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Viscosity Explicit Midpoint Methods for Nonexpansive Mappings in Hadamard Spaces

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Abstract. In this paper, the viscosity explicit midpoint method for nonexpansive mappings in Hadamard spaces is introduced. Under certain appropriate conditions on the sequence of parameters, it is proved that the limit of the approximating sequence generated by proposed method converges strongly to a fixed point of T which solves some variational inequalities. Moreover, we give an application to the equilibrium problem.

1. Introduction

Initial value problems (IVP) are those for which the solution is entirely known at some time, say $t = 0$, and the question is to solve the ordinary differential equation (ODE)

$$y'(t) = \Phi(t, y(t)), \qquad y(0) = y_0, \tag{1}$$

where Φ is a continuous function from $\mathbb{R}^N$ to $\mathbb{R}^N$. Many different methods have been proposed and used to solve various kinds of ODEs. The midpoint method is the one of the most popular method for numerically solving ODEs. The midpoint method improves the Euler's method by adding a midpoint in the step which increases the accuracy by one order; see [1, 2, 6, 17, 18] and the references therein. For instance, the implicit and explicit midpoint methods are given by two formulas as follows.

$$y_{n+1} = y_n + h\Phi\left(\frac{y_n + y_{n+1}}{2}\right), \;\; n \geq 0 \tag{2}$$

and

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + h\Phi(y_n) \\ y_{n+1} &= y_n + h\Phi\left(\frac{y_n + \tilde{y}_{n+1}}{2}\right), \;\; n \geq 0, \end{aligned} \tag{3}$$

where $h > 0$ is a stepsize. It is known that if $\Phi : R^N \to R^N$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{y_n\}$ defined by (2) and (3) converge to the exact solution of (1) as $h \to 0$ uniformly over $t \in [0, \bar{t}]$ for any fixed $\bar{t} > 0$.

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Let C be a nonempty closed convex subset of a Hilbert space H, let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$), and let $f : C \to C$ be a contraction (i.e., $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in C$ and some $\alpha \in [0, 1)$). The combination of midpoint method and iterative mathod approximating fixed points for nonlinear mappings was first introduced by Alghamdi et al. [1]. They considered the following implicit iterative scheme.

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \ n \ge 0, \tag{4}$$

where the initial guess $x_0 \in H$ is arbitrarily chosen, $\alpha_n \in (0, 1)$ for all n. The weak convergence for (4) was proved under some suitable condition on $\{\alpha_n\}$.

In 2015, Xu et al. [21] combined the viscosity technique and the implicit midpoint rule for nonexpansive mappings. They introduced the following semi-implicit algorithm, so-called, viscosity implicit midpoint rule.

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \ n \ge 0. \tag{5}$$

They proved that the sequence $\{x_n\}$ defined by (5) converges strongly to a fixed point of T which, in addition, also solves the variational inequality:

$$\langle (I - f)q, p - q\rangle \ge 0, \ p \in F(T), \tag{6}$$

where $F(T)$ is the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Recently, Marino et al. [15] introduced the viscosity explicit midpoint method for quasi-nonexpansive mappings as follows.

$$\begin{aligned} &x_0 \in C \text{ arbitarily chosen} \\ &y_{n+1} = \gamma_n x_n + (1 - \gamma_n)Tx_n \\ &x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)y_{n+1}), \ n \ge 0. \end{aligned} \tag{7}$$

Under suitable assumptions on the sequence of parameters, they proved that the explicit iteration strongly converges to a fixed point of a quasi-nonexpansive mapping T which solves the variational inequality (6).

In the general Hadamard space setting, Preechasilp [16], extended the iterations (5) for a nonexpansive mapping T:

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T\left(\frac{x_n \oplus x_{n+1}}{2}\right), \ n \ge 0, \tag{8}$$

where $\{\alpha_n\} \subset (0, 1)$. Researcher proved $\{x_n\}$ defined by (8) converges strongly to $q \in F(T)$ which solves the variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{pq}\rangle \ge 0, \quad p \in F(T). \tag{9}$$

All of the above bring us the following conjectures.

Question 1.1. *Could we obtain the strong convergence for the viscosity explicit midpoint method in the framework of Hadamard space?*

The purpose of this paper is to study the following iterative schemes in a Hadamard space.

$$\begin{aligned} &x_0 \in C \text{ arbitarily chosen} \\ &y_{n+1} = \gamma_n x_n \oplus (1 - \gamma_n)Tx_n \\ &x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(\beta_n x_n \oplus (1 - \beta_n)y_{n+1}), \ n \ge 0. \end{aligned} \tag{10}$$

We prove the iterative scheme (10) converges strongly to q such that $q = P_{F(T)}f(q)$ which is the unique solution of the variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \geq 0, \quad p \in F(T). \tag{11}$$

The structure of the paper is as follows. Section 2 introduce some basic knowledge of inequalities and convergence types in Hadamard spaces. We obtain, in Section 3, the strong convergence theorem of viscosity explicit midpoint rule for nonexpansive mappings in Hadamard spaces. An application to equilibrium problems is given in Section 4.

2. Preliminaries

Let (X, d) be a metric space. X is called Hadamard space if it complete and if for each pair of points $x, y \in X$ there exists a point $w \in X$ such that for all $z \in X$

$$d^2(z, w) \leq \frac{1}{2}d^2(z, x) + \frac{1}{2}d^2(z, y) - \frac{1}{4}d^2(x, y).$$

A Hadamard space is sometimes called global nonpositive curvature space or complete CAT(0) space. In the rest of paper, if not otherwise specified , we denote X by Hadamard space. It is proved in Lemma 2.1 of [7] that, in Hadamard spaces, for each $x, y \in X$ there exists the unique point z in the geodesic segment joining from x to y with

$$d(z, x) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

We also denote by $[[x, y]]$ the geodesic segment joining from x to y, that is, $[[x, y]] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$. A subset C of X is convex if $[[x, y]] \subseteq C$ for all $x, y \in C$. The following lemmas play an important role in our paper.

Lemma 2.1. *[7] Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then*

(i) $d(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$ *and*
(ii) $d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)$.

Lemma 2.2. *[4] Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda p \oplus (1 - \lambda)q, \lambda r \oplus (1 - \lambda)s) \leq \lambda d(p, r) + (1 - \lambda)d(q, s).$$

In 2008, Berg and Nikolaev [3] introduced the concept of quasilinearization as follows:

Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad (a, b, c, d \in X). \tag{12}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \tag{13}$$

for all $a, b, c, d \in X$. It known [3, Corollary 3] that the Hadamard space satisfies the Cauchy-Schwarz inequality.

The following two lemmas can be found in [19].

Lemma 2.3. *[19] Let X be a CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds*

$$d^2(x, u) \leq d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.4. *[19] Let X be a CAT(0) space. For any $u, v \in X$ and $t \in [0,1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Let C be a nonempty closed convex subset of a Hadamard space X. It is proved in [4] that for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x, u) = \min_{y \in C} d(x, y).$$

The mapping $P_C : X \to C$ defined by $P_C(x) = u$ is called the metric projection from X onto C. A characterization of metric projection by using quasilinearization was first studied by Dehghan and Rooin [5].

Theorem 2.5. *[5, Theorem 2.4] Let C be a nonempty convex subset of a Hadamard space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle \overrightarrow{uy}, \overrightarrow{ux} \rangle \leq 0, \quad \forall y \in C.$$

Now, we collect the concept of two types of convergence.

Δ-convergence: The concept of Δ-convergence introduced by Lim [13] in 1976 was shown by Kirk and Panyanak [12] in CAT(0) spaces as follows.

A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

w**-convergence:** Using the concept of quasilinearization, Kakavandi and Amini [10] introduced the following notion of w-convergence.

A sequence $\{x_n\}$ in the complete CAT(0) space (X, d), x_n w-converges to $x \in X$ if $\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$, i.e. $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

Lemma 2.6. *([12],p.3690) Every bounded sequence in a Hadamard space always has a Δ-convergent subsequence.*

Lemma 2.7. *[8] If C is a closed convex subset of a Hadamard space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.8. *[9] Let X be a Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if* $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ *for all $y \in X$.*

Lemma 2.9. *[8] If C is a closed convex subset of X and $T : C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ-convergence to x and $d(x_n, Tx_n) \to 0$, and imply $x \in C$ and $Tx = x$.*

Recall that a continuous linear functional μ on l_∞, the Banach space of bounded real sequence, is called a Banach limit if $\|\mu\| = \mu(1, 1, \ldots)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in l_\infty$.

The following lemma is an important tool for proving the strong convergence of a sequence $\{d^2(x_n, q)\}$.

Lemma 2.10. *[20] Let $\{a_n\}$ be a sequence of non-negative real number satisfying the property*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subseteq (0,1)$ and $\{\beta_n\}, \{\gamma_n\} \subseteq \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;

(iii) $\gamma_n \geq 0$, $\sum_{n=0}^{\infty} \gamma_n < +\infty$.

Then $\{a_n\}$ converges to zero, as $n \to \infty$.

Lemma 2.11. *[14] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \text{ for all } j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and, for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

3. Main Results

In this section, we consider the viscosity technique for the explicit midpoint rule of nonexpansive mappings which generates a sequence $\{x_n\}$ in the following manner: $x_0 \in C$

$$\begin{aligned} y_{n+1} &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n \\ x_{n+1} &= \alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1}\right), \quad n \geq 0. \end{aligned} \tag{14}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ for all $n \geq 0$.

Theorem 3.1. *Let C be a closed convex subset of a Hadamard space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\alpha \in [0,1)$. Let $\{x_n\}$ be generated by* (14)*. Assume the following conditions hold:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \gamma_n(1-\gamma_n)(1-\beta_n) > 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to q such that $q = P_{F(T)}f(q)$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \geq 0, \quad p \in F(T). \tag{15}$$

Proof. We first show that the sequence $\{x_n\}$ is bounded. For any $p \in F(T)$, we have that

$$\begin{aligned} d(y_{n+1}, p) &= d(\gamma_n x_n \oplus (1-\gamma_n)Tx_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n)d(Tx_n, p) \\ &= d(x_n, p). \end{aligned}$$

And, also

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1}\right), p\right) \\ &\leq \alpha_n d\left(f(x_n), p\right) + (1-\alpha_n)d\left(T\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1}\right), p\right) \\ &\leq \alpha_n d\left(f(x_n), p\right) + (1-\alpha_n)d\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1}, p\right) \\ &\leq \alpha_n d\left(f(x_n), f(p)\right) + \alpha_n d\left(f(p), p\right) + (1-\alpha_n)\beta_n d\left(x_n, p\right) + (1-\alpha_n)(1-\beta_n)d\left(y_{n+1}, p\right) \\ &\leq \alpha_n \alpha d\left(x_n, p\right) + \alpha_n d\left(f(p), p\right) + (1-\alpha_n)\beta_n d\left(x_n, p\right) + (1-\alpha_n)(1-\beta_n)d(x_n, p) \\ &= (1-\alpha_n(1-\alpha))d\left(x_n, p\right) + \alpha_n d\left(f(p), p\right). \end{aligned}$$

Therefore,

$$d(x_{n+1},p) \leq \max\left\{d(x_n,p), \frac{1}{1-\alpha}d(f(p),p)\right\}.$$

By induction, we have

$$d(x_n,p) \leq \max\left\{d(x_0,p), \frac{1}{1-\alpha}d(f(p),p)\right\},$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$.

It follows from Lemma 2.3 and Lemma 2.4 (i), (ii) that

$$\begin{aligned}
d^2(x_{n+1},p) &= d\left(\alpha_n f(x_n) \oplus (1-\alpha_n)T\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1}\right),p\right) \\
&\leq (1-\alpha_n)^2 d^2\left(T(\beta_n x_n \oplus (1-\beta_n)y_{n+1},p)\right) + 2\alpha_n\langle\overrightarrow{f(x_n)p},\overrightarrow{x_{n+1}p}\rangle \\
&= (1-\alpha_n)^2 d^2\left(T(\beta_n x_n \oplus (1-\beta_n)y_{n+1},p)\right) + 2\alpha_n\langle\overrightarrow{f(x_n)f(p)},\overrightarrow{x_{n+1}p}\rangle + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&\leq (1-\alpha_n)^2 d^2\left(\beta_n x_n \oplus (1-\beta_n)y_{n+1},p\right) + 2\alpha_n d(f(x_n),f(p))d(x_{n+1},p) + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&\leq (1-\alpha_n)^2\left[\beta_n d^2(x_n,p) + (1-\beta_n)d^2(y_{n+1},p) - \beta_n(1-\beta_n)d^2(x_n,y_{n+1})\right] \\
&\quad +2\alpha_n\alpha d(x_n,p)d(x_{n+1},p) + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&\leq (1-\alpha_n)^2\beta_n d^2(x_n,p) + (1-\alpha_n)^2(1-\beta_n)d^2(y_{n+1},p) - (1-\alpha_n)^2\beta_n(1-\beta_n)d^2(x_n,y_{n+1}) \\
&\quad +\alpha_n\alpha\left[d^2(x_n,p) + d^2(x_{n+1},p)\right] + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&\leq (1-\alpha_n)^2\beta_n d^2(x_n,p) + (1-\alpha_n)^2(1-\beta_n)\gamma_n d^2(x_n,p) \\
&\quad +(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)d^2(Tx_n,p) - (1-\alpha_n)^2(1-\beta_n)\gamma_n(1-\gamma_n)d^2(x_n,Tx_n) \\
&\quad -(1-\alpha_n)^2\beta_n(1-\beta_n)(1-\gamma_n)^2 d^2(x_n,Tx_n) \\
&\quad +\alpha_n\alpha\left[d^2(x_n,p) + d^2(x_{n+1},p)\right] + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&\leq (1-\alpha_n)^2\beta_n d^2(x_n,p) + (1-\alpha_n)^2(1-\beta_n)\gamma_n d^2(x_n,p) \\
&\quad +(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)d^2(x_n,p) \\
&\quad -(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)\left[\gamma_n + \beta_n(1-\gamma_n)\right]d^2(x_n,Tx_n) \\
&\quad +\alpha_n\alpha\left[d^2(x_n,p) + d^2(x_{n+1},p)\right] + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle \\
&= \left((1-2\alpha_n) + \alpha\alpha\right)d^2(x_n,p) + \alpha_n^2 d^2(x_n,p) \\
&\quad -(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)\left[\gamma_n + \beta_n(1-\gamma_n)\right]d^2(x_n,Tx_n) \\
&\quad +\alpha_n\alpha d^2(x_{n+1},p) + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle. \qquad (16)
\end{aligned}$$

We then have that

$$\begin{aligned}
&d^2(x_{n+1},p) \\
&\quad \leq \left((1-2\alpha_n) + \alpha\alpha\right)d^2(x_n,p) + \alpha_n^2 d^2(x_n,p) + \alpha_n\alpha d^2(x_{n+1},p) + 2\alpha_n\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle, \qquad (17)
\end{aligned}$$

and so

$$d^2(x_{n+1},p) \leq \left(1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)d^2(x_n,p) + \left(\frac{\alpha_n^2\alpha}{1-\alpha_n\alpha}\right)M + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle\overrightarrow{f(p)p},\overrightarrow{x_{n+1}p}\rangle, \qquad (18)$$

where $M \geq \sup_{n\geq 0}\{d^2(x_n,p)\}$. Thanks to (16), we get

$$\begin{aligned}
&\frac{(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)\left[\gamma_n + \beta_n(1-\gamma_n)\right]}{1-\alpha_n\alpha}d^2(x_n,Tx_n) \\
&\qquad \leq \left(d^2(x_n,p) - d^2(x_{n+1},p)\right) + \left(\frac{\alpha_n^2\alpha}{1-\alpha_n\alpha}\right)M + \left(\frac{2\alpha_n}{1-\alpha_n\alpha}\right)\overline{M}, \qquad (19)
\end{aligned}$$

where $\overline{M} \geq \sup_{n\geq 0}\{\langle \overrightarrow{f(p)p}, \overrightarrow{x_n p}\rangle\}$. To reach the boundedness of $\{x_n\}$, we consider the following two cases regarding the sequence $\{d^2(x_n, p)\}_{n\geq 0}$:

Case 1 Assume that there exists n_0 such that $\{d^2(x_n, p)\}_{n\geq n_0}$ is non-increasing. Then $\lim_{n\to\infty} d^2(x_n, p)$ exists. It follows from (19) and $\lim_{n\to\infty} \alpha_n = 0$ that

$$0 \leq \limsup_{n\to\infty} \frac{(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)\left[\gamma_n + \beta_n(1-\gamma_n)\right]}{1-\alpha_n\alpha} d^2(x_n, Tx_n)$$
$$\limsup_{n\to\infty}\left[\left(d^2(x_n, p) - d^2(x_{n+1}, p)\right) + \left(\frac{\alpha_n^2\alpha}{1-\alpha_n\alpha}\right)M + \left(\frac{2\alpha_n}{1-\alpha_n\alpha}\right)\overline{M}\right] = 0.$$

By virtue of (iii), one has

$$\limsup_{n\to\infty} \frac{(1-\alpha_n)^2(1-\beta_n)(1-\gamma_n)\left[\gamma_n + \beta_n(1-\gamma_n)\right]}{1-\alpha_n\alpha} > 0.$$

Thus, $\lim_{n\to\infty} d^2(x_n, Tx_n) = 0$. Since $y_{n+1} = \gamma_n x_n \oplus (1-\gamma_n)Tx_n$, one has

$$d(y_{n+1}, x_n) = (1-\gamma_n)d(Tx_n, x_n) \to 0 \ \text{ as } \ n \to \infty.$$

Putting $z_n = \beta_n x_n \oplus (1-\beta_n)y_{n+1}$. Then, we have that

$$d(Tz_n, Tx_n) \leq d(z_n, x_n) = (1-\beta_n)d(y_{n+1}, x_n) \to 0 \ \text{ as } \ n \to \infty,$$

and so

$$d(Tz_n, x_n) \leq d(Tz_n, Tx_n) + d(Tx_n, x_n) \to 0 \ \text{ as } \ n \to \infty. \tag{20}$$

We now prove that

$$\limsup_{n\to\infty}\langle \overrightarrow{f(q)q}, \overrightarrow{x_n q}\rangle \leq 0,$$

where q is the unique solution of variational inequality (15) on the fixed point set of T. Due to the boundedness of $\{x_n\}$, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \overrightarrow{pf(p)}, \overrightarrow{px_n}\rangle = \lim_{j\to\infty}\langle \overrightarrow{pf(p)}, \overrightarrow{px_{n_j}}\rangle \quad \text{for all } p \in F(T). \tag{21}$$

Thanks to the boundedess of $\{x_n\}$ again, there exists a subsequence $\{x_{n_k}\}$ of $\{x_{n_j}\}$ which Δ-converges to a point q. Since $d^2(x_n, Tx_n) \to 0$ as $n \to \infty$ and demiclosedness of $I - T$ at 0, we have that $q \in F(T)$. We claim that

$$\limsup_{n\to\infty}\langle \overrightarrow{qf(q)}, \overrightarrow{qx_n}\rangle \leq 0.$$

By Lemma 2.8 we get that

$$\limsup_{j\to\infty}\langle \overrightarrow{qf(q)}, \overrightarrow{qx_{n_j}}\rangle \leq 0.$$

Combine the last inequality with (21), we have the claim. Applying Lemma 2.10 with (18), one has $x_n \to q$ as $n \to \infty$ which is the fixed point element of T.

We next show that q is a solution of (15). Applying Lemma 2.1(ii), for any $p \in F(T)$,

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2\left(\alpha_n f(x_n) \oplus (1-\alpha_n)Tz_n, p\right) \\ &\leq \alpha_n d^2(f(x_n), p) + (1-\alpha_n)d^2(Tz_n, p) - \alpha_n(1-\alpha_n)d^2(f(x_n), Tz_n) \\ &\leq \alpha_n d^2(f(x_n), p) + (1-\alpha_n)d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(f(x_n), Tz_n). \end{aligned} \tag{22}$$

Let μ be a Banach limit. Then linearity and positivity of Banach limit implies

$$\mu_n d^2(x_{n+1}, p) \quad \leq \quad \alpha_n \mu_n d^2(f(x_n), p) + (1-\alpha_n)\mu_n d^2(x_n, p) - \alpha_n(1-\alpha_n)\mu_n d^2\left(f(x_n), Tz_n\right).$$

It follows from the shift-invariance of Banach limit that

$$\mu_n d^2(x_{n+1}, p) \leq \mu_n d^2(f(x_n), p) - (1-\alpha_n)\mu_n d^2\left(f(x_n), Tz_n\right).$$

Since $x_n \to q$ and $\alpha_n \to 0$ as $n \to \infty$, we obtain that

$$d^2(q, p) \leq d^2(f(q), p) - d^2\left(f(q), q\right).$$

Hence

$$0 \leq \frac{1}{2}\left[d^2(q,q) + d^2(f(q),p) - d^2(q,p) - d^2(f(q),q)\right] = \langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle, \quad \forall p \in F(T).$$

That is, q is the unique solution of the variational inequality (15).

Case 2 There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p) < d(x_{n_k+1}, p) \quad \text{for all } k \geq 0.$$

It follows from Lemma 2.11 that there exists a nondecreasing sequence of integers $\{\tau(n)\}$ such that, for all $n \geq n_0$, the following statments:

$$\lim_{n\to\infty} \tau(n) = +\infty$$

and

$$d(x_{\tau(n)}, p) < d(x_{\tau(n)+1}, p) \quad \text{and} \quad d(x_n, p) < d(x_{\tau(n)+1}, p).$$

By virtue of inequality (18), one has

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty}\left(d^2(x_{\tau(n)+1}, p) - d^2(x_{\tau(n)}, p)\right) \\
&\leq \limsup_{n\to\infty}\left(d^2(x_{\tau(n)+1}, p) - d^2(x_{\tau(n)}, p)\right) \\
&\leq \limsup_{n\to\infty}\left(d^2(x_{n+1}, p) - d^2(x_n, p)\right) \\
&\leq \limsup_{n\to\infty}\left(\left(\frac{\alpha_n^2\alpha}{1-\alpha_n\alpha}\right)M + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle \overrightarrow{f(p)p}, \overrightarrow{x_{n+1}p} \rangle\right) \leq 0.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\left(d^2(x_{\tau(n)+1}, p) - d^2(x_{\tau(n)}, p)\right) = 0. \tag{23}$$

Hence, as in Case 1, we obtain that $d^2(x_{\tau(n)}, Tx_{\tau(n)}) \to 0$ as $n \to \infty$ and

$$\limsup_{n\to\infty}\langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)}q} \rangle \leq 0,$$

where q is the unique solution of variational inequality (15) on the fixed point set of T.

Replacing n with $\tau(n)$ in (18), we get

$$\begin{aligned} d^2(x_{\tau(n)+1}, q) &\leq \left(1 - \frac{2\alpha_{\tau(n)}(1-\alpha)}{1-\alpha_{\tau(n)}\alpha}\right) d^2(x_{\tau(n)}, q) + \left(\frac{\alpha^2_{\tau(n)}\alpha}{1-\alpha_{\tau(n)}\alpha}\right) M \\ &\quad + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\alpha} \langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)+1}q} \rangle \\ &\leq \left(1 - \frac{2\alpha_{\tau(n)}(1-\alpha)}{1-\alpha_{\tau(n)}\alpha}\right) d^2(x_{\tau(n)+1}, q) + \left(\frac{\alpha^2_{\tau(n)}\alpha}{1-\alpha_{\tau(n)}\alpha}\right) M \\ &\quad + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\alpha} \langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)+1}q} \rangle, \end{aligned}$$

that is,

$$\frac{2\alpha_{\tau(n)}(1-\alpha)}{1-\alpha_{\tau(n)}\alpha} d^2(x_{\tau(n)+1}, q) \leq \left(\frac{\alpha^2_{\tau(n)}\alpha}{1-\alpha_{\tau(n)}\alpha}\right) M + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\alpha} \langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)+1}q} \rangle.$$

It turn out that

$$\frac{2(1-\alpha)}{1-\alpha_{\tau(n)}\alpha} d^2(x_{\tau(n)+1}, q) \leq \left(\frac{\alpha_{\tau(n)}\alpha}{1-\alpha_{\tau(n)}\alpha}\right) M + \frac{2}{1-\alpha_{\tau(n)}\alpha} \langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)+1}q} \rangle.$$

Thank to assumption (i) and $\limsup_{n\to\infty} \langle \overrightarrow{f(q)q}, \overrightarrow{x_{\tau(n)+1}q} \rangle \leq 0$, one has $d(x_{\tau(n)}, q) \to 0$ as $n \to \infty$. By Lemma 2.11, we can conclude that $d(x_n, q) \to 0$ as $n \to \infty$. The proof is complete. □

If we set $f(x) = u$ for all $x \in C$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let C be a closed convex subset of a Hadamard space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by $u, x_0 \in C$*

$$\begin{aligned} y_{n+1} &= \gamma_n x_n \oplus (1-\gamma_n) T x_n \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n) T (\beta_n x_n \oplus (1-\beta_n) y_{n+1}), n \geq 0. \end{aligned} \tag{24}$$

Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \gamma_n(1-\gamma_n)(1-\beta_n) > 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to q such that $q = P_{F(T)}u$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{qu}, \overrightarrow{pq} \rangle \geq 0, \quad p \in F(T).$$

4. Applications

Let (X, d) be a Hadamard space and $g : C \times C \to \mathbb{R}$ be a bifunction. The following equilibrium problem is to find $\tilde{x} \in C$ such that

$$g(\tilde{x}, y) \geq 0 \ \ \forall y \in C. \tag{25}$$

We denote $EP(g, C)$ by the set of equilibrium point of g over a nonempty closed convex subset C. For $r > 0$, define the resolvent mapping $J_r^g : X \to 2^C$, briefly J_r, of g in Hadamard spaces as

$$J_r(x) = \{z \in C : g(z, y) + r\left\langle \overrightarrow{xz}, \overrightarrow{zy} \right\rangle \ \ \forall y \in C\} \tag{26}$$

for all $x \in X$. Khatibzadeh and Mohebbi [11] proved that the resolvent is well defined for all $r > 0$ under the standing assumption that the bifunction g satisfies the following set of standard properties.

Assumtion 4.1. *The bifunction $g : C \times C \to \mathbb{R}$ is such that:*

(g_1) *For any $x \in C$, $g(x, x) = 0$.*
(g_2) *For any $y \in C$, $g(\cdot, y) : C \to \mathbb{R}$ is upper-semicontinuous.*
(g_3) *For any $x \in C$, $g(x, \cdot) : C \to \mathbb{R}$ is convex and lower-semicontinuous.*
(g_4) *g is monotone on C, i.e., $g(x, y) + g(y, x) \le 0$ for all $x, y \in C$.*

Theorem 4.2. *[11] Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies Assumption 4.1. Then J_r is well defined for all $r > 0$.*

The following lemma shows that the mapping J_r is firmly nonexpansive and the fixed point set $F(J_r)$ and equilibrium point set $EP(g, C)$ are coincide.

Lemma 4.3. *[11] Let $g : C \times C \to \mathbb{R}$ be a bifunction and $r > 0$ such that $J_r(x)$ exists.*

(i) *If g is monotone, then the resolvent J_r is firmly nonexpansive, that is,*

$$d^2(J_r(x), J_r(y)) \le \langle \overrightarrow{xy}, \overrightarrow{J_r(x)J_r(y)} \rangle$$

for all $x, y \in X$.

(ii) *$F(J_r) = EP(g, C)$.*

Remark 4.4. *Every firmly nonexpansive mapping is nonexpansive.*

The following two corollaries follow from Theorem 3.1 and Corollary 3.2.

Corollary 4.5. *Let C be a closed convex subset of a Hadamard space X. Let $f : C \to C$ be a contraction with coefficient $\alpha \in [0, 1)$. Assume that $g : C \times C \to \mathbb{R}$ satisfies Assumption 4.1. Let $r > 0$ and define the sequence $\{x_n\}$ as follows: $x_0 \in C$ and*

$$y_{n+1} = \gamma_n x_n \oplus (1 - \gamma_n) J_r(x_n)$$
$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) J_r (\beta_n x_n \oplus (1 - \beta_n) y_{n+1}), \quad n \ge 0.$$

Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \gamma_n(1 - \gamma_n)(1 - \beta_n) > 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to q, the solution of equilibrium problem (25), such that $q = P_{F(J_r)} f(q)$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \ge 0, \quad p \in F(J_r).$$

Corollary 4.6. *Let C be a closed convex subset of a Hadamard space X. Assume that $g : C \times C \to \mathbb{R}$ satisfies Assumption 4.1. Let $r > 0$ and define the sequence $\{x_n\}$ as follows: $u, x_0 \in C$ and*

$$y_{n+1} = \gamma_n x_n \oplus (1 - \gamma_n) J_r(x_n)$$
$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_r (\beta_n x_n \oplus (1 - \beta_n) y_{n+1}), \quad n \ge 0.$$

Assume the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\limsup_{n\to\infty} \gamma_n(1 - \gamma_n)(1 - \beta_n) > 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to q, the solution of equilibrium problem (25), such that $q = P_{F(J_r)} u$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{qu}, \overrightarrow{pq} \rangle \ge 0, \quad p \in F(J_r).$$

5. Conclusions

In this paper, the viscosity explicit midpoint iteration (10) is introduced in the framework of Hadamard space. Under certain appropriate conditions on the sequence of parameters, it is proved that the limit of the approximating sequence generated by proposed method converges strongly to a fixed point which solves the variational inequality (11). The presented result in the paper extends the result in [15] in the general Hadamard space setting. However, the strong convergence theorem is established for nonexpansive mappings. Therefore, extending the result to the discontinuous mapping, for example, quasi-nonexpansive mappings, is an interesting open problem.

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