(A) \cup f_2(A))$$
$$\leq p_2 \leq p_1 \textstyle\bigcup_A (f_1(A) f_2(A)) \cup p_1' \bigcup_A (f_1'(A) f_2(A))$$
$$\cdots$$
$$\textstyle\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$
$$\leq p_k \leq \textstyle\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup_A (f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A))),$$

where $X = \Phi(p_1, \dots, p_k, T)$ expresses the general solution of the equation

$$(f_1(X) + p_1) \cup \ldots \cup (f_k(X) + p_k) = 0.$$

5. Systems of k Boolean inequations and a Boolean equation

In this section we shall consider the system

$$g(X)=0\ \wedge f_1(X)\neq 0\ \wedge\cdots\wedge\ f_k(X)\neq 0 \tag{2}$$

where $g, f_1, \ldots, f_k : B^n \to B$ are Boolean functions. When k=1 then

$$g(X)=0\ \wedge\ f(X)\neq 0. \tag{7}$$

Schröder give the condition of the consistency of the system (7). This condition can be found in [6].

Theorem 5.1. *(Proposition 10.1. in [6]) System (7) has solution if and only if*

$$\prod_A g(A)=0\ \wedge\ \bigcup_A f(A)g'(A)\neq 0. \tag{8}$$

Banković describe all solutions of the system (7) when the system is consistent.

Theorem 5.2. *(Theorem 9 in [3]) Let $g, f : B^n \to B$ be Boolean functions. Suppose that the system*

$$g(X)=0\ \wedge\ f(X)\neq 0$$

has solution i.e.

$$\prod_A g(A)=0\ \wedge\ \bigcup_A f(A)g'(A)\neq 0.$$

Let $X=\Phi(T,p)$ expresses the general solution of the equation

$$(f(X)+p)\cup g(X)=0.$$

Then for every $X\in B^n$,

$$g(X)=0\ \wedge\ f(X)\neq 0 \Leftrightarrow$$
$$(\exists p)(\exists T)(p\neq 0\wedge \prod_A (f(A)\cup g(A))\leq p\leq \bigcup_A f(A)g'(A)\wedge X=\Phi(T,p)).$$

Marriott and Odersky determinated satisfiability of system (2) in [8] . They applied this sistem for query optimization in databases [9]. These results are presented in [6].

Theorem 5.3. *(Proposition 5.5 in [6]) Suppose $g, f_1 \ldots f_k$ are single Boolean fuctions and $card(B)\geq 2^{k-1}$. Then the following conditions are equivalent:*

1. *$g(X)=0\ \wedge f_1(X)\neq 0\ \wedge\cdots\wedge\ f_k(X)\neq 0$ is satisfiable;*
2. *each atomic GSBE $g(X)=0\ \wedge f_i(X)\neq 0$ $(i=1\ldots k)$ is satisfiable;*
3. *each negated Boolean equation $f_i(X)\not\leq g(X)$ $(i=1\ldots k)$ is satisfiable;*
4. *$\bigvee_A g'(A)f_i(A)\neq 0$ $(i=1\ldots k)$.*

Lemma 5.4. *Let $g, f_1, \ldots, f_k : B^n \to B$ be Boolean functions. Then*

$$g(X)=0\ \wedge f_1(X)\neq 0\ \wedge\cdots\wedge\ f_k(X)\neq 0 \Leftrightarrow \tag{9}$$
$$(\exists p_1)\cdots(\exists p_k)(p_1\neq 0\wedge\cdots\wedge p_k\neq 0\wedge(g(X)\cup(f_1(X)+p_1)\cup\cdots\cup(f_k(X)+p_k))=0).$$

Proof. Using (5) and formula $(\exists X)A(x)\wedge B \Leftrightarrow (\exists x)(A(x)\wedge B)$ (x is not free in B) we get

$$\begin{aligned}
&g(X)=0\ \wedge f_1(X)\neq 0\ \wedge\ \cdots\wedge\ f_k(X)\neq 0\\
\Leftrightarrow\ & g(X)=0\ \wedge(\exists p_1)(p_1\neq 0\wedge f_1(X)=p_1)\wedge\cdots\wedge(\exists p_k)(p_k\neq 0\wedge f_k(X)=p_k)\\
\Leftrightarrow\ & (\exists p_1)\cdots(\exists p_k)(g(X)=0\ \wedge p_1\neq 0\wedge f_1(X)=p_1\wedge\cdots\wedge p_k\neq 0\wedge f_k(X)=p_k)\\
\Leftrightarrow\ & (\exists p_1)\cdots(\exists p_k)(p_1\neq 0\wedge\cdots\wedge p_k\neq 0\wedge g(X)=0\ \wedge f_1(X)+p_1=0\wedge\cdots\wedge f_k(X)+p_k=0)\\
\Leftrightarrow\ & (\exists p_1)\cdots(\exists p_k)(p_1\neq 0\wedge\cdots\wedge p_k\neq 0\wedge g(X)\cup(f_1(X)+p_1)\cup\cdots\cup(f_k(X)+p_k)=0\big).
\end{aligned}$$

□

Lemma 5.5. *Let $g, f_1, \dots, f_k : B^n \to B$ be Boolean functions and $p_1, \dots, p_k \in B$. Then*

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = \bigcup_{(c_1,\dots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A (g(A) \cup {f_1}^{c_1'}(A) \cup \cdots \cup {f_k}^{c_k'}(A)). \tag{10}$$

Proof. Let F be the Boolean function defined by

$$F(p_1, \dots, p_k) = \prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)).$$

Then, by Theorem 1.3,

$$F(p_1, \dots, p_k) = \bigcup_{(c_1,\dots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} F(c_1, \dots, c_k)$$

$$= \bigcup_{(c_1,\dots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A (g(A) \cup (f_1(A) + c_1) \cup \cdots \cup (f_k(A) + c_k)).$$

Since $f_i(A) + c_i = f_i(A) = {f_i}^{c_i'}(A)$ for $c_i = 0$ and $f_i(A) + c_i = f_i'(A) = {f_i}^{c_i'}(A)$ for $c_i = 1$, for every $i \in (1, \dots, k)$, it follows that

$$F(p_1, \dots, p_k) = \bigcup_{(c_1,\dots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A (g(A) \cup {f_1}^{c_1'}(A) \cup \cdots \cup {f_k}^{c_k'}(A)).$$

□

Let $C_i = (c_1, \dots, c_i)$.

Lemma 5.6. *Let $g, f_1, \dots, f_k : B^n \to B$ be Boolean functions. Then the equation*

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0 \tag{11}$$

in $p_1, \dots, p_k$ has a solution if $g(X) = 0$ has a solution.

Proof. The equation (11) has a solution if and only if

$$\prod_{C_k\in\{0,1\}^k} \prod_A (g(A) \cup (f_1(A) + c_1) \cup \cdots \cup (f_k(A) + c_k)) = 0 \tag{12}$$

by Theorem 3.1. The equality (12) can be written as

$$\prod_{C_k\in\{0,1\}^k} \prod_A (g(A)) \cup \prod_{C_k\in\{0,1\}^k} \prod_A ((f_1(A) + c_1) \cup \cdots \cup (f_k(A) + c_k)) = 0.$$

Since equation (6) has solution, by Lemma 4.4, it follows that

$$\prod_{C_k\in\{0,1\}^k} \prod_A ((f_1(A) + c_1) \cup \cdots \cup (f_k(A) + c_k)) = 0$$

by Theorem 3.1. If the equation $g(X) = 0$ has a solution then $\prod_A (g(A)) = 0$. Thus

$$\prod_{C_k\in\{0,1\}^k} \prod_A (g(A)) = 0.$$

Therefore (12) holds i.e. the equation (11) has a solution. □

Lemma 5.7. *Let $g, f_1, \ldots, f_k : B^n \to B$ be Boolean functions and $p_1, \ldots, p_k \in B$. Then*

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0 \Leftrightarrow$$

$$\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$

$$\leq p_k \leq \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup_A (g'(A) f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A))$$

$$\wedge \prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})) = 0.$$

Proof. Using (10) we have

$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k))$

$= \bigcup_{C_k \in \{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A))$

$= p_k (\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k'(A)))$

$\cup p_k' (\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A)))$.

Let us introduce the following notation

$a = \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k'(A))$

$b = \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$.

Applying Theorem 1.5 we get

$$ab = \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} (\prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k'(A))) (\prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))).$$

Using the equality $(x \cup y)(x \cup y') = x$, we get

$$(g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))(g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k'(A)) = g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A).$$

Thus

$$ab = \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} (\prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A))).$$

In accordance with Lemma 5.5 we have

$$ab = \prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})).$$

The equation $ap_k \cup bp_k' = 0$ has a solution if and only if $ab = 0$, by Lemma 3.4. The equality $ab = 0$ can be written as

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \ldots \cup (f_{k-1}(A) + p_{k-1})) = 0.$$

This equation has a solution, by Lemma 5.6 if $g(X) = 0$ has a solution. In accordance with Lemma 3.4, the equation $ap_k \cup bp_k' = 0$ is equivalent to $b \leq p_k \leq a'$, i.e.

$$\bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$

$$\leq p_k \leq \bigcup_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup_A (g'(A) f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A)).$$

□

Theorem 5.8. *Let $g, f_1, \ldots, f_k : B^n \to B$ be Boolean function. Then*

$$g(X) = 0 \wedge f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0 \Leftrightarrow$$
$$(\exists p_1)\cdots(\exists p_k)(\exists T)(p_1 \neq 0 \wedge \cdots \wedge p_k \neq 0 \wedge X = \Phi(p_1, \ldots, p_k, T)$$
$$\wedge\ \prod_A (g(A) \cup f_1(A)) \leq p_1 \leq \bigcup_A (g'(A) f_1(A))$$
$$\wedge\ p_1 \prod_A (g(A) \cup f_1'(A) \cup f_2(A)) \cup p_1' \prod_A (g(A) \cup f_1(A) \cup f_2(A))$$
$$\leq p_2 \leq p_1 \bigcup_A (g'(A) f_1(A) f_2(A)) \cup p_1' \bigcup_A (g'(A) f_1'(A) f_2(A))$$
$$\cdots$$
$$\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$
$$\leq p_k \leq \bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup_A (g'(A) f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A)),$$

where $X = \Phi(p_1, \ldots, p_k, T)$ expresses the general solution of the equation

$$g(X) \cup (f_1(X) + p_1) \cup \ldots \cup (f_k(X) + p_k) = 0.$$

Proof. By Lemma 5.4 equivalence (9) holds. Let $X = \Phi(p_1, \ldots, p_k, T)$ be a general solution of the equation

$$g(X) \cup (f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k) = 0. \tag{13}$$

Then, by Definition 3.3,

$$g(X) \cup (f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k) = 0 \Leftrightarrow \tag{14}$$
$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0\ \wedge\ (\exists T) X = \Phi(p_1, \ldots, p_k, T).$$

The condition

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0$$

is an equation in $p_1, \ldots p_k$, which has a solution, by Lemma 5.6. According to Lemma 5.7, this equation is equivalent to

$$\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$
$$\leq p_k \leq \bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup_A (g'(A) f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A))$$
$$\wedge\ \prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})) = 0.$$

Similarly, according to Lemma 5.7 it follows that

$$\prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})) = 0 \Leftrightarrow$$
$$\bigcup_{C_{k-2}\in\{0,1\}^{k-2}} {p_1}^{c_1} \cdots {p_{k-2}}^{c_{k-2}} \prod_A (g(A) \cup f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-2}'}(A) \cup f_{k-1}(A))$$
$$\leq p_{k-1} \leq \bigcup_{C_{k-2}\in\{0,1\}^{k-2}} {p_1}^{c_1} \cdots p_{k-2}^{c_{k-2}} \bigcup_A (g'(A) f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-2}}(A) f_{k-1}(A))$$
$$\wedge\ \prod_A (g(A) \cup (f_1(A) + p_1) \cup \cdots \cup (f_{k-2}(A) + p_{k-2})) = 0.$$

Applying Lemma 5.7 k times we get $\prod_A (g(A) \cup (f_1(A) + p_1)) = 0$, which can be written as

$$\prod_A (g(A) \cup f_1'(A)) p_1 \cup \prod_A (g(A) \cup f_1(A)) p_1' = 0. \tag{15}$$

This equation in p_1 has a solution if and only if $\prod_A (g(A) \cup f_1'(A)) \prod_A (g(A) \cup f_1(A)) = 0$, by Theorem 3.1. Since $g(X) = 0$ is consistent we have $\prod_A (g(A) \cup f_1'(A)) \prod_A (g(A) \cup f_1(A)) = \prod_A (g(A) = 0$. Thus the equation (15) is consistent and its solutions are

$$\prod_A (g(A) \cup f_1(A)) \leq p_1 \leq \bigcup_A (g'(A) f_1(A))$$

by Lemma 3.4.
From (9), (14) and the previous conditions for $p_1, \dots, p_k$ we get Theorem 5.8. □

Let $m = 2^n - 1$ and $h(X, p_1, \dots, p_k) = g(X) \cup (f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k)$. According to Theorem 3.5 the general solution of the equation (13) can be obtained as follows:

(16)
$$\Phi(p_1, \dots, p_k, T)) = \bigcup_{i=0}^{m} (h(A_i, p_1, \dots, p_k)' A_i \cup$$
$$h(A_i, p_1, \dots, p_k) h(A_{i_1}, p_1, \dots, p_k)' A_{i_1} \cup \cdots \cup$$
$$h(A_i, p_1, \dots, p_k) h(A_{i_1}, p_1, \dots, p_k) \cdots h(A_{i_m}, p_1, \dots, p_k)' A_{i_m}) T^{A_i}$$

where for every $i \in \{0, \dots, m\}$, $(A_i, A_{i_1}, \dots, A_{i_m})$ is a permutation of $\{0, 1\}^n$.

Example 1. Let $a, b, c, d, e, f \in$ B.Solve the system

$$ax \cup bx' = 0 \wedge cx \cup dx' \neq 0 \wedge ex \cup fx' \neq 0.$$

Using Theorem 5.8 and (16) for $n = 1$ we get

$$ax \cup bx' = 0 \wedge cx \cup dx' \neq 0 \wedge ex \cup fx' \neq 0 \Leftrightarrow$$
$$(\exists p)(\exists q)(\exists t)(p \neq 0 \wedge q \neq 0 \wedge (a \cup c)(b \cup d) \leq p \leq a'c \cup b'd \wedge$$
$$p((a \cup c' \cup e)(b \cup d' \cup f)) \cup p'((a \cup c \cup e)(b \cup d \cup f))) \leq q \leq$$
$$p((a'ce) \cup (b'df)) \cup p'((a'c'e) \cup (b'd'f)) \wedge$$
$$x = (a \cup (c + p) \cup (e + q))' t \cup (b \cup (d + p) \cup (f + q)) t).$$

Example 2. Let $B = \{0, 1, m, l, k, m', l', k'\}$. Solve the system

$$m'x' = 0 \wedge m'x \neq 0 \wedge kx \cup lx' \neq 0.$$

Using Example 1, where $a = 0,\ b = m',\ c = m',\ d = 0,\ e = k,\ f = l$, we get

$$m'x' = 0 \wedge m'x \neq 0 \wedge kx \cup lx' \neq 0 \Leftrightarrow$$
$$(\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge p = m' \wedge q = k \wedge$$
$$x = ((m' + p) \cup (k + q))' t \cup (m' \cup p \cup (l + q)) t).$$

Thus $x = t \cup m't'$. Taking $t \in \{0, 1, m, l, k, m', l', k'\}$ we get $x \in \{m', 1\}$.

Example 3. Let $B = \{0, 1, m, l, k, m', l', k'\}$. Solve the system

$$mx \cup lx' = 0 \wedge kx \cup lx' \neq 0 \wedge m'x' \neq 0.$$

Using Example 1, where $a = m,\ b = l,\ c = k,\ d = l,\ e = 0,\ f = m'$, we get

$$mx \cup lx' = 0 \wedge kx \cup lx' \neq 0 \wedge m'x' \neq 0 \Leftrightarrow$$
$$(\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge p = k \wedge q = 0 \wedge$$
$$x = (m \cup (k + p) \cup q)' t \cup (l \cup (l + p) \cup (m' + q)) t).$$

We get a contradiction $g \neq 0 \wedge q = 0$ and hence the system has no solution.

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