



Periodic Solution and Stationary Distribution for Stochastic Predator-Prey Model With Modified Leslie-Gower and Holling Type II Schemes

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Abstract. In this paper, a stochastic predator-prey system with modified Leslie-Gower and Holling type II schemes is studied. For the autonomous case, we prove that the system has a stationary distribution under some parametric restrictions. We also obtain conditions for the non-persistence of the system, and the results are illustrated by computer simulations. For the non-autonomous system with continuous periodic coefficients, sufficient conditions which guarantee the existence of periodic solution of the system are established.

1. Introduction

One of the dominant themes in both ecology and mathematical ecology is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics. The predator-prey system, incorporating a modified Leslie-Gower function response and the Holling-type II function response, can be written as follows [1, 2]:

$$\begin{cases} \dot{x}(t) = x(t) \left(a - bx(t) - \frac{cy(t)}{m_1 + x(t)} \right), \\ \dot{y}(t) = y(t) \left(r - \frac{fy(t)}{m_2 + x(t)} \right), \end{cases} \quad (1)$$

where $x(t)$ and $y(t)$ represent the population densities at time t , the model parameters a, b, c, r, f, m_1 and m_2 are assuming only positive values. a is the growth rate of prey, b is the strength of competition among individuals of species x , c is the maximum value of the per capita reduction rate of x due to y , m_1 and

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m_2 measure the extent to which the environment provides protection to prey x and to the predator y respectively, r describes the growth rate of y and f has a similar meaning to c . Aziz-Alaoui and Daher Okiye [1] point out that:

- (i) System (1) has a unique interior equilibrium $E^*(x^*, y^*)$ if $\frac{rm_2}{f} < \frac{am_1}{c}$.
- (ii) The interior equilibrium $E^*(x^*, y^*)$ is globally asymptotically stable if the following conditions hold

$$L_1 < \frac{am_1}{2c}, \quad m_1 < 2m_2, \quad 4(a + bm_1) < c,$$

where $L_1 = \frac{1}{4fb}(af(a + 4) + (r + 1)^2(a + bm_2))$. More related results on (1) can be seen in [1–5].

In fact, population dynamics is inevitably affected by environmental white noise. Some researchers have paid their attention to the environmentally perturbed system, and many interesting results are obtained [6–10]. In this paper, we will focus on the case that a and r in model (1) are perturbed with white noise, that is

$$\begin{aligned} a &\rightarrow a + \alpha \dot{B}_1(t), \\ r &\rightarrow r + \beta \dot{B}_2(t). \end{aligned}$$

Then we get the following stochastic system

$$\begin{cases} dx(t) = x(t) \left(a - bx(t) - \frac{cy(t)}{m_1 + x(t)} \right) dt + \alpha x(t) dB_1(t), \\ dy(t) = y(t) \left(r - \frac{fy(t)}{m_2 + x(t)} \right) dt + \beta y(t) dB_2(t), \end{cases} \tag{2}$$

where $B_1(t)$ and $B_2(t)$ are independent one-dimensional Wiener processes, α^2 and β^2 represent the intensities of the white noise.

Ji and Jiang [6, 7] investigate the special case of system (2), that is, $m_1 = m_2 = m$:

$$\begin{cases} dx(t) = x(t) \left(a - bx(t) - \frac{cy(t)}{m + x(t)} \right) dt + \alpha x(t) dB_1(t), \\ dy(t) = y(t) \left(r - \frac{fy(t)}{m + x(t)} \right) dt + \beta y(t) dB_2(t). \end{cases} \tag{3}$$

Condition for the system to be extinct is given and persistent condition is established. Moreover, they show that there is a stationary distribution for the system by constructing the Lyapunov function. However, in these work, the existence of stationary distribution depends heavily on the positive equilibrium of corresponding deterministic system. In this paper, one of our aims is to establish sufficient conditions for the existence of stationary distribution of system (2) by constructing the suitable Lyapunov function, which does not depend on the existence and the stability of the positive equilibrium of system (1).

Since there are number of factors in the environment, which vary periodically with changing seasons, affect various parameters in the ecological models, therefore the study of ecological systems driven by periodic external forces is of importance. However, there is little work about periodically stochastic differential equations. In [11], Li and Xu obtain some sufficient conditions for the existence of periodic solution of the delay equations by using the properties of periodic Markov processes. Lin and Jiang [12] investigate stochastic SIR epidemic model with variation in all parameters, they obtain the threshold for the epidemic to occur. Moreover, the existence of nontrivial positive periodic solution is obtained. Motivated by these, we consider a stochastic non-autonomous predator-prey system with modified Leslie-Gower and Holling type II schemes with periodic coefficients, which takes the following form:

$$\begin{cases} dx(t) = x(t) \left(a(t) - b(t)x(t) - \frac{c(t)y(t)}{m_1(t) + x(t)} \right) dt + \alpha(t)x(t) dB_1(t), \\ dy(t) = y(t) \left(r(t) - \frac{f(t)y(t)}{m_2(t) + x(t)} \right) dt + \beta(t)y(t) dB_2(t), \end{cases} \tag{4}$$

where $a(t), b(t), c(t), m_1(t), r(t), f(t), m_2(t), \alpha(t)$ and $\beta(t)$ are positive continuous functions with period θ , $B_1(t)$ and $B_2(t)$ are standard one-dimensional Wiener processes (the independence of $B_i(t)$ is not necessary in model (4)), and $\alpha^2(t), \beta^2(t)$ are the intensity of the white noise at time t . We will study the sufficient conditions for existence of positive periodic solution of system (4) by using the periodic theory of Has'minskii.

The structure of this paper is as follows. In Section 2, we establish the existence of unique positive global solution for system (2). If the noise is relatively small, there is a stationary distribution. Conditions for non-persistence of the system are established. In Section 3, we explore the existence of periodic solution of system (4) provided the coefficients of the system are continuous periodic functions.

Throughout this paper, let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while \mathcal{F}_0 contains all P-null sets). We denote by R_+^n the positive cone in R^n , that is $R_+^n = \{x \in R^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$.

If $f(t)$ is a continuous bounded function on $[0, +\infty)$, we define

$$f^u = \sup_{t \in [0, +\infty)} f(t), \quad f^l = \inf_{t \in [0, +\infty)} f(t).$$

In general, consider a d -dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad \text{on } t \geq t_0 \tag{5}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^d$. Define the differential operator L associated with equation (5) by

$$L = \frac{\partial}{\partial t} + \sum_{k=1}^d f_k(x, t) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,j=1}^d [g^T(x, t)g(x, t)]_{kj} \frac{\partial^2}{\partial x_k \partial x_j}.$$

2. Stationary distribution and non-persistence of system (2)

2.1. Existence and uniqueness of the global positive solution

In order for the model to make sense, we need to show the solution is global and nonnegative. However, theorem of existence and uniqueness (cf. Mao [15]) is not satisfied in (1.2). Using the Lyapunov analysis method ([13]), we will show the existence and uniqueness of the global positive solution of (1.2).

Theorem 2.1. *For any given initial value $(x_0, y_0) \in R_+^2$, there exists a unique solution $(x(t), y(t))$ to (2) and the solution will remain in R_+^2 with probability 1, that is, $(x(t), y(t)) \in R_+^2$ for all $t \geq 0$ almost surely.*

Proof. First, consider the equation

$$\begin{cases} du(t) = \left[a - \frac{\alpha^2}{2} - be^{u(t)} - \frac{ce^{v(t)}}{m_1 + e^{u(t)}} \right] dt + \alpha dB_1(t), \\ dv(t) = \left[r - \frac{\beta^2}{2} - \frac{fe^{v(t)}}{m_2 + e^{u(t)}} \right] dt + \beta dB_2(t). \end{cases} \tag{6}$$

For any given initial value $u(0) = \log x(0), v(0) = \log y(0)$, there exists a unique local solution $(u(t), v(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [14, 15]. By Itô's formula, it is easy to see that $x(t) = e^{u(t)}, y(t) = e^{v(t)}$ is the unique locally positive solution of (2) with initial value $(x(0), y(0)) \in R_+^2$. To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 > 0$ be sufficient large so that x_0 and y_0 lie within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, we define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_e) : x(t) \notin (\frac{1}{m}, m) \text{ or } y(t) \notin (\frac{1}{m}, m)\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual \emptyset denotes the empty set). It is clearly that τ_m is increasing as $m \rightarrow \infty$. Denote $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$, whence $\tau_\infty \leq \tau_e$. It is easy to show that $\tau_\infty = \infty$ a.s. implies

$\tau_\epsilon = \infty$ a.s. Therefore, to complete this proof, it is enough to show that $\tau_\infty = \infty$ a.s. If this statement is not true, there will exist a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_m \leq T\} \geq \epsilon. \tag{7}$$

Define a C^2 function $V(x, y)$ as follows:

$$V(x, y) = x + 1 - \log x + (y + 1 - \log y).$$

By Itô's formula, we get

$$dV(x, y) = LV(x, y)dt + \alpha(x - 1)dB_1(t) + \beta(y - 1)dB_2(t), \tag{8}$$

where

$$\begin{aligned} LV(x, y) &= (x - 1)\left(a - bx - \frac{cy}{m_1 + x}\right) + \frac{\alpha^2}{2} + (y - 1)\left(r - \frac{fy}{m_2 + x}\right) + \frac{\beta^2}{2} \\ &= -bx^2 + (a + b)x - a - \frac{cxy - cy}{m_1 + x} + ry - \frac{fy^2}{m_2 + x} - r + \frac{fy}{m_2 + x} + \frac{\alpha^2}{2} + \frac{\beta^2}{2} \\ &\leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} + (a + b)x + \left(r + \frac{c}{m_1} + \frac{f}{m_2}\right)y \\ &= c_1 + c_2x + c_3y \\ &\leq c_1 + 2c_2(x + 1 - \log x) + 2c_3(y + 1 - \log y), \end{aligned}$$

where $c_1 = \frac{\alpha^2}{2} + \frac{\beta^2}{2}$, $c_2 = a + b$, $c_3 = r + \frac{c}{m_1} + \frac{f}{m_2}$, and inequality $z \leq 2(z + 1 - \log z) - (4 - 2 \log 2)$, ($z \geq 0$) is used in the last inequality. Therefore,

$$LV(x, y) \leq c_1 + c_4V(x, y) \leq c_5(1 + V(x, y)), \tag{9}$$

where $c_4 = \max\{2c_2, 2c_3\}$, $c_5 = \max\{c_1, c_4\}$. Substituting (9) into (8), we get

$$dV(x, y) \leq c_5(1 + V(x, y)) + \alpha(x - 1)dB_1(t) + \beta(y - 1)dB_2(t).$$

Hence for any $0 \leq t_1 \leq T$, we have

$$\begin{aligned} &\int_0^{\tau_m \wedge t_1} dV(x(t), y(t)) \\ &\leq \int_0^{\tau_m \wedge t_1} c_5(1 + V(x(t), y(t)))dt + \int_0^{\tau_m \wedge t_1} \alpha(x - 1)dB_1(t) + \int_0^{\tau_m \wedge t_1} \beta(y - 1)dB_2(t). \end{aligned}$$

Taking expectation of both sides, yields

$$\begin{aligned} E[V(x(\tau_m \wedge t_1)), (y(\tau_m \wedge t_1))] &\leq V(x(0), y(0)) + c_5E \int_0^{\tau_m \wedge t_1} V(x(t), y(t))dt \\ &\leq V(x(0), y(0)) + c_5T + c_5 \int_0^{t_1} EV(x(t \wedge \tau_m), y(t \wedge \tau_m))dt. \end{aligned}$$

The Gronwall's inequality yields that

$$E[V(x(\tau_m \wedge T)), (y(\tau_m \wedge T))] \leq c_6,$$

where $c_6 = (V(x(0), y(0)) + c_5T)e^{c_5T}$. Set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_0$, due to (7), we have $P(\Omega_m \geq \epsilon)$. Note that for every $\omega \in \Omega_m$, at least one of $x(\tau_m, \omega)$, $y(\tau_m, \omega)$ equal to m or $\frac{1}{m}$. It follows from that

$$\begin{aligned} c_6 &\geq E[I_{\Omega_m} V(x(\tau_m), y(\tau_m))] \\ &\geq \epsilon \left[(m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 - \log \frac{1}{m} \right) \right], \end{aligned}$$

here I_{Ω_m} is the indicator function of Ω_m . Letting $m \rightarrow \infty$ yields the contradiction $\infty > c_6 = \infty$. Therefore we obtain that $\tau_\infty = \infty$ a.s. This completes the proof of Theorem 2.1. \square

2.2. Existence of stationary distribution

In this section, we prove the existence of stationary distribution of system (2). First, we cite a result from [16] as a lemma.

Let $X(t)$ be a homogeneous Markov Process in E_l (E_l denotes l dimensional Euclidean space), and is described by the following stochastic equation:

$$dX(t) = b(X)dt + \sum_{r=1}^k g_r(X)dB_r(t). \tag{10}$$

The diffusion matrix is defined as follows:

$$\Lambda(x) = (\lambda_{ij}(x)), \quad \lambda_{ij}(x) = \sum_{r=1}^k g_r^i(x)g_r^j(x).$$

Assumption (A): Assume that there exists a bounded domain $U \in E_l$ with regular boundary Γ , having the following properties:

A_1 : In the domain U and some neighborhood thereof, the smallest eigenvalue of the the diffusion matrix $\Lambda(x)$ is bounded away from zero.

A_2 : If $x \in E_l \setminus U$, the mean time τ at which a path emerging from x reaches the set U is finite, and $\sup_{x \in G} E_x \tau < \infty$

for every compact subset $G \in E_l$.

Lemma 2.2. [16] If Assumption (A) holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ . Then

$$P_x \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(X(t))dt = \int_{E_l} f(x)\mu(dx) \right\} = 1$$

for all $x \in E_l$.

Remark 2.3. In order to verify (A_1) , it is sufficient to prove that F is uniformly elliptical in U , where $Fu = b(x) \cdot u_x + \frac{1}{2}tr(A(x)u_{xx})$, that is, there is a positive number M such that $\sum_{i,j=1}^l a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2$, $x \in U$, $\xi \in R^l$ (see

[17], Chapter 3, p.103). To validate (A_2) , it is sufficient to verify that there exist a nonnegative C^2 -function V such that LV is negative for any $E_l \setminus U$ (see [18], p.1163).

Theorem 2.4. Assume that $0 < \frac{(r-\frac{\beta^2}{2})m_2}{f} < \frac{(a-\frac{\alpha^2}{2})m_1}{c}$, then for any initial value $(x(0), y(0)) \in R_+^2$, there is a stationary distribution $\mu(\cdot)$ for system (2) and it has ergodic property.

Proof. Introduce a nonnegative C^2 -function

$$V(x, y) = g(x, y) + cy - g(x^*, y^*),$$

where

$$g(x, y) = rx - \frac{km_1}{c} \log x + \frac{km_2}{f} \log y + \frac{1}{y^\rho},$$

$(x^*, y^*) = (\frac{km_1}{rc}, (\frac{km_2}{\rho f})^{-\frac{1}{\rho}})$ is the unique minimum point of function $g(x, y)$, and k and ρ are positive constants chosen in the following proof. Applying Itô's formula, we get

$$LV = rx \left(a - bx - \frac{cy}{m_1 + x} \right) + cy \left(r - \frac{fy}{m_2 + x} \right) + \frac{kbm_1}{c}x + \frac{k(m_1 - m_2)xy}{(m_1 + x)(m_2 + x)} - \rho \left(r - \frac{fy}{m_2 + x} \right) y^{-\rho} + \frac{1}{2} \rho(\rho + 1) \beta^2 y^{-\rho} - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right]. \tag{11}$$

Consider the following bounded subset

$$U = \{(x, y) \in \mathbb{R}_+^2, \epsilon \leq x \leq \frac{1}{\epsilon}, \epsilon \leq y \leq \frac{1}{\epsilon}\}.$$

Then

$$\mathbb{R}_+^2 \setminus U = U_1 \cup U_2 \cup U_3 \cup U_4,$$

where

$$U_1 = \left\{ (x, y) \in \mathbb{R}_+^2, x > \frac{1}{\epsilon} \right\}, U_2 = \{(x, y) \in \mathbb{R}_+^2, 0 < x < \epsilon\},$$

$$U_3 = \{(x, y) \in \mathbb{R}_+^2, 0 < y < \epsilon\}, U_4 = \left\{ (x, y) \in \mathbb{R}_+^2, x \geq \epsilon, y > \frac{1}{\epsilon} \right\},$$

ϵ ($0 < \epsilon < 1$) is sufficiently small number satisfying the following conditions

$$M_1 + \frac{|M_2|}{m_2} - \frac{rb}{2\epsilon^2} < -1, \quad (12)$$

$$\frac{(2 + |M_3|)bm_1}{c \left[\frac{(a - \frac{a^2}{2})m_1}{c} - \frac{(r - \frac{\rho^2}{2})m_2}{f} \right]} \epsilon < 1, \quad (13)$$

$$M_4 + \frac{|M_2|}{m_2} - \rho\epsilon^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] < -1, \quad (14)$$

and

$$M_4 + \frac{|M_5|}{m_2} - \frac{cf}{2m_2\epsilon^2 + 2\epsilon} < -1, \quad (15)$$

where M_1, M_2, M_3, M_4 and M_5 are defined in the following proof.

Case 1. If $(x, y) \in U_1$, we have

$$\begin{aligned}
 LV &\leq \left(ra + \frac{kbm_1}{c} \right) x - rbx^2 + \frac{1}{m_2 + x} \left[-cfy^2 + \frac{crm_1(m_2 + x)}{m_1 + x} y + \frac{k(m_1 - m_2)x}{m_1 + x} y \right] \\
 &\quad + \frac{\rho fy^{1-\rho}}{m_2} - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right] \\
 &= \left(ra + \frac{kbm_1}{c} \right) x - rbx^2 + \frac{1}{m_2 + x} \left[-cfy^2 + crm_1 \left(\frac{m_2 - m_1}{m_1 + x} + 1 \right) y + \frac{k(m_1 - m_2)x}{m_1 + x} y \right] \\
 &\quad + \frac{1}{m_2 + x} \left[\rho fy^{1-\rho} + \frac{\rho fxy^{1-\rho}}{m_2} \right] - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right] \\
 &\leq \left(ra + \frac{kbm_1}{c} \right) x - rbx^2 + \frac{1}{m_2 + x} \{ -cfy^2 + [cr(m_1 + m_2) + km_1]y \} \\
 &\quad + \frac{1}{m_2 + x} \left[\rho fy^{1-\rho} + \frac{\rho f}{2m_2} (x^2 + y^{2-2\rho}) \right] - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] \\
 &\quad - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right] \\
 &\leq \left(ra + \frac{kbm_1}{c} \right) x - \frac{1}{2} \left(rb - \frac{\rho f}{m_2^2} \right) x^2 + \frac{-cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2} y^{2-2\rho}}{m_2 + x} \\
 &\quad - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right] - \frac{1}{2} rbx^2.
 \end{aligned} \tag{16}$$

We can choose $\rho (0 < \rho < 1)$ sufficiently small such that

$$rb - \frac{\rho f}{m_2^2} > 0 \tag{17}$$

and

$$r - \frac{1}{2}(\rho + 1)\beta^2 > 0. \tag{18}$$

Inequality (18) is allowed by the assumption $r > \frac{\beta^2}{2}$. Substituting (18) and condition $\frac{(r - \frac{\beta^2}{2})m_2}{f} < \frac{(a - \frac{\alpha^2}{2})m_1}{c}$ into (16) yields

$$\begin{aligned}
 LV &\leq \left(ra + \frac{kbm_1}{c} \right) x - \frac{1}{2} \left(rb - \frac{\rho f}{m_2^2} \right) x^2 + \frac{-cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2} y^{2-2\rho}}{m_2 + x} \\
 &\quad - \frac{1}{2} rbx^2 \\
 &\leq \left(ra + \frac{kbm_1}{c} \right) x - \frac{1}{2} \left(rb - \frac{\rho f}{m_2^2} \right) x^2 + \frac{M_2}{m_2 + x} - \frac{rb}{2\epsilon^2} \\
 &\leq M_1 + \frac{|M_2|}{m_2} - \frac{rb}{2\epsilon^2},
 \end{aligned}$$

where

$$M_1 = \sup_{x \in (0, \infty)} \{ (ra + \frac{kbm_1}{c})x - \frac{1}{2} (rb - \frac{\rho f}{m_2^2})x^2 \} < \infty$$

and

$$M_2 = \sup_{y \in (0, \infty)} \{-cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2}y^{2-2\rho}\} < \infty.$$

It then follows from (12) that

$$LV < -1.$$

Case 2. If $(x, y) \in U_2$, that is $0 < x < \epsilon < 1$, then

$$\begin{aligned} LV &\leq ra + cry - \frac{cfy^2}{m_2 + 1} + \frac{k}{m_2}y + \frac{\rho f}{m_2}y^{1-\rho} - k \left[\frac{(a - \frac{\alpha^2}{2})}{c} - \frac{(r - \frac{\beta^2}{2})}{f} \right] + \frac{kbm_1\epsilon}{c} \\ &\leq |M_3| - k \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right] + \frac{kbm_1\epsilon}{c}, \end{aligned}$$

where

$$M_3 = \sup_{y \in (0, \infty)} (ra + cry - \frac{cfy^2}{m_2 + 1} + \frac{k}{m_2}y + \frac{\rho f}{m_2}y^{1-\rho}).$$

Choosing $k = \frac{2 + |M_3|}{\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f}}$, we have

$$LV \leq -2 + \frac{(2 + |M_3|)bm_1}{c \left[\frac{(a - \frac{\alpha^2}{2})m_1}{c} - \frac{(r - \frac{\beta^2}{2})m_2}{f} \right]} \epsilon.$$

It then follows from (31) that

$$LV < -1.$$

Case 3. If $(x, y) \in U_3$,

$$\begin{aligned} LV &\leq \left(ra + \frac{kbm_1}{c} \right) x - rbx^2 + \frac{1}{m_2 + x} \{-cfy^2 + [cr(m_1 + m_2) + km_1]y\} \\ &\quad + \frac{1}{m_2 + x} \left[\rho fy^{1-\rho} + \frac{\rho f}{2m_2}(x^2 + y^{2-2\rho}) \right] - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] \\ &\leq \left(ra + \frac{kbm_1}{c} \right) x - \left(rb - \frac{\rho f}{2m_2^2} \right) x^2 + \frac{-cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2}y^{2-2\rho}}{m_2 + x} \\ &\quad - \rho y^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] \\ &\leq M_4 + \frac{M_2}{m_2 + x} - \rho \epsilon^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right] \\ &\leq M_4 + \frac{|M_2|}{m_2} - \rho \epsilon^{-\rho} \left[r - \frac{1}{2}(\rho + 1)\beta^2 \right], \end{aligned}$$

where

$$M_4 = \sup_{x \in (0, \infty)} \left\{ \left(ra + \frac{kbm_1}{c} \right) x - \left(rb - \frac{\rho f}{2m_2^2} \right) x^2 \right\} < \infty.$$

By (14), we have

$$LV < -1.$$

Case 4. If $(x, y) \in U_4$,

$$\begin{aligned} LV &\leq \left(ra + \frac{kbm_1}{c} \right) x - \left(rb - \frac{\rho f}{2m_2^2} \right) x^2 + \frac{-cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2} y^{2-2\rho}}{m_2 + x} \\ &\leq M_4 + \frac{1}{m_2 + x} \left(-\frac{cf}{2} y^2 + M_5 \right) \\ &\leq M_4 + \frac{|M_5|}{m_2} - \frac{cf}{2(m_2 + x)} y^2 \\ &\leq M_4 + \frac{|M_5|}{m_2} - \frac{\frac{cf}{\epsilon^2}}{2m_2 + \frac{2}{\epsilon}} \\ &= M_4 + \frac{|M_5|}{m_2} - \frac{cf}{2m_2\epsilon^2 + 2\epsilon}, \end{aligned}$$

where

$$M_5 = \sup_{y \in (0, \infty)} \left\{ -\frac{1}{2}cfy^2 + [cr(m_1 + m_2) + km_1]y + \rho fy^{1-\rho} + \frac{\rho f}{2m_2} y^{2-2\rho} \right\}.$$

By (15), we derive that

$$LV < -1.$$

According to the discussion above, we have

$$LV < -1$$

for any $(x, y) \in R_+^2 \setminus U$. Hence condition (A_2) in Lemma 2.1 is satisfied.

Besides, choosing

$$M = \min\{\alpha^2 x^2, \beta^2 y^2, (x, y) \in \bar{U}\} > 0,$$

then we have

$$\sum_{i,j=1}^2 a_{ij}(x, y) \xi_i \xi_j = \alpha^2 x^2 \xi_1^2 + \beta^2 y^2 \xi_2^2 \geq M |\xi|^2,$$

which means that condition (A_1) is satisfied. By Lemma 2.1, the desired results can be obtained. \square

Remark 2.5. In Ref [6, 7], Ji and Jiang investigated the dynamic of (3), which is a special case of system (2). They show that there is a stationary distribution $\mu(\cdot)$ for system (3) and it has ergodic property provided the following conditions hold:

(H1) $\alpha > 0, \beta > 0$;

$$(H2) \delta < \min \left\{ \frac{bfm-cr}{f} \left[x^* + \frac{f}{4(bfm-cr)} (x^* \alpha^2 + \frac{cy^* \beta^2}{r}) \right]^2, \frac{cf(y^*)^2}{r} \right\},$$

where (x^*, y^*) is the interior equilibrium of the corresponding deterministic system of (3) and $\delta = \frac{f}{16(bfm-cr)} (x^* \alpha^2 + \frac{cy^* \beta^2}{r})^2 + \frac{1}{2} (x^* + m) (x^* \alpha^2 + \frac{cy^* \beta^2}{r})$. Theorem 2.2 in our investigation shows that if the intensities of the white noise are small, only condition $\frac{(r-\frac{\beta^2}{2})m_2}{f} < \frac{(a-\frac{\alpha^2}{2})m_1}{c}$ is required, without other conditions imposed on the coefficients. Therefore, Theorem 2.2 in large improves Theorem 2.1 in [7]. Moreover, we see that if $\alpha = 0, \beta = 0$, the above condition is reduced to $\frac{rm_2}{f} < \frac{am_1}{c}$, which is the condition for the existence of interior equilibrium of system (1). This means that the existence of white noise is beneficial to the stability of the system.

2.3. Non-persistence

In this section, we will discuss the non-persistence of system (2).

First, we consider the following stochastic equation

$$\begin{cases} dX(t) = X(t)(a - bX(t)) dt + \alpha X(t)dB_1(t), \\ X(0) = x(0). \end{cases} \tag{19}$$

Lemma 2.6. [6] Suppose that $a > \frac{\alpha^2}{2}$, then for any initial value $x(0) > 0$, the solution of (19) has the following properties:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\log X(t)}{t} &= 0, \text{ a.s.}, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X(s)ds &= \frac{a - \frac{\alpha^2}{2}}{b}, \text{ a.s.} \end{aligned}$$

Theorem 2.7. Suppose that $(x(t), y(t))$ is the solution of (2) with initial value $(x_0, y_0) \in R_+^2$.

- (1) If $a - \frac{\alpha^2}{2} < 0, r - \frac{\beta^2}{2} > 0$, then $\lim_{t \rightarrow \infty} x(t) = 0, \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s)ds = \frac{m_2(r - \frac{\beta^2}{2})}{f}$ a.s.;
- (2) If $a - \frac{\alpha^2}{2} > 0, r - \frac{\beta^2}{2} < 0$, then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s)ds = \frac{a - \frac{\alpha^2}{2}}{b}, \lim_{t \rightarrow \infty} y(t) = 0$ a.s.;
- (3) If $a - \frac{\alpha^2}{2} < 0, r - \frac{\beta^2}{2} < 0$, then $\lim_{t \rightarrow \infty} x(t) = 0, \lim_{t \rightarrow \infty} y(t) = 0$ a.s.

Proof. Clearly,

$$dx(t) \leq x(t)(a - bx(t)) dt + \alpha x(t)dB_1(t).$$

By the comparison theorem, we derive that

$$x(t) \leq \Phi(t), \tag{20}$$

where $\Phi(t)$ is the solution of the following equation

$$\begin{cases} d\Phi(t) = \Phi(t)(a - b\Phi(t)) dt + \alpha\Phi(t)dB_1(t), \\ \Phi(0) = x(0). \end{cases} \tag{21}$$

According to Theorem 2.2 in [19], we derive that

$$\Phi(t) = \frac{e^{(a - \frac{\alpha^2}{2})t + \alpha B_1(t)}}{\frac{1}{x_0} + b \int_0^t e^{(a - \frac{\alpha^2}{2})s + \alpha B_1(s)} ds}. \tag{22}$$

Case 1: (22) and (20) show that

$$x(t) \leq \Phi(t) \leq x(0)e^{a - \frac{\alpha^2}{2}t + B_1(t)}.$$

If $a - \frac{\alpha^2}{2} < 0$, obviously

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Then for any $\varepsilon > 0$, there exists Ω_ε such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$, for every $\omega \in \Omega_\varepsilon$, there exists $t_0 = t_0(\omega) > 0$ such that

$$\frac{x(t)}{m_2 + x(t)} \leq \varepsilon \text{ whenever } t \geq t_0(\omega).$$

Hence we have

$$\begin{aligned} dy(t) &= y(t) \left(r - \frac{fy(t)}{m_2 + x(t)} \right) dt + \beta y(t) dB_2(t) \\ &= y(t) \left[r - \frac{f}{m_2} y(t) + \frac{fx(t)y(t)}{m_2(m_2 + x(t))} \right] dt + \beta y(t) dB_2(t) \\ &\leq y(t) \left[r - \frac{f}{m_2} (1 - \epsilon) y(t) \right] dt + \beta y(t) dB_2(t). \end{aligned}$$

On the other hand

$$dy(t) \geq y(t) \left(r - \frac{f}{m_2} y(t) \right) dt + \beta y(t) dB_2(t).$$

If $r - \frac{\beta^2}{2} > 0$, according to Lemma 2.2 and comparison theorem, we have

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds &\geq \frac{m_2(r - \frac{\beta^2}{2})}{f} \text{ a.s.} \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds &\leq \frac{m_2(r - \frac{\beta^2}{2})}{f(1 - \epsilon)} \text{ a.s.} \end{aligned}$$

For the arbitrary of ϵ , we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds = \frac{m_2(r - \frac{\beta^2}{2})}{f} \text{ a.s.}$$

Case 2: Take (20) into the second equation of (2), we have

$$dy(t) \leq y(t) \left(r - \frac{fy(t)}{m_2 + \Phi(t)} \right) dt + \beta y(t) dB_2(t),$$

which implies

$$y(t) \leq \Psi(t), \tag{23}$$

where $\Psi(t)$ is the solution of the following equation

$$\begin{cases} d\Psi(t) = \Psi(t) \left(r - \frac{f\Psi(t)}{m_2 + \Phi(t)} \right) dt + \beta \Psi(t) dB_2(t), \\ \Psi(0) = y(0). \end{cases} \tag{24}$$

The solution of (24) is

$$\Psi(t) = \frac{e^{(r - \frac{\beta^2}{2})t + \beta B_2(t)}}{\frac{1}{y_0} + f \int_0^t \frac{1}{m_2 + \Phi(s)} e^{(r - \frac{\beta^2}{2})s + \beta B_2(s)} ds}. \tag{25}$$

(23) and (25) show that

$$y(t) \leq \Psi(t) \leq y(0) e^{(r - \frac{\beta^2}{2})t + \beta B_2(t)}.$$

If $r - \frac{\beta^2}{2} < 0$, we get

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

Then for any $\varepsilon > 0$, there exists Ω_ε such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$, for every $\omega \in \Omega_\varepsilon$, there exists $t_1 = t_1(\omega) > 0$ such that

$$\frac{cy(t)}{m_1 + x(t)} \leq \varepsilon, t \geq t_1(\omega).$$

Hence we have

$$dx(t) \leq x(t)(a - bx(t))dt + \alpha x(t)dB_1(t),$$

and

$$dx(t) \geq x(t)(a - bx(t) - \varepsilon)dt + \alpha x(t)dB_1(t).$$

If $a - \frac{\alpha^2}{2} > 0$, according to Lemma 2.2 and comparison theorem, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x(s)ds = \frac{a - \frac{\alpha^2}{2}}{b} \text{ a.s.}$$

Case 3: Basing on the discussion in case 1 and case 2, if $a - \frac{\alpha^2}{2} < 0, r - \frac{\beta^2}{2} < 0$, it follows that

$$\lim_{t \rightarrow \infty} x(t) = 0, \lim_{t \rightarrow \infty} y(t) = 0 \text{ a.s.}$$

□

2.4. Simulations

In this section, we will illustrate our findings by the method developed in [20]. Consider the corresponding discretization equation:

$$\begin{cases} x_{k+1} = x_k + x_k \left[\left(a - bx_k - \frac{cy_k}{m_1 + x_k} \right) \Delta t + \alpha \varepsilon_{1,k} \sqrt{\Delta t} + \frac{\alpha^2}{2} (\varepsilon_{1,k}^2 \Delta t - \Delta t) \right], \\ y_{k+1} = y_k + y_k \left[\left(r - \frac{fy_k}{m_2 + x_k} \right) \Delta t + \beta \varepsilon_{2,k} \sqrt{\Delta t} + \frac{\beta^2}{2} (\varepsilon_{2,k}^2 \Delta t - \Delta t) \right], \end{cases}$$

where $\varepsilon_{1,k}$ and $\varepsilon_{2,k}$ are the Gaussian random variables $N(0, 1)$. Parameters are listed in the following Table 1 and initial values are $x(0) = 1, y(0) = 1.5$, we get simulations by the help of Matlab.

Table 1: List of parameters

Parameters	Description	Values	Source
a	Intrinsic growth rate of the prey	1	[3, 4]
b	Strength of competition among the prey	0.06	[3, 5]
c	The maximum value of reduction rate of prey	1	[3]
m_1	The extent of protection to prey by environment	10	[4]
r	Growth rate of the predator	0.03	Assumed
f	The maximum value of reduction rate of predator	1	[3]
m_2	The extent of protection to predator by environment	20	[3, 4]

First, we start the numerical simulation with environmental intensities $\alpha = 0.05, \beta = 0.05$, parameters satisfy conditions $a > \frac{\alpha^2}{2}, r > \frac{\beta^2}{2}$ and $\frac{(r - \frac{\beta^2}{2})m_2}{f} < \frac{(a - \frac{\alpha^2}{2})m_1}{c}$. Hence Theorem 2.2 is satisfied. In the left figure, we show that the population densities fluctuate around the the equilibrium (x_1^*, x_2^*) of the deterministic system. Stationary distribution of x and y is provided in the right figure (see the histogram on the right in Fig.1).

In Fig.2, we choose the same parameters as in Figure 1, but change the intensities of the white noise ($\alpha = 1.5, \beta = 0.01$), case 1 in Theorem 2.3 is satisfied. We can see that species $x(t)$ will die out, and species $y(t)$ will be persistent.

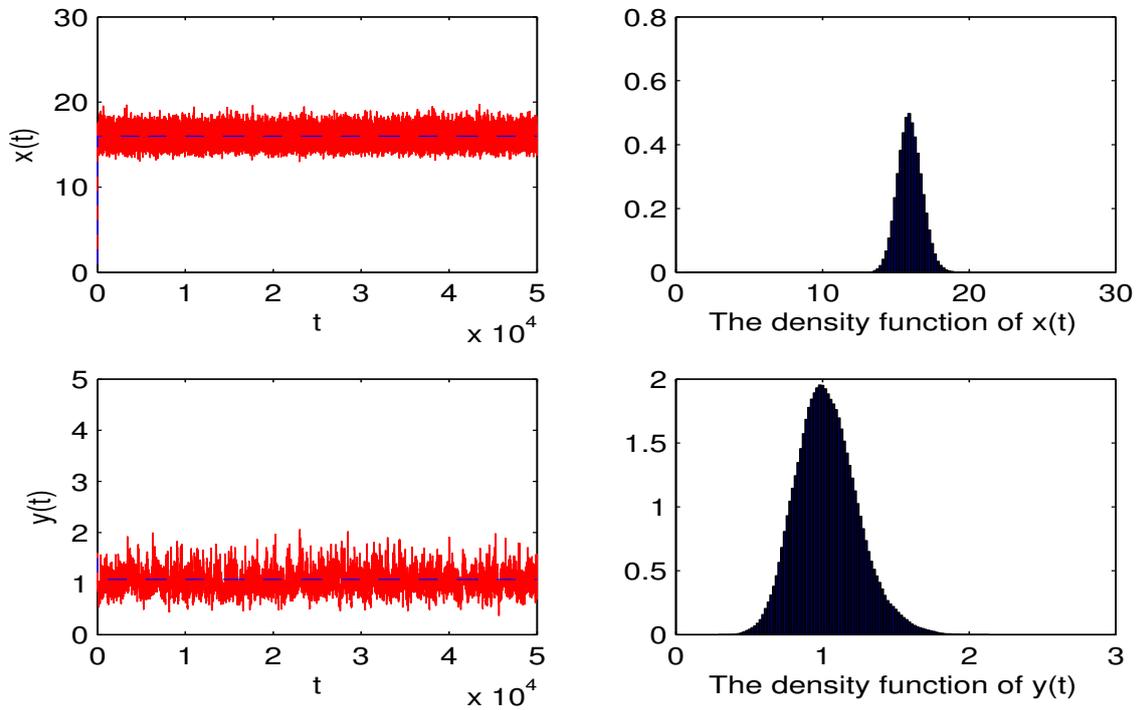


Figure 1: The solution $x(t)$ of model (2) compared to the deterministic system with $(\alpha, \beta) = (0.05, 0.05)$ and its histogram. The red line represents the solution of stochastic system, and the blue line represents the solution of the corresponding deterministic system.

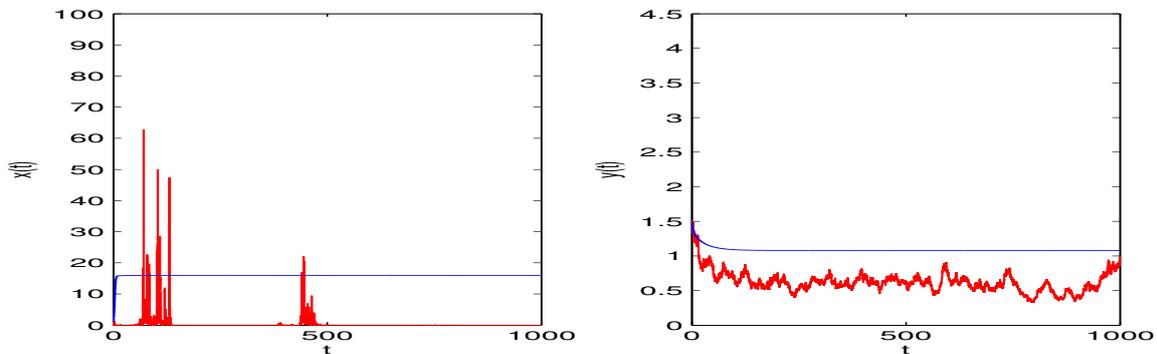


Figure 2: The solution $x(t)$ of model (2) compared to the deterministic system with $(\alpha, \beta) = (1.5, 0.01)$. The red line represents the solution of stochastic system, and the blue line represents the solution of the corresponding deterministic system.

In Fig.3, we choose intensities of the white noise $\alpha = 0.1, \beta = 1.3$, thus case 2 in Theorem 2.3 is satisfied. We see that species $y(t)$ will die out, and species $x(t)$ will be persistent.

In Fig.4, we increase the intensities of the white noise ($\alpha = 1.5, \beta = 1.3$), parameters satisfy case 3 in Theorem 2.3. In this case, both of the two species go to extinction after some initial large amplitude oscillation. However, the corresponding deterministic system is persistent. These show that the strong white noise may make a persistent system to be extinct.

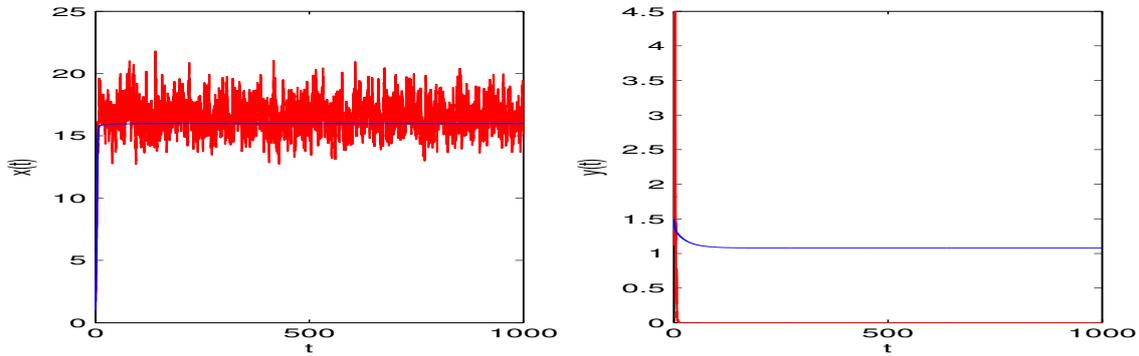


Figure 3: The solution $x(t)$ of model (2) compared to the deterministic system with $(\alpha, \beta) = (0.1, 1.3)$. The red line represents the solution of stochastic system, and the blue line represents the solution of the corresponding deterministic system.

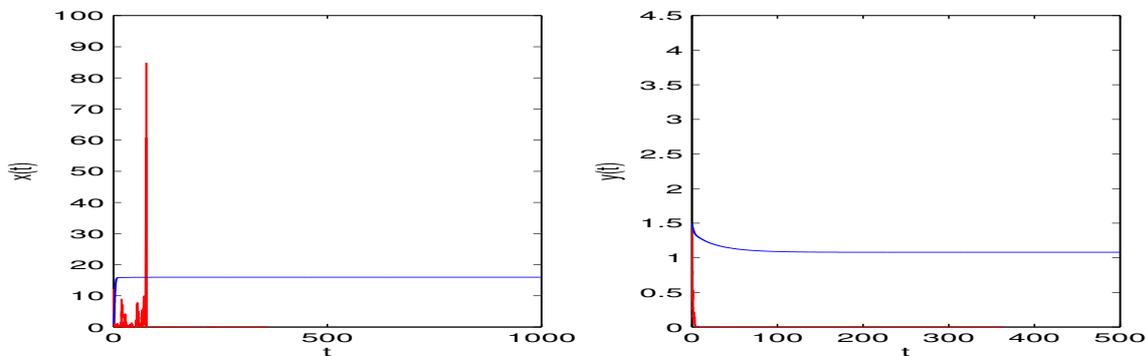


Figure 4: The solution $x(t)$ of model (2) compared to the deterministic system with $(\alpha, \beta) = (1.5, 1.3)$. The red line represents the solution of stochastic system, and the blue line represents the solution of the corresponding deterministic system.

3. Existence of periodic solution of system (4)

3.1. Existence and uniqueness of the global positive solution

Theorem 3.1. For any given initial value $(x_0, y_0) \in R_+^2$, there exists a unique solution $(x(t), y(t))$ to (4) and the solution will remain in R_+^2 with probability 1, that is, $(x(t), y(t)) \in R_+^2$ for all $t \geq 0$ almost surely.

The proof is a modification of the autonomous case (Theorem 2.1) hence is omitted.

3.2. Existence of θ -periodic solution

In this section, we will discuss the existence of positive periodic solution of system (4). First, we assume that

(H) : $a(t), b(t), c(t), m_1(t), r(t), f(t), m_2(t), \alpha(t)$ and $\beta(t)$ are positive continuous θ -periodic functions.

Now, we shall present some definitions, lemmas which are used in the follows.

Definition 3.2. [21] A stochastic process $\xi(t) = \xi(t, \omega) (-\infty < t < \infty)$ is said to be periodic with period θ if for every finite sequence of numbers t_1, t_2, \dots, t_n the joint distribution of random variables $\xi(t_1 + h), \dots, \xi(t_n + h)$ is independent of h , where $h = k\theta (k = \pm 1, \pm 2, \dots)$.

Remark 3.3. In [21], they show that a Markov process $x(t)$ is θ -periodic if and only if its transition probability function is θ -periodic and the function $P_0(t, A) = P\{X(t) \in A\}$ satisfies the equation

$$P_0(s, A) = \int_{R^l} P_0(s, dx)P(s, x, s + \theta, A) \equiv P_0(s + \theta, A)$$

for every $A \in \mathfrak{B}$, where \mathfrak{B} denotes the σ -algebra of Borel sets

Consider the following equation

$$X(t) = X(t_0) + \int_{t_0}^t b(s, X(s))ds + \sum_{r=1}^k \int_{t_0}^t \sigma_r(s, X(s))dB_r(s), \tag{26}$$

assume that the coefficients $b(s, x), \sigma_1(s, x), \sigma_2(s, x), \dots, \sigma_r(s, x)$ satisfy the following conditions:

$$|b(s, x) - b(s, y)| + \sum_{r=1}^k |\sigma_r(s, x) - \sigma_r(s, y)| \leq B|x - y|, \quad |b(s, x)| + \sum_{r=1}^k |\sigma_r(s, x)| \leq B(1 + |x|), \tag{27}$$

where B is a constant.

Lemma 3.4. [21] Suppose that the coefficients of (26) are θ -periodic in t and satisfy condition (27) in every cylinder $I \times U$, and suppose further that there exists a function $V(t, x) \in C^2$ which is θ -periodic in t , and satisfies the following conditions

$$\inf_{|x|>R} V(t, x) \rightarrow \infty \text{ as } R \rightarrow \infty, \tag{28}$$

$$LV(t, x) \leq -1 \tag{29}$$

outside some compact set. Then there exists a solution of (26) which is a θ -periodic Markov process.

Remark 3.5. According to the proof of Lemma 3.1, we note that linear growth condition is only used to guarantee the existence and uniqueness of the solution of (26).

Theorem 3.6. Suppose that condition (H) holds. Assume further that

- (A) $\int_0^\theta (r(s) - \frac{1}{2}\beta^2(s))ds > 0,$
- (B) $\int_0^\theta \left[\frac{m_1^l}{c^u} (a(s) - \frac{\alpha^2(s)}{2}) - \frac{m_2^u}{f^l} (r(s) - \frac{\beta^2(s)}{2}) \right] ds > 0,$

then there exists a θ periodic solution of (4).

Proof. Since the coefficients of system (4) satisfy the local Lipschitz condition and the existence and uniqueness of the positive solution of (4) has been guaranteed by Theorem 3.1, to prove Theorem 3.2 we only need to show that conditions (28) and (29) are satisfied. Define a C^2 function $V(t, x, y)$ as follows:

$$\begin{aligned} V(t, x, y) &= (x - k \frac{m_1^l}{c^u} \log x + y) + (k \frac{m_2^u}{f^l} \log y + \frac{e^{\rho\omega_1(t)}}{y^\rho} + k\omega_2(t)) \\ &:= V_1(t, x, y) + V_2(t, x, y), \end{aligned}$$

where $\omega_i(t) \in C^1(R_+, R)(i = 1, 2)$ is a θ -periodic function which will be determined in the following proof, and ρ is a sufficient small positive number satisfying

$$\frac{1}{\theta} \int_0^\theta (r(s) - \frac{1}{2}\beta^2(s))ds - \frac{\rho}{2}\beta^{2u} > 0. \tag{30}$$

We note that

$$\liminf_{k \rightarrow \infty, (x,y) \in R_+^2 \setminus D_k} V(t, x, y) = \infty, \tag{31}$$

where

$$D_k = \left\{ (x, y), (x, y) \in \left(\frac{1}{k}, k \right) \times \left(\frac{1}{k}, k \right) \right\}.$$

Applying Itô's formula, we get

$$\begin{aligned} LV_1(t, x, y) &= x \left(a(t) - b(t)x - \frac{c(t)y}{m_1(t) + x} \right) - \frac{km_1^l}{c^u} \left[\left(a(t) - \frac{\alpha^2(t)}{2} \right) - b(t)x - \frac{c(t)y}{m_1(t) + x} \right] \\ &\quad + y \left(r(t) - \frac{f(t)y}{m_2(t) + x} \right) \\ &\leq \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{km_1^l y}{m_1(t) + x(t)} + \frac{1}{m_2(t) + x} (-f^l y^2 + m_2^u r^u y + r^u xy) \\ &\quad - \frac{km_1^l}{c^u} \left(a(t) - \frac{\alpha^2(t)}{2} \right) \end{aligned} \tag{32}$$

and

$$\begin{aligned} LV_2(t, x, y) &= k \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} - \frac{f(t)y}{m_2(t) + x} \right) + \rho e^{\rho \omega_1(t)} \omega_1'(t) y^{-\rho} \\ &\quad - \rho e^{\rho \omega_1(t)} y^{-\rho} \left[r(t) - \frac{1}{2}(\rho + 1)\beta^2(t) - \frac{f(t)y}{m_2(t) + x} \right] + k \omega_2'(t) \\ &\leq k \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - k \frac{m_2^u y}{m_2(t) + x} + \frac{\tilde{\rho}}{m_2(t) + x} y^{1-\rho} \\ &\quad - \rho e^{\rho \omega_1(t)} y^{-\rho} \left[r(t) - \frac{1}{2}(\rho + 1)\beta^2(t) - \omega_1'(t) + \rho(r(t) - \frac{1}{2}(\rho + 1)\beta^2(t)) \right] + k \omega_2'(t) \\ &\leq k \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - k \frac{m_2^u y}{m_2(t) + x} + \rho(\rho + e^{\rho \omega_1(t)}) \frac{f^u}{m_2(t) + x} y^{1-\rho} \\ &\quad - \rho e^{\rho \omega_1(t)} y^{-\rho} \left[r(t) - \frac{1}{2}\beta^2(t) - \frac{\rho}{2}\beta^{2u} - \omega_1'(t) + \rho(r^u + \frac{1}{2}(\rho + 1)\beta^{2u}) \right] + k \omega_2'(t), \end{aligned} \tag{33}$$

where $\tilde{\rho} = \rho e^{\rho \omega_1^u} f^u$.

Let

$$\omega_1'(t) = \left(r(t) - \frac{1}{2}\beta^2(t) \right) - \frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2}\beta^2(s) \right) ds. \tag{34}$$

Then $\omega_1(t)$ is a θ periodic function. In fact,

$$\begin{aligned} \omega_1(t + \theta) - \omega_1(t) &= \int_t^{t+\theta} \omega_1'(s) ds \\ &= \int_t^{t+\theta} \left(r(s) - \frac{1}{2}\beta^2(s) \right) ds - \int_0^\theta \left(r(s) - \frac{1}{2}\beta^2(s) \right) ds \\ &= 0. \end{aligned} \tag{35}$$

Substituting (34) into (33) yields

$$\begin{aligned}
 LV_2(t, x, y) \leq & k \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - k \frac{m_2^u y}{m_2(t) + x} + \frac{\tilde{\rho}}{m_2(t) + x} y^{1-\rho} \\
 & - \rho e^{\rho \omega_1^l} y^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right] + k \omega_2'(t).
 \end{aligned}
 \tag{36}$$

Then

$$\begin{aligned}
 LV(t, x, y) \leq & \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{km_1^l y}{m_1(t) + x(t)} + \frac{1}{m_2(t) + x} (-f^l y^2 + m_2^u r^u y + r^u xy) \\
 & - \frac{km_1^l}{c^u} \left(a(t) - \frac{\alpha^2(t)}{2} \right) + k \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - k \frac{m_2^u y}{m_2(t) + x} + \frac{\tilde{\rho}}{m_2(t) + x} y^{1-\rho} \\
 & - \rho e^{\rho \omega_1^l} y^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right] + k \omega_2'(t) \\
 \leq & \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{1}{m_2(t) + x} \left[-f^l y^2 + m_2^u r^u y + r^u xy + \frac{k(m_1^l - m_2^u)xy}{m_1(t) + x} \right] \\
 & + \frac{\tilde{\rho}}{m_2(t) + x} y^{1-\rho} - \rho e^{\rho \omega_1^l} y^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right] \\
 & - k \left[\frac{m_1^l}{c^u} \left(a(t) - \frac{\alpha^2(t)}{2} \right) - \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - \omega_2'(t) \right].
 \end{aligned}
 \tag{37}$$

Let

$$\omega_2'(t) = \frac{m_1^l}{c^u} \left(a(t) - \frac{\alpha^2(t)}{2} \right) - \frac{m_2^u}{f^l} \left(r(t) - \frac{\beta^2(t)}{2} \right) - \frac{1}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds.
 \tag{38}$$

By the similar computation as (35), we show that $\omega_2(t)$ is a θ periodic function. Substituting (38) into (37) gives

$$\begin{aligned}
 LV(t, x, y) \leq & \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{1}{m_2(t) + x} \left[-f^l y^2 + m_2^u r^u y + r^u xy + \frac{k(m_1^l - m_2^u)xy}{m_1(t) + x} \right] \\
 & + \frac{\tilde{\rho}}{m_2(t) + x} y^{1-\rho} - \rho e^{\rho \omega_1^l} y^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right] \\
 & - k \left\{ \frac{1}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds \right\}.
 \end{aligned}
 \tag{39}$$

Consider the following closed set

$$\tilde{U} = \{ (x, y) \in R_+^2, \lambda \leq x \leq \frac{1}{\lambda}, \lambda \leq y \leq \frac{1}{\lambda} \}.$$

Then

$$R_+^2 \setminus \tilde{U} = \tilde{U}_1 \cup \tilde{U}_2 \cup \tilde{U}_3 \cup \tilde{U}_4,$$

where

$$\tilde{U}_1 = \left\{ (x, y) \in R_+^2, x > \frac{1}{\lambda} \right\}, \quad \tilde{U}_2 = \{ (x, y) \in R_+^2, 0 < x < \lambda \},$$

$$\tilde{U}_3 = \{(x, y) \in R_+^2, 0 < y < \lambda\}, \tilde{U}_4 = \left\{ (x, y) \in R_+^2, \lambda < x < \frac{1}{\lambda}, y > \frac{1}{\lambda} \right\},$$

$\lambda(0 < \lambda < 1)$ is sufficiently small number such that

$$H_1 + \frac{|H_2|}{m_2^l} - \frac{b^l}{2\lambda^2} \leq -1, \tag{40}$$

$$-2 + \frac{(2 + |H_3|)m_1^l b^u}{c^u \left[\frac{1}{\theta} \int_0^\theta \left(\frac{m_1^l}{c^u} (a(s) - \frac{\alpha^2(s)}{2}) - \frac{m_2^u}{f^l} (r(s) - \frac{\beta^2(s)}{2}) \right) ds \right]} \lambda \leq -1, \tag{41}$$

$$H_4 + \frac{|H_2|}{m_2^l} - \rho e^{\rho\omega_1} \lambda^{-\rho} \left[\frac{1}{\theta} \int_0^\theta (r(s) - \frac{1}{2}\beta^2(s)) ds - \frac{\rho}{2}\beta^{2u} \right] \leq -1, \tag{42}$$

$$H_4 + \frac{|H_5|}{m_2^l} - \frac{f^l}{4m_2^u \lambda^2 + 4\lambda} \leq -1. \tag{43}$$

Case 1. If $(x, y) \in \tilde{U}_1$, substituting conditions (A) and (B) into (39) yields

$$\begin{aligned} LV(t, x, y) &\leq \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{1}{m_2(t) + x} \left[-f^l y^2 + m_2^u r^u y + r^u x y + \frac{k(m_1^l - m_2^u)xy}{m_1(t) + x} \right] \\ &\leq \left(a^u + \frac{km_1^l b^u}{c^u} \right) x - b^l x^2 + \frac{-f^l y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + r^u (\frac{1}{4\lambda_0} x^2 + \lambda_0 y^2) + k|m_1^l - m_2^u|y}{m_2(t) + x}. \end{aligned} \tag{44}$$

where Young inequality is used in the second inequality. Choosing $\lambda_0 = \frac{f^l}{2r^u}$ yields

$$\begin{aligned} LV(t, x, y) &\leq \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - b^l x^2 + \frac{-\frac{f^l}{2} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + k|m_1^l - m_2^u|y}{m_2(t) + x} \\ &\leq \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - \frac{b^l}{2} x^2 + \frac{-\frac{f^l}{2} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + k|m_1^l - m_2^u|y}{m_2(t) + x} - \frac{b^l}{2} x^2 \\ &\leq H_1 + \frac{|H_2|}{m_2^l} - \frac{b^l}{2\lambda^2}, \end{aligned} \tag{45}$$

where

$$H_1 = \sup_{x \in (0, \infty)} \left\{ \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - \frac{b^l}{2} x^2 \right\} < \infty$$

and

$$H_2 = \sup_{y \in (0, \infty)} \left\{ -\frac{f^l}{2} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + k|m_1^l - m_2^u|y \right\} < \infty.$$

It then follows from (40) that

$$LV \leq -1. \tag{46}$$

Case 2. If $(x, y) \in \tilde{U}_2$, that is $0 < x < \lambda < 1$, we get

$$\begin{aligned}
 LV(t, x, y) &\leq a^u - \frac{f^l}{m_2^u} y^2 + \left[\left(\frac{m_2^u}{m_2^l} + 1 \right) r^u + \frac{k|m_1^l - m_2^u|}{m_2^l} \right] y + \frac{\tilde{\rho}}{m_2^l} y^{1-\rho} + \frac{km_1^l b^u}{c^u} \lambda \\
 &\quad - k \left\{ \frac{1}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds \right\} \\
 &\leq |H_3| - k \left\{ \frac{1}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds \right\} + \frac{km_1^l b^u}{c^u} \lambda.
 \end{aligned} \tag{47}$$

where

$$H_3 = \sup_{y \in (0, \infty)} \left\{ a^u - \frac{f^l}{m_2^u} y^2 + \left[\left(\frac{m_2^u}{m_2^l} + 1 \right) r^u + \frac{k|m_1^l - m_2^u|}{m_2^l} \right] y + \frac{\tilde{\rho}}{m_2^l} y^{1-\rho} \right\}.$$

Choosing $k = \frac{2+|H_3|}{\frac{1}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds}$, we have

$$LV \leq -2 + \frac{(2 + |H_3|)m_1^l b^u}{\frac{c^u}{\theta} \int_0^\theta \left[\frac{m_1^l}{c^u} \left(a(s) - \frac{\alpha^2(s)}{2} \right) - \frac{m_2^u}{f^l} \left(r(s) - \frac{\beta^2(s)}{2} \right) \right] ds} \lambda.$$

It then follows from (41) that

$$LV \leq -1. \tag{48}$$

Case 3. If $(x, y) \in \tilde{U}_3$, making use of (45) we obtained that

$$\begin{aligned}
 LV &\leq \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - b^l x^2 + \frac{1}{m_2(t) + x} \left(-\frac{f^l}{2} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + k|m_1^l - m_2^u| y \right) \\
 &\quad - \rho e^{\rho\omega_1} y^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right] \\
 &\leq H_4 + \frac{|H_2|}{m_2^l} - \rho e^{\rho\omega_1} \lambda^{-\rho} \left[\frac{1}{\theta} \int_0^\theta \left(r(s) - \frac{1}{2} \beta^2(s) \right) ds - \frac{\rho}{2} \beta^{2u} \right],
 \end{aligned}$$

where

$$H_4 = \sup_{x \in (0, \infty)} \left\{ \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - b^l x^2 \right\} < \infty.$$

By (42), we have

$$LV \leq -1. \tag{49}$$

Case 4. If $(x, y) \in \tilde{U}_4$, from the the proof of Case 3, together with (30), we derive

$$\begin{aligned}
 LV &\leq \left(a^u + \frac{km_1^l b^u}{c^u} + \frac{r^u}{2f^l} \right) x - b^l x^2 + \frac{1}{m_2(t) + x} \left(-\frac{f^l}{2} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-\rho} + k|m_1^l - m_2^u| y \right) \\
 &\leq H_4 + \frac{1}{m_2(t) + x} \left(-\frac{f^l}{4} y^2 + H_5 \right) \leq H_4 + \frac{|H_5|}{m_2^l} - \frac{f^l}{4(m_2^u + x)} y^2 \\
 &\leq H_4 + \frac{|H_5|}{m_2^l} - \frac{f^l}{4m_2^u + \frac{4}{\lambda}} = H_4 + \frac{|H_5|}{m_2^l} - \frac{f^l}{4m_2^u \lambda^2 + 4\lambda'}
 \end{aligned}$$

where

$$H_5 = \sup_{y \in (0, \infty)} \left\{ -\frac{f^l}{4} y^2 + m_2^u r^u y + \tilde{\rho} y^{1-p} + k |m_1^l - m_2^u| y \right\}.$$

By (43), we derive that

$$LV \leq -1. \tag{50}$$

It then follows from (46), (48), (49) and (50) that

$$LV(t, x, y) \leq -1, (x, y) \in R_+^2 \setminus \tilde{U}.$$

According to Lemma 3.1, the proof is completed. \square

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