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Spectral Inclusion by the Quadratic Numerical Range of 2×2 Operator Matrices with Unbounded Entries

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Abstract. This paper deals with the spectral inclusion properties of 2×2 operator matrices with unbounded entries in Hilbert space. The conditions for spectral inclusion by the quadratic numerical range are described. In addition, some examples are given to illustrate the main results.

1. Introduction

Operator matrices arise in various areas of mathematics and mathematical physics such as elastic mechanics and quantum mechanics. As a result, their spectral properties play an important role in reflecting the time evolution and hence the stability of the underlying physical systems (see [1, 2]). Thus, many authors are attracted to focus on the spectral properties of operator matrices (see [3–5] and the references there in).

As is known, for an operator T in Hilbert space, the spectrum $\sigma(T)$ can be located by its numerical range $W(T)$. However, the numerical range can not provide an accurate description for the spectrum of an operator T, if $\sigma(T)$ consists of two separate parts. Therefore, the quadratic numerical range $W^2(M)$, which gives a preferable characterization of the spectrum than the numerical range, was introduced in [5] for operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Note that H. Langer and C. Tretter proved in [5] that $\sigma_p(M) \subset W^2(M)$ and $\sigma_{app}(M) \subset \overline{W^2(M)}$ hold while diagonal entries A, D are densely defined closed and off-diagonal elements B, C are bounded. After that H. Langer, A. Markus, V. Matsaev and C. Tretter verified in [6] that $W^2(M) \subset W(M)$ and $\sigma(M) \subset \overline{W^2(M)}$ are true for bounded operator matrix M, and found that $W^2(M)$ may not be convex. However, the spectral inclusion $\sigma(M) \subset \overline{W^2(M)}$ do not hold naturally for operator matrices with unbounded entries. As follows, we provide a simple illustrating example.

Example 1.1. Let $\mathcal{H} = L^2[0, \infty)$, and let $A = D = i\dfrac{d}{dx} : \mathcal{D}(A) = \mathcal{D}(D) = \{u \in \mathcal{H} : u$ is absolutely continuous, $u' \in \mathcal{H}, u(0) = 0\}$. Consider the operator matrix $M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, and it is easy to see $W^2(M) \subset \mathbb{R}$ since A and D

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are symmetric operators. On the other hand, through calculations we have $\sigma_r(M) = \sigma_r(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda < 0\}$, and hence $\sigma(M) \not\subset \overline{W^2(M)}$.

Thus, assuming all entries A, B, C and D are unbounded, C. Tretter investigated the spectral inclusion of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in [7], and proved the following conclusions:

(i) $W^2(M) \subset W(M)$ and $\sigma_p(M) \subset W^2(M)$ are still true.

(ii) If one of the following statements is satisfied:

(a) M is diagonally dominant of order 0;

(b) M is off-diagonally dominant of order 0 and B, C are boundedly invertible.

Then $\sigma_{app}(M) \subset \overline{W^2(M)}$.

(iii) If one of the conditions (ii)(a) and (ii)(b) is fulfilled, and every component of $\mathbb{C} \setminus \overline{W^2(M)}$ contains a point $\mu \in \rho(M)$, then $\sigma(M) \subset \overline{W^2(M)}$.

Based on the above works, Y. Qi, J. Huang and A. Chen studied in [8] the spectral inclusion properties of Hamiltonian operator matrix $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$, and verified that $\sigma(H) \subset \overline{W^2(H)}$ holds (the general 2×2 case) under either of the following assumptions:

(i) H is diagonally dominant of order 0, and $\mathcal{D}(A) = \mathcal{D}(A^*)$;

(ii) H is off-diagonally dominant of order 0, $\mathcal{D}(B) = \mathcal{D}(C)$ and $0 \notin \sigma_p(B) \cap \sigma_p(C)$.

In this paper, we will weaken the tight assumption in [7, 8] that the order of dominance is 0, and drop the rigorous condition that imposed on the domain of dominant operators in [8]. It is one of our main techniques to discuss the spectral inclusion properties on the core of dominant operators.

2. Preliminaries

Throughout this paper, $\mathcal{H}$ is always a complex Hilbert space.

Definition 2.1. (See [3, P.92]) Let T be a closable linear operator in $\mathcal{H}$, then the *resolvent set* and the *spectrum* of T are defined as

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is an injection}, (T - \lambda)^{-1} \text{ is bounded}\}, \quad \sigma(T) = \mathbb{C} \setminus \rho(T),$$

respectively, and the *point spectrum* $\sigma_p(T)$, *residual spectrum* $\sigma_r(T)$ and *continuous spectrum* $\sigma_c(T)$ as

$$\begin{aligned} \sigma_p(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not an injection}\}, \\ \sigma_r(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is injective}, \overline{\mathcal{R}(T - \lambda)} \neq \mathcal{H}\}, \\ \sigma_c(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is injective}, \overline{\mathcal{R}(T - \lambda)} = \mathcal{H}, \mathcal{R}(T - \lambda) \neq \mathcal{H}\}. \end{aligned}$$

In addition, if T is a closed operator, then by the closed graph theorem, we have

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is a bijection}\},$$

and hence

$$\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T).$$

Beside, the set

$$\sigma_{app}(T) = \{\lambda \in \mathbb{C} : \exists (v_n)_{n=1}^{+\infty} \subset \mathcal{D}(T), \|v_n\| = 1, (T - \lambda)v_n \to 0, n \to \infty\}$$

is called the *approximate point spectrum* of T.

Definition 2.2. (See [3, P.92–100]) Let T and S be operators with the same domain Banach space $\mathcal{B}$ such that

$$\|Sv\| \le a_S\|v\| + b_S\|Tv\|, v \in \mathcal{D}(T), \tag{1}$$

where $a_S, b_S \ge 0$. Then we say that S is *relatively bounded with respect to T* (or *T-bounded*). The infimum δ_S of all b_S so that (1) holds for some a_S is called *relative bound of S with respect to T* (or *T-bound of S*).

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces. The operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the product space $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ is called

(i) *diagonally dominant of order δ*, if C is A-bounded with A-bound δ_C, B is D-bounded with D-bound δ_B, and $\delta = \max\{\delta_B, \delta_C\}$,

(ii) *off-diagonally dominant of order δ*, if A is C-bounded with C-bound δ_A, and D is B-bounded with B-bound δ_D, and $\delta = \max\{\delta_A, \delta_D\}$.

Definition 2.3. (See [6, P.91]) Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block operator matrix with $\mathcal{D}(M) = \mathcal{D}_1 \oplus \mathcal{D}_2 := (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus (\mathcal{D}(B) \cap \mathcal{D}(D))$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$. For $f \in \mathcal{D}_1$, $g \in \mathcal{D}_2$ with $\|f\| = \|g\| = 1$, define the 2×2 complex matrix

$$M_{f,g} = \begin{pmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{pmatrix}.$$

Then the set

$$W^2(M) = \bigcup_{\substack{f \in \mathcal{D}_1, g \in \mathcal{D}_2, \\ \|f\| = \|g\| = 1}} \sigma_p(M_{f,g})$$

is called the *quadratic numerical range* of M (with respect to $\mathcal{H}_1 \oplus \mathcal{H}_2$).

Definition 2.4. (See [9, P.166]) Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a closed operator. If $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}$ is a closable operator and $\overline{S} = T$, then $\mathcal{D}(S)$ is called the *core* of T.

Proposition 2.5. (See [10, P.88]) Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a closable linear operator, then

$$\mathcal{D}(\overline{T}) = \left\{ v \in \mathcal{H} : \begin{array}{l} \exists (v_n) \subset \mathcal{D}(T) \text{ such that } v_n \to v \\ \text{and for which } (Tv_n) \text{ is also convergent} \end{array} \right\},$$

$$\overline{T}v = \lim_{n\to\infty} Tv_n, \quad v \in \mathcal{D}(\overline{T}).$$

Lemma 2.6. (See [11, P.1132]) If T is a densely defined closed linear operator in $\mathcal{H}$, then $\sigma(T) \subset \overline{W(T)}$ holds if and only if $\sigma_{r,1}(T) \subset \overline{W(T)}$, where

$$\sigma_{r,1}(T) = \{\lambda \in \sigma_r(T) : \mathcal{R}(T - \lambda) \text{ is closed}\}.$$

Lemma 2.7. (See [9, P.190]) Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces, let T, S be linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$, and let S be T-bounded with T-bounded < 1. Then $T + S$ is closable if and only if so is T, and $\mathcal{D}(\overline{T + S}) = \mathcal{D}(\overline{T})$. In particular, $T + S$ is closed if and only if so is T.

Lemma 2.8. (See [12, P.524]) Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces, and let $T : \mathcal{D}(T) \subset \mathcal{B}_1 \to \mathcal{B}_2$ be densely defined closed operator. Suppose S is a T-bounded operator such that S^* is T^*-bounded with both relative bounds smaller than one. Then $T + S$ is closed and $(T + S)^* = T^* + S^*$.

Lemma 2.9. (See [4, P.919–920]) Let $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : \mathcal{D}(H) \subset \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ be a symplectic self-adjoint Hamiltonian operator matrix(i.e., $(JH)^* = JH$ holds with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$). Then $\sigma(H)$, $\sigma_p(H) \cup \sigma_r(H)$ and $\sigma_c(H)$ are symmetric with respect to the imaginary axis, respectively.

3. Main results

In this section, the spectral inclusion of the densely defined operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{D}(M) \subset \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

is discussed. Here, if no other statement, A, B, C and D are densely defined closable operators in $\mathcal{H}$.

First, we consider the upper-triangular operator matrix $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$.

Theorem 3.1. Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \to \mathcal{H} \oplus \mathcal{H}$ be a densely defined upper-triangular operator matrix. If A and D are closed operators, $\sigma_{r,1}(A) \subset \overline{W(A)}$ and $\sigma_{r,1}(D) \subset \overline{W(D)}$ are satisfied, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

Proof. From the Definition 2.3, we see that

$$\overline{W^2(M)} = \overline{W(A)} \cup \overline{W(D)}.$$

On the other hand, it is easy to proof $\sigma(M) \subset (\sigma(A) \cup \sigma(D))$. By Lemma 2.6, it follows from $\sigma_{r,1}(A) \subset \overline{W(A)}$(resp. $\sigma_{r,1}(D) \subset \overline{W(D)}$) that $\sigma(A) \subset \overline{W(A)}$(resp. $\sigma(D) \subset \overline{W(D)}$), and hence

$$\sigma(M) \subset (\sigma(A) \cup \sigma(D)) \subset \overline{W(A)} \cup \overline{W(D)} = \overline{W^2(M)}.$$

□

Corollary 3.2. Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(D) \to \mathcal{H} \oplus \mathcal{H}$ be a densely defined upper-triangular operator matrix. If A and D are both self-adjoint operators, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

Proof. Since A and D are self-adjoint operators, we have $\sigma_r(A) = \sigma_r(D) = \emptyset$. Thus $\sigma(M) \subset \overline{W^2(M)}$ by Theorem 3.1. □

Next, we discuss the spectral inclusion of general operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Theorem 3.3. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a densely defined operator matrix in $\mathcal{H} \oplus \mathcal{H}$. Then

(I) Assume that M is diagonally dominant with order $\delta < 1$, A and D are closed operators. If $\mathcal{D}(A) \cap \mathcal{D}(D)$ is a core of A and D, and every component of $\mathbb{C} \setminus \overline{W^2(M)}$ contains a point $\mu \in \rho(M)$, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

(II) Assume that M is off-diagonally dominant with order $\delta < 1$, B and C are closed operators. If $\mathcal{D}(B) \cap \mathcal{D}(C)$ is a core of B and C, and every component of $\mathbb{C} \setminus \overline{W^2(M)}$ contains a point $\mu \in \rho(M)$, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

Proof. According to [3, Theorem 1.3.1], $\sigma(M) \subset \overline{W^2(M)}$ holds immediately in finite dimensional Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Hence, we only need to discuss in the infinite dimensional space. Here, we prove (I), the proof of (II) is analogous.

Since A, D are closed operators and M is diagonally dominant with order $\delta < 1$, we decompose M as $M = T + S$, where

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then, T is closed and S is T bounded with T-bound < 1, clearly. Hence, M is closed by Lemma 2.7.

We claim that

$$\sigma_{app}(M) \subset \overline{W^2(M)}. \tag{2}$$

To this end, let $\lambda_0 \in \sigma_{app}(M)$, then there exists $(v_n)_{n=1}^{\infty} = ((f_n \ \ g_n)^t)_{n=1}^{\infty} \subset \mathcal{D}(M)$ with $\|f_n\|^2 + \|g_n\|^2 = 1$ such that

$$(M - \lambda_0)v_n \to 0 \ \ (n \to \infty). \tag{3}$$

Write $\mathcal{D}_0 = \mathcal{D}(A) \cap \mathcal{D}(D)$, and then $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$ is a core of M clearly. Since $\mathcal{D}$ is a core of M, according to Proposition 2.5, for each $v_n = (f_n \ \ g_n)^t \in \mathcal{D}(M)$, there exists $(v_n^{(k)})_{k=1}^{\infty} = ((f_n^{(k)} \ \ g_n^{(k)})^t)_{k=1}^{\infty} \subset \mathcal{D}$ with $\|f_n^{(k)}\|^2 + \|g_n^{(k)}\|^2 = 1$ such that

$$v_n^{(k)} \to v_n \ \ (k \to \infty), \qquad M v_n^{(k)} \to M v_n \ \ (k \to \infty).$$

In each sequence $(v_n^{(k)})_{k=1}^{\infty} (n \in \mathbb{N})$, we choose an element $v_n^{(n)} = (f_n^{(n)} \ \ g_n^{(n)})^t \in (v_n^{(k)})_{k=1}^{\infty}$, then get a sequence $(v_n^{(n)})_{n=1}^{\infty} = ((f_n^{(n)} \ \ g_n^{(n)})^t)_{n=1}^{\infty}$ with $\|f_n^{(n)}\|^2 + \|g_n^{(n)}\|^2 = 1$, and it follows from (3) that

$$(M - \lambda_0)v_n^{(n)} \to 0 \ \ (n \to \infty), \tag{4}$$

i.e.,

$$\begin{array}{l} (A - \lambda_0)f_n^{(n)} + Bg_n^{(n)} := h_n^{(n)} \to 0, \\ Cf_n^{(n)} + (D - \lambda_0)g_n^{(n)} := k_n^{(n)} \to 0, \end{array} \quad (n \to \infty). \tag{5}$$

As follows, we discuss in three cases.

Case 1: $\liminf\limits_{n\to\infty} \|f_n^{(n)}\| > 0$ and $\liminf\limits_{n\to\infty} \|g_n^{(n)}\| > 0$. Without loss of generality, we may assume $\|f_n^{(n)}\| > 0, \|g_n^{(n)}\| > 0$ $(n \in \mathbb{N})$. It follows from (5) that

$$\frac{(Af_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} - \lambda_0 + \frac{(Bg_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} = \frac{(h_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})}, \tag{6}$$

$$\frac{(Cf_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} + \frac{(Dg_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} - \lambda_0 = \frac{(k_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})}. \tag{7}$$

Let

$$d_n(\lambda) = \det \begin{pmatrix} \dfrac{(Af_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} - \lambda & \dfrac{(Bg_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} \\ \dfrac{(Cf_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} & \dfrac{(Dg_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} - \lambda \end{pmatrix} = \det \begin{pmatrix} \dfrac{(Af_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} - \lambda & \dfrac{(Bg_n^{(n)}, f_n^{(n)})}{\|f_n^{(n)}\|\|g_n^{(n)}\|} \\ \dfrac{(Cf_n^{(n)}, g_n^{(n)})}{\|f_n^{(n)}\|\|g_n^{(n)}\|} & \dfrac{(Dg_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} - \lambda \end{pmatrix}.$$

As $d_n(\lambda)$ is a monic quadratic polynomial, we can write

$$d_n(\lambda) = (\lambda - \lambda_n^1)(\lambda - \lambda_n^2),$$

where λ_n^1, λ_n^2 are the solutions of the quadratic equation $d_n(\lambda) = 0$, and hence $\lambda_n^1, \lambda_n^2 \in W^2(M|_{\mathcal{D}})$. Substitute (6) and (7) into the first column of $d_n(\lambda_0)$, then

$$d_n(\lambda_0) = \det\begin{pmatrix} \dfrac{(h_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} & \dfrac{(Bg_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} \\ \dfrac{(k_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} & \dfrac{(Dg_n^{(n)}, g_n^{(n)})}{(g_n^{(n)}, g_n^{(n)})} - \lambda_0 \end{pmatrix}.$$

Since M is diagonally dominant of order $\delta < 1$, the operator $S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ is M-bounded. From (4), it follows that $(Sv_n^{(n)})_{n=1}^\infty$ and hence $(Bg_n^{(n)})_{n=1}^\infty$, $(Cf_n^{(n)})_{n=1}^\infty$ are bounded, which together with (5) implies $((D-\lambda_0)g_n^{(n)})_{n=1}^\infty$ is bounded as well. Hence, the facts $h_n^{(n)} \to 0 \ (n \to \infty)$ and $k_n^{(n)} \to 0 \ (n \to \infty)$ imply $d_n(\lambda_0) \to 0 \ (n \to \infty)$, and thus $\lambda_n^1 \to \lambda_0$ or $\lambda_n^2 \to \lambda_0 \ (n \to \infty)$. Therefore, $\lambda_0 \in \overline{W^2(M|_{\mathcal{D}})}$ and hence $\lambda_0 \in \overline{W^2(M)}$.

Case 2: $\liminf_{n\to\infty} \|g_n^{(n)}\| = 0$. It follows from $\|f_n^{(n)}\|^2 + \|g_n^{(n)}\|^2 = 1$ that $\liminf_{n\to\infty} \|f_n^{(n)}\| > 0$. Without loss of generality, we assume $\lim_{n\to\infty} g_n^{(n)} = 0, \|f_n^{(n)}\| > \gamma \ (n \in \mathbb{N})$ for some $\gamma \in (0,1]$. Since $Cf_n^{(n)} \in \mathcal{H} \ (n \in \mathbb{N})$ and $\dim \mathcal{H} = \infty$, there exists $z_n \in \mathcal{D}(D), z_n \neq 0 \ (n \in \mathbb{N})$ such that $(Cf_n^{(n)}, z_n) = 0 \ (n \in \mathbb{N})$. Let $\lambda_n = \dfrac{(Af_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})}$, then

$$\det\begin{pmatrix} \dfrac{(Af_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} - \lambda_n & \dfrac{(Bz_n, f_n^{(n)})}{\|z_n\|\|f_n^{(n)}\|} \\ \dfrac{(Cf_n^{(n)}, z_n)}{\|f_n^{(n)}\|\|z_n\|} & \dfrac{(Dz_n, z_n)}{(z_n, z_n)} - \lambda_n \end{pmatrix} = \det\begin{pmatrix} 0 & \dfrac{(Bz_n, f_n^{(n)})}{\|z_n\|\|f_n^{(n)}\|} \\ 0 & \dfrac{(Dz_n, z_n)}{(z_n, z_n)} - \lambda_n \end{pmatrix} = 0,$$

and hence $\lambda_n \in W^2(M)$. It follows from (6) that

$$\lambda_0 = \lambda_n + \frac{(Bg_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})} - \frac{(h_n^{(n)}, f_n^{(n)})}{(f_n^{(n)}, f_n^{(n)})}.$$

From $g_n^{(n)} \to 0 \ (n \to \infty)$ and $\mathcal{D}_0$ is a core of D, it follows that $Dg_n^{(n)} \to 0 \ (n \to \infty)$. As B is D-bounded, we have $Bg_n^{(n)} \to 0 \ (n \to \infty)$, which combing with $h_n^{(n)} \to 0 \ (n \to \infty)$ imply $\lambda_n \to \lambda_0 \ (n \to \infty)$, and thus $\lambda_0 \in \overline{W^2(M)}$.

Case 3: $\liminf_{n\to\infty} \|f_n\| = 0$. This case can be treated analogously, if we see that C is A-bounded and $\mathcal{D}_0$ is a core of A. Then, the proof of claim (2) is completed.

Besides, according to [9, Theorem V.3.2], for $\lambda \in \mathbb{C} \setminus \sigma_{app}(M)$, $\mathcal{R}(M-\lambda)$ is closed and the mapping $\lambda \to \dim \mathcal{R}(M-\lambda)^\perp$ is a constant on every component of $\mathbb{C} \setminus \sigma_{app}(M)$. Thus, by (2), the same is true on each component of $\mathbb{C} \setminus \overline{W^2(M)} \subset \mathbb{C} \setminus \sigma_{app}(M)$. It follows that $\mathcal{R}(M-\lambda) = \mathcal{H} \oplus \mathcal{H}$ holds for all $\lambda \in \mathbb{C} \setminus \overline{W^2(M)}$. Therefore, we obtain $\mathbb{C} \setminus \overline{W^2(M)} \subset \rho(M)$, and hence $\sigma(M) \subset \overline{W^2(M)}$. □

Theorem 3.4. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a densely defined operator matrix in $\mathcal{H} \oplus \mathcal{H}$. Assume that A is closed, B and C are both self-adjoint or both anti-self-adjoint operators(i.e., $B^* = -B$ and $C^* = -C$), and $D = \pm A^*$. Then

(I) If M is diagonally dominant with order $\delta < 1$, and $\mathcal{D}(A) \cap \mathcal{D}(D)$ is a core of A and D, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

(II) If M is off-diagonally dominant with order $\delta < 1$, and $\mathcal{D}(B) \cap \mathcal{D}(C)$ is a core of B and C, then

$$\sigma(M) \subset \overline{W^2(M)}.$$

Proof. We only need to prove (II), the proof of (I) is analogous.

Write $\mathcal{D}_0 = \mathcal{D}(B) \cap \mathcal{D}(C)$, and we claim that

$$\overline{W^2(M)} = \overline{W^2(M|_{\mathcal{D}})}, \tag{8}$$

where $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$. It suffices to show $W^2(M) \subset \overline{W^2(M|_{\mathcal{D}})}$. To this end, let $\lambda \in W^2(M)$. Then, there exists $(f \ \ g)^t \in \mathcal{D}(M) = \mathcal{D}(C) \oplus \mathcal{D}(B)$ with $\|f\| = \|g\| = 1$ such that

$$\det(M_{f,g} - \lambda) = \det\begin{pmatrix} (Af, f) - \lambda & (Bg, f) \\ (Cf, g) & (Dg, g) - \lambda \end{pmatrix} = 0,$$

and hence λ has one of the following expressions:

$$\lambda^{\pm} = \frac{1}{2}\left((Af, f) + (Dg, g) \pm \sqrt{((Af, f) - (Dg, g))^2 + 4(Bg, f)(Cf, g)}\right).$$

Since $\mathcal{D}_0$ is a core of B and C, there are two sequences $(\hat{f}_n)_{n=1}^{\infty}$ and $(\hat{g}_n)_{n=1}^{\infty}$ in $\mathcal{D}_0$ such that

$$\hat{f}_n \to f \ (n \to \infty), \ \ C\hat{f}_n \to Cf \ (n \to \infty),$$
$$\hat{g}_n \to g \ (n \to \infty), \ \ B\hat{g}_n \to Bg \ (n \to \infty).$$

Obviously, $\|\hat{f}_n\| \to 1 \ (n \to \infty)$, $\|\hat{g}_n\| \to 1 \ (n \to \infty)$. We may assume $\hat{f}_n \neq 0, \hat{g}_n \neq 0$, and let $f_n = \hat{f}_n/\|\hat{f}_n\|, g_n = \hat{g}_n/\|\hat{g}_n\|$, then $\|f_n\| = \|g_n\| = 1$ and

$$f_n \to f \ (n \to \infty), \ \ Cf_n \to Cf \ (n \to \infty),$$
$$g_n \to g \ (n \to \infty), \ \ Bg_n \to Bg \ (n \to \infty).$$

Since A is C-bounded and D is B-bounded, we see that $(Af_n)_{n=1}^{\infty}$ and $(Dg_n)_{n=1}^{\infty}$ are both Cauchy sequences. From the closedness of A and D, it follows that

$$Af_n \to Af \ (n \to \infty), \ \ Dg_n \to Dg \ (n \to \infty).$$

Hence

$$(Af_n, f_n) + (Dg_n, g_n) \to (Af, f) + (Dg, g) \ (n \to \infty), \quad \Delta_n \to \Delta \ (n \to \infty),$$

where

$$\Delta_n = ((Af_n, f_n) - (Dg_n, g_n))^2 + 4(Bg_n, f_n)(Cf_n, g_n),$$
$$\Delta = ((Af, f) - (Dg, g))^2 + 4(Bg, f)(Cf, g).$$

Thus

$$\lambda_n^{\pm} = \frac{1}{2}((Af_n, f_n) + (Dg_n, g_n) \pm \sqrt{\Delta_n}) \to \lambda^{\pm} \ (n \to \infty).$$

In other words, $\lambda \in \overline{W^2(M|_{\mathcal{D}})}$. This proves (8).

Next, we show that $D = \pm A^*$ implies

$$\lambda \in \overline{W^2(M)} \iff \pm\overline{\lambda} \in \overline{W^2(M)}. \tag{9}$$

To this end, we prove

$$\lambda \in W^2(M|_{\mathcal{D}}) \iff \pm\overline{\lambda} \in W^2(M|_{\mathcal{D}}). \tag{10}$$

In fact, since $D = \pm A^*$, for each $(f \;\; g)^t \in \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$ with $\|f\| = \|g\| = 1$, we have

$$\begin{aligned} &\det(M_{f,g} - \lambda) \\ &= \det\begin{pmatrix} (Af,f) - \lambda & (Bg,f) \\ (Cf,g) & (Dg,g) - \lambda \end{pmatrix} \\ &= \det\begin{pmatrix} (Af,f) - \lambda & (Bg,f) \\ (Cf,g) & (\pm A^* g,g) - \lambda \end{pmatrix} = \det\begin{pmatrix} (A^* g,g) \mp \lambda & (Cf,g) \\ (Bg,f) & (\pm Af,f) \mp \lambda \end{pmatrix}. \end{aligned} \tag{11}$$

On the other hand, since M is off-diagonally dominant with order $\delta < 1$, we decompose M as $M = T + S$, where

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then, T is S bounded and S-bound < 1, and thus it is easy to see that T^* is S^* bounded and S^*-bound < 1 since $D = \pm A^*$, B and C are both self-adjoint or anti-self-adjoint. By Lemma 2.8, we have

$$(T + S)^* = T^* + S^*,$$

and thus $D = A^*$(resp. $D = -A^*$) implies

$$M^* = JMJ \tag{12}$$

with $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$(resp. $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$). Hence, combing (11) and (12), we obtain

$$\lambda \in W^2(M|_{\mathcal{D}}) \iff \pm\lambda \in W^2(M^*|_{\mathcal{D}}).$$

Then it follows from

$$\begin{aligned} (M_{f,g})^* &= \begin{pmatrix} (Af,f) & (Bg,f) \\ (Cf,g) & (Dg,g) \end{pmatrix}^* \\ &= \begin{pmatrix} (Af,f) & (Bg,f) \\ (Cf,g) & (\pm A^* g,g) \end{pmatrix}^* = \begin{pmatrix} \overline{(Af,f)} & \overline{(Cf,g)} \\ \overline{(Bg,f)} & \overline{(\pm A^* g,g)} \end{pmatrix} \\ &\qquad\qquad = \begin{pmatrix} (A^* f,f) & (\pm Cg,f) \\ (\pm Bf,g) & (\pm Ag,g) \end{pmatrix} = (M^*|_{\mathcal{D}_{f,g}}) \end{aligned}$$

that $W^2(M|_{\mathcal{D}})^* = W^2(M^*|_{\mathcal{D}})$, and hence (10) holds. Combing with (8), we see (9) is true. It means that $D = A^*$(resp. $D = -A^*$) implies $\overline{W^2(M)}$ is symmetric with respect to the real axis(resp. imaginary axis).

Similarly as proof in Theorem 3.3, the order of the dominance $\delta < 1$ implies M is a closed operator. Besides, by Lemma 2.9 and (12), from $D = A^*$(resp. $D = -A^*$), it is easy to see that $\sigma(M)$, $\sigma_p(M) \cup \sigma_r(M)$ and $\sigma_c(M)$ are symmetric with respect to the real axis(resp. the imaginary axis), respectively. According to Theorem 3.3, we have $\sigma_{app}(M) \subset \overline{W^2(M)}$, and thus $(\sigma_p(M) \cup \sigma_c(M)) \subset \sigma_{app}(M) \subset \overline{W^2(M)}$. Since $\sigma_p(M) \cup \sigma_r(M)$ and $\overline{W^2(M)}$ have the same symmetric property, we see that $(\sigma_p(M) \cup \sigma_r(M)) \subset \overline{W^2(M)}$. Therefore, $\sigma(M) \subset \overline{W^2(M)}$. □

Finally, based on the results obtained above, we discuss some refined spectral distribution of operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Lemma 3.5. (See [7, P.3821]) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with $a, b, c, d \in \mathbb{C}$, then the eigenvalues λ_1, λ_2 of M have the following properties:

(i) If $\mathrm{Re}d < 0 < \mathrm{Re}a$ and $bc \geq 0$, then $\mathrm{Re}\lambda_2 \leq \mathrm{Re}d < 0 < \mathrm{Re}a \leq \mathrm{Re}\lambda_1$.
(ii) If $\mathrm{Re}d < \mathrm{Re}a$ and $bc \leq 0$, then
(a) $\mathrm{Re}d \leq \mathrm{Re}\lambda_2 \leq \mathrm{Re}\lambda_1 \leq \mathrm{Re}a$,
(b) $\mathrm{Re}\lambda_2 \leq \mathrm{Re}d + \sqrt{|bc|} < \mathrm{Re}a - \sqrt{|bc|} \leq \mathrm{Re}\lambda_1$ if $\sqrt{|bc|} < (\mathrm{Re}a - \mathrm{Re}d)/2$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ if, in addition, $a, d \in \mathbb{R}$.
(c) $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 = (a+d)/2, |\mathrm{Im}\lambda_1| = |\mathrm{Im}\lambda_2| = \sqrt{|bc| - (a-d)^2/4}$ if $\sqrt{|bc|} \geq (a-d)/2$ and $a, d \in \mathbb{R}$.

Theorem 3.6. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a densely defined operator matrix in $\mathcal{H} \oplus \mathcal{H}$, where A is closed, B and C are both self-adjoint or both anti-self-adjoint operators, and $D = \pm A^*$. Assume that one of the following assumptions is fulfilled:
(a) M is diagonally dominant with order $\delta < 1$, and $\mathcal{D}(A) \cap \mathcal{D}(D)$ is a core of A and D.
(b) M is off-diagonally dominant with order $\delta < 1$, and $\mathcal{D}(B) \cap \mathcal{D}(C)$ is a core of B and C.
Denote by $\mathcal{D}_0$ the subspace $\mathcal{D}(A) \cap \mathcal{D}(D)$ (resp. $\mathcal{D}(B) \cap \mathcal{D}(C)$) in Case (a) (resp. in Case (b)), and write the sector

$$\Sigma_\omega = \{re^{i\phi} : r \geq 0, |\phi| \leq \omega\}$$

for $\omega \in [0, \pi)$. Then the following statements hold:
(I) If $C \subset \gamma B$ with $\gamma > 0$ and there exist $\delta > 0$ and $\theta \in [0, \pi/2]$ such that

$$W(A|_{\mathcal{D}_0}) \subset \{z \in \Sigma_\theta : \mathrm{Re}z \geq \delta\},$$

then

$$\sigma(M) \subset \{z \in -\Sigma_\theta : \mathrm{Re}z \leq -\delta\} \cup \{z \in \Sigma_\theta : \mathrm{Re}z \geq \delta\}.$$

(II) If $C \subset \gamma B$ with $\gamma < 0$ and A is self-adjoint, then we have
(i′) In Case (b), if A is bounded, we define

$$\alpha = \inf W(A), \quad \beta = \sup W(A),$$

then

$$\sigma(M) \cap \mathbb{R} \subset [\alpha, \infty), \quad \sigma(M) \setminus \mathbb{R} \subset \{z \in \mathbb{C} : \frac{\alpha+\beta}{2} \leq \mathrm{Re}z\};$$

(ii′) In Case (a), if B, C are bounded, then $C = \gamma B$ and

$$\sigma(M) \setminus \mathbb{R} \subset \{z \in \mathbb{C} : |\mathrm{Im}z| \leq \max\{\|B\|, \|C\|\}\}.$$

Proof. According to Theorem 3.4, either of the assumptions (a) and (b) implies that (8) holds and hence $\sigma(M) \subset \overline{W^2(M|_{\mathcal{D}})}$ with $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_0$. Replacing M by $M_{f,g}$ in Lemma 3.5 (i) yields

$$\sigma_p(M_{f,g}) \subset \{z \in -\Sigma_\theta : \mathrm{Re}z \leq -\delta\} \cup \{z \in \Sigma_\theta : \mathrm{Re}z \geq \delta\}$$

for all $(f \ \ g)^t \in \mathcal{D}$ with $\|f\| = \|g\| = 1$. Thus

$$\sigma(M) \subset \overline{W^2(M|_{\mathcal{D}})} \subset \{z \in -\Sigma_\theta : \mathrm{Re}z \leq -\delta\} \cup \{z \in \Sigma_\theta : \mathrm{Re}z \geq \delta\}.$$

This proves (I).

Next we prove (II). Since A is self-adjoint, Lemma 3.5 (ii), applied to the 2×2 matrices $M_{f,g}$, implies that the desired inclusions hold with $W^2(M)$ instead of $\sigma(M)$. This proves (i′). If B, C are bounded, then we consider the block operator matrices

$$T = \begin{pmatrix} A & 0 \\ 0 & \pm A^* \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

In view of T is self-adjoint and S is bounded, we see that M is a bounded perturbation of the self-adjoint operator T. Since $\|S\| \leq \max\{\|B\|, \|C\|\}$, the claim (ii′) follows from standard perturbation theorems (see [9, Problem V.4.8]). □

4. Example

Example 4.1. Let $\mathcal{H} = L^2(-\infty, +\infty)$. Consider the operator matrix

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} \frac{d^2}{dt^2} & i\frac{d}{dt} \\ 0 & -\frac{d^2}{dt^2} \end{pmatrix} : \mathcal{D}(M) = \mathcal{D}(A) \oplus \mathcal{D}(D) \subset \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H},$$

where

$$\begin{aligned} \mathcal{D}(A) = \mathcal{D}(D) &= \{v \in \mathcal{H} : v', v'' \in \mathcal{H}, v' \text{ is absolutely continuous}\}, \\ \mathcal{D}(B) &= \{v \in \mathcal{H} : v' \in \mathcal{H}, v \text{ is absolutely continuous}\}. \end{aligned}$$

It is easy to see that A and D are self-adjoint operators, and thus $\sigma(M) \subset \overline{W^2(M)}$ holds by Corollary 3.1.

On the other hand, the self-adjointness of A implies $\sigma(A) \subset \overline{W(A)} \subset \mathbb{R}$. Hence, from $\overline{W^2(M)} = \overline{W(A)} \cup \overline{W(D)}$, it follows that $\overline{W^2(M)}$ is symmetric with respect to the imaginary axis, and

$$\sigma(M) \subset (\sigma(A) \cup \sigma(D)) \subset \overline{W^2(M)} \subset \mathbb{R}.$$

Example 4.2. Let $\mathcal{H} = L^2(0, 1)$. Investigate the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{d^2}{dx^2} & \frac{d^2}{dx^2} - I \\ -\frac{d^2}{dx^2} & -\frac{d^2}{dx^2} \end{pmatrix} : \mathcal{D}(M) = \mathcal{D}(C) \oplus \mathcal{D}(B) \subset \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H},$$

where

$$\mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}(C) = \mathcal{D}(D) = \left\{ v \in \mathcal{H} : \begin{array}{l} v', v'' \in \mathcal{H}, v' \text{ is absolutely continuous,} \\ \text{and } v(0) = v(1) = 0 \end{array} \right\}.$$

Obviously, M satisfies the conditions of Theorem 3.4, and hence $\sigma(M) \subset \overline{W^2(M)}$ holds.

On the other hand, through direct calculations, we will show the result is correct.

First, it is easy to see that C is boundedly invertible. Then, according to [3, Definition 2.3.1 and Theorem 2.3.7], the resolvent set and spectrum of M can be characterized by the quadratic complement $T_2(\lambda)$, here

$$\begin{aligned} T_2(\lambda) &= B - (A - \lambda)C^{-1}(D - \lambda) \\ &= \frac{d^2}{dx^2} - I - (\frac{d^2}{dx^2} - \lambda)(-\frac{d^2}{dx^2})^{-1}(-\frac{d^2}{dx^2} - \lambda) \\ &= \frac{d^2}{dx^2} - I - (\frac{d^2}{dx^2} - \lambda)(I + \lambda(\frac{d^2}{dx^2})^{-1}) \\ &= \frac{d^2}{dx^2} - I - \frac{d^2}{dx^2} + \lambda^2(\frac{d^2}{dx^2})^{-1} \\ &= \lambda^2(\frac{d^2}{dx^2})^{-1} - I, \quad \lambda \in \mathbb{C}. \end{aligned}$$

We consider the equation $T_2(\lambda)v = 0, v \in \mathcal{D}(B)$, i.e.,

$$\lambda^2(\frac{d^2}{dx^2})^{-1}v - v = 0. \tag{13}$$

Let $\tilde{v} = (\frac{d^2}{dx^2})^{-1}v$, then $v = \tilde{v}''$, and (13) turns into

$$\frac{d^2\tilde{v}}{dx^2} - \lambda^2\tilde{v} = 0. \tag{14}$$

From $v \in \mathcal{D}(B)$, it follows that $\tilde{v}(0) = \tilde{v}(1) = 0$ and $\tilde{v}''(0) = \tilde{v}''(1) = 0$. Since C is boundedly invertible, $\tilde{v}(0) = \tilde{v}(1) = 0$ implies $\tilde{v}''(0) = \tilde{v}''(1) = 0$, and thus we ignore the latter. Then, solving the equation (14), we see that (13) has non-zero solutions if and only if $\lambda = \pm k\pi i$ $(k = 1, 2, \dots)$. Thus,

$$\sigma_p(M) = \sigma_p(T_2(\lambda)) = \{\lambda \in \mathbb{C} : \lambda = \pm k\pi i, k = 1, 2, \dots\}.$$

Clearly, if $\lambda \neq \pm k\pi i$ $(k = 1, 2, \dots)$, bounded linear operator $T_2(\lambda)$ is a bijection. Hence,

$$\rho(M) = \rho(T_2(\lambda)) = \{\lambda \in \mathbb{C} : \lambda \neq \pm k\pi i, k = 1, 2, \dots\},$$

and thus $\sigma(M) = \sigma_p(M)$.

Besides, by Definition 2.3, for each $\lambda \in W^2(M)$, there exist $(f \ \ g)^t \in \mathcal{D}(M)$ with $\|f\| = \|g\| = 1$ such that the quadratic equation

$$\begin{aligned}\det(\lambda) :&= \det\begin{pmatrix}(Af, f) - \lambda & (Bg, f) \\ (Cf, g) & (Dg, g) - \lambda\end{pmatrix} \\ &= \Big((Af, f) - \lambda\Big)\Big((Dg, g) - \lambda\Big) - (Bg, f)(Cf, g) \\ &= \Big((f'', f) - \lambda\Big)\Big((-g'', g) - \lambda\Big) - (g'' - g, f)(-f'', g) \\ &= \lambda^2 + \Big((g'', g) - (f'', f)\Big)\lambda + \Big((f'', g)(g'', f) - (f'', g)(g, f) - (f'', f)(g'', g)\Big) \\ &= 0\end{aligned} \tag{15}$$

holds. And, through calculations, we see that the eigenvalues and the corresponding eigenfunctions of operator A are

$$\tau_k = -k^2\pi^2, \quad \xi_k = \sqrt{2}\sin k\pi x \ (k = 1, 2, 3, \dots).$$

Thus, taking

$$f_k = g_k = \xi_k/\|\xi_k\| = \sin k\pi x/\|\sin k\pi x\| \ (k = 1, 2, 3, \dots)$$

in (15), we have

$$\lambda_k^2 = (f_k'', f_k) = \tau_k(f_k, f_k) = -k^2\pi^2 \ (k = 1, 2, 3, \dots),$$

and hence $\lambda_k = \pm k\pi i$ $(k = 1, 2, 3, \dots)$. Therefore, the conclusion $\sigma(M) \subset \overline{W^2(M)}$ is true.

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