Filomat 34:5 (2020), 1571–1579
https://doi.org/10.2298/FIL2005571A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pointwise Well-Posedness and Scalarization of Optimization Problems for Locally Convex Cone-Valued Functions

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Abstract. We investigate the pointwise well-posedness of optimization problems for locally convex cone-valued functions and establish some relations between the kinds of well-posedness. Via the neighborhoods and elements, we define the scalarization functions for locally convex cones and discuss their properties. We consider the scalar optimization problems and obtain some results about the well-posedness of the optimization problems.

1. Introduction

Well-posedness of an optimization problem is to study the behavior of the objective function, when its value is close to the optimal value. The classical well-posedness for a scalar optimization problem was first introduced by Tykhonov [22] in 1966. Since then, various notions of well-posedness were introduced and studied for scalar optimization problems (see [4], [6]). In the last decades, some extensions of this concept to the vector optimization problems were appeared (see [2, 3], [5], [8], [12, 13]) and recently we have some extensions of that to the set optimization problems (see [7], [10, 11], [23]). Our aim in the present work is twofold. In section 2, we define the notions of pointwise well-posedness for locally convex cone-valued functions and discuss the relations among them; in particular, we remark that these notions are real extensions of the corresponding ones by Miglierina et al. in [13]. In Section 3, we introduce scalarization functions for locally convex cones and consider some of its properties. Finally, we study the pointwise well-posedness of optimization problems for locally convex cone-valued functions by the scalar optimization problems.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a+b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a preordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply

2010 *Mathematics Subject Classification*. Primary 46N10; Secondary 58E30; 49K40; 46A30
Keywords. Locally convex cones, Optimization problems, Well-posedness, Scalarization functions
Received: 23 July 2018; Accepted: 20 January 2020
Communicated by Naseer Shahzad
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to cones without order structures as well. A *full locally convex cone* is a preordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of $\mathcal{P}$ with the following properties:

(v_1) $0 < v$ for all $v \in \mathcal{V}$,

(v_2) for all $u, v \in \mathcal{V}$ there is $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$,

(v_3) $u + v \in \mathcal{V}$ and $\lambda v \in \mathcal{V}$ for all $u, v \in \mathcal{V}$ and $\lambda > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} \mid b \leq a + v\}, \text{ respectively } (a)v = \{b \in \mathcal{P} \mid a \leq b + v\},$$

to be a neighborhood of a in the *upper*, respectively *lower topology* on $\mathcal{P}$. Their common refinement is called *symmetric topology*. We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. The cone $\overline{\mathbb{R}}$ with abstract neighborhood system $\varepsilon = \{\epsilon \in \mathbb{R} \mid \epsilon > 0\}$ forms a full locally convex cone. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$.

2. Pointwise well-posedness

Let X be a Hausdorff topological space, G a non-empty subset of X and $f : G \to (\mathcal{P}, \mathcal{V})$ be a mapping. The *general optimization problem* for f is formalized by $(G, f, \leq)$ $\mathrm{Min}(f(G), \leq)$, where $f(G) = \{f(x) : x \in G\}$ and $\leq$ is the original preorder on $\mathcal{P}$. An element $\bar{x} \in G$ is called a *minimal solution* of the problem $(G, f, \leq)$, if $f(\bar{x})$ is a minimal element of $f(G)$, i.e., if $f(x) \leq f(\bar{x})$ for $x \in G$ then $f(\bar{x}) \leq f(x)$. We denote by $\mathrm{Eff}(G, f, \leq)$ the set of all minimal solutions of $(G, f, \leq)$. The image of the set $\mathrm{Eff}(G, f, \leq)$ under the function f is denoted by $\mathrm{Min}(G, f, \leq)$ and its elements are called *minimal points* of the set $f(G)$. A net $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ in X *clusters* at x_0 if it is frequently in every neighborhood of x_0, i.e., for every $\alpha \in \mathcal{I}$ and every open neighborhood U of x_0, there exists $\beta \geq \alpha$ such that $x_\beta \in U$. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ be a net in $\mathcal{P}$ and $x \in \mathcal{P}$. We write $x_\alpha \downarrow x$ ($x_\alpha \uparrow x$) if $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ converges to x with respect to the lower (respectively, upper) topology of $\mathcal{P}$. Also $x_\alpha \to x$ means that $x_\alpha \downarrow x$ and $x_\alpha \uparrow x$, i.e., $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ converges to x with respect to the symmetric topology. For more information about the convergence of nets in lower, upper and symmetric topologies see [14-16]).

If $\bar{y} \in \mathrm{Min}(G, f, \leq)$, then we say that $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ is a *$\bar{y}$-minimizing net* to $(G, f, \leq)$ if there exists a $v \in \mathcal{V}$ and a positive real net $\{\lambda_\alpha\}_{\alpha \in \mathcal{I}}$ such that $\lambda_\alpha \to 0$ and $f(x_\alpha) \leq \bar{y} + \lambda_\alpha v$. The problem $(G, f, \leq)$ is called *L-well-posed* at $\bar{y}$ if every $\bar{y}$-minimizing net clusters to an element of $S(f, \leq, \bar{y}) = \{x \in G : f(x) \leq \bar{y}\}$, also it is called *weakly L-well-posed* at $\bar{y} \in \mathrm{Min}(G, f, \leq)$ if every net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ with $f(x_\alpha) \uparrow \bar{y}$, clusters to an element of $S(f, \leq, \bar{y})$.

Remark 2.1. (i) For $y \in G$, we consider the sublevel set $S(f, \leq, f(y)) = \{x \in G : f(x) \leq f(y)\}$. If $\bar{x} \in \mathrm{Eff}(G, f, \leq)$, then $S(f, \leq, f(\bar{x})) \subseteq \mathrm{Eff}(G, f, \leq)$.

(ii) It is easy to see that, if $(G, f, \leq)$ is (weakly) L-well-posed at $\bar{y} \in \mathrm{Min}(G, f, \leq)$ then $S(f, \leq, \bar{y})$ is compact.

Consider $\mathcal{P}$ with the upper topology and let $F : S \subset \mathcal{P} \to 2^X$ be a set-valued map. We recall that F is *upper semicontinuous* at $y \in S$ if for every open set U containing $F(y)$, there exists $v \in \mathcal{V}$ such that $F(x) \subseteq U$ for all $x \in v(y) \cap S$. A problem $(G, f, \leq)$ is said to be *B-well-posed* at $\bar{y} \in \mathrm{Min}(G, f, \leq)$ if the set-valued map $F : f(G) \to 2^G$ such that $F(y) = S(f, \leq, y)$ for all $y \in f(G)$ is upper semicontinuous at $\bar{y}$.

Proposition 2.2. *If $\bar{y} \in Min(G, f, \leq)$ and every net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subseteq G \setminus S(f, \leq, \bar{y})$ with $f(x_\alpha) \uparrow \bar{y}$ has a cluster point in $S(f, \leq, \bar{y})$, then $(G, f, \leq)$ is B-well-posed at $\bar{y}$. The converse is true if $S(f, \leq, \bar{y})$ is a singleton.*

Proof. Suppose F is not upper semicontinuous at $\bar{y}$. There exists an open set W containing $S(f, \leq, \bar{y})$ such that $F(v(\bar{y}) \cap f(G)) \not\subseteq W$ for all $v \in \mathcal{V}$. For every $v \in \mathcal{V}$, let $x_v \in G$ such that $x_v \in F(v(\bar{y}) \cap f(G))$ and $x_v \notin W$. Since $v(\bar{y}) \cap f(G) = \{y \in f(G) : y \leq \bar{y} + v\}$ and for $y \in v(\bar{y})) \cap f(G)$ we have $F(y) = \{x \in G : f(x) \leq y\}$, then $F(v(\bar{y}) \cap f(G)) = \{x \in G : f(x) \leq \bar{y} + v\}$. This implies that $f(x_v) \uparrow \bar{y}$. But $\{x_v\}_{v \in \mathcal{V}} \subseteq G \setminus W$ by the assumption, so $\{x_v\}_{v \in \mathcal{V}}$ has a cluster point in $S(f, \leq, \bar{y})$ which is a contradiction. Conversely, let $(G, f, \leq)$ be B-well-posed at $\bar{y}$ and let $S(f, \leq, \bar{y})$ be a singleton. Suppose that there exists a net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subseteq G \setminus S(f, \leq, \bar{y})$ with $f(x_\alpha) \uparrow \bar{y}$ which admits no subnet converging to $S(f, \leq, \bar{y})$. Then, there exists an open set U such that $S(f, \leq, \bar{y}) \subset U$ and $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ is not frequently in U. Since F is upper semicontinuous at $\bar{y}$, there exists $v \in \mathcal{V}$ such that $F(v(\bar{y}) \cap f(G)) \subset U$. If we choose $\alpha_0 \in \mathcal{I}$ such that $f(x_\alpha) \in v(\bar{y})$ for all $\alpha \geq \alpha_0$, then $x_\alpha \in F(v(\bar{y}) \cap f(G)) \subset U$ for all $\alpha \geq \alpha_0$ which is a contradiction. $\square$

A problem $(G, f, \leq)$ is said to be *H-well-posed* at $\bar{x} \in \mathrm{Eff}(G, f, \leq)$ if for every net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ such that $f(x_\alpha) \to f(\bar{x})$, we have $x_\alpha \to \bar{x}$.

Remark 2.3. (i) For $v \in \mathcal{V}$, an element $a \in \mathcal{P}$ is called *v-bounded* if $a \leq \lambda v$ for some $\lambda > 0$ and a is called *bounded* if it is v-bounded for all $v \in \mathcal{V}$. If the abstract neighborhood system $\mathcal{V}$ has some bounded elements, then nets H-well-posedness is equivalent to H-well-posedness in sequential sense. Indeed, nets H-well-posedness entails H-well-posedness in the sequential sense. For the converse, let $(G, f, \leq)$ be sequential H-well-posed at $\bar{x}$ and it fails to be nets H-well-posed. Then, there exists some net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subseteq G$ such that $f(x_\alpha) \to f(\bar{x})$ and $x_\alpha \not\to \bar{x}$. Thus we can find a neighborhood U of $\bar{x}$ such that for every $\alpha \in \mathcal{I}$ there exists $\beta \geq \alpha$ with $x_\beta \notin U$. Let $v \in \mathcal{V}$ be bounded. There exists a sequence $\{x_{\beta_n}\}_{n \in \mathbb{N}}$ such that $f(x_{\beta_n}) \in v/n(f(\bar{x})) \cap (f(\bar{x}))v/n$ and $x_{\beta_n} \notin U$ for all $n \in \mathbb{N}$. Since v is bounded, for every $v' \in \mathcal{V}$ we have $f(x_{\beta_n}) \in v'/n(f(\bar{x})) \cap (f(\bar{x}))v'/n$ which implies $f(x_{\beta_n}) \to f(\bar{x})$ when $n \to \infty$, so $x_{\beta_n} \to \bar{x}$ which is a contradiction.

(ii) If $(G, f, \leq)$ is H-well-posed at $\bar{x} \in \mathrm{Eff}(G, f, \leq)$ then $S(f, \leq, f(\bar{x})) = \{\bar{x}\}$. Indeed, for $\hat{x} \in S(f, \leq, f(\bar{x}))$, we have $f(\hat{x}) \leq f(\bar{x})$ which implies $f(\bar{x}) \leq f(\hat{x})$. Then, for the constant net $x_\alpha := \hat{x}$, $f(\hat{x}) \uparrow f(\bar{x})$ and $f(\hat{x}) \downarrow f(\bar{x})$. Therefore $\hat{x} \to \bar{x}$, i.e., $\hat{x} = \bar{x}$.

For every $\bar{x} \in \mathrm{Eff}(G, f, \leq)$, $v \in \mathcal{V}$ and $\lambda > 0$, let

$$L_f(\bar{x}, v, \lambda) = \{x \in G : f(x) \leq f(\bar{x}) + \lambda v\}.$$

We say that $(G, f, \leq)$ is *DH-well-posed* at $\bar{x} \in \mathrm{Eff}(G, f, \leq)$ if for every $v \in \mathcal{V}$ and every open neighborhood $U \subseteq X$ of $\bar{x}$ there exists $\lambda > 0$ such that $L_f(\bar{x}, v, \lambda) \subseteq U$. Also, $(G, f, \leq)$ is said to be *weakly DH-well-posed* at $\bar{x}$ if for every $v \in \mathcal{V}$ and every open set $U \subseteq X$ containing $S(f, \leq, f(\bar{x}))$, there exists $\lambda > 0$ such that $L_f(\bar{x}, v, \lambda) \subseteq U$.

Remark 2.4. (i) Obviously, DH-well-posedness implies weakly DH-well-posedness.

(ii) If $v \in \mathcal{V}$ is bounded, then $(G, f, \leq)$ is DH-well-posed (weakly DH-well-posed) at $\bar{x} \in \mathrm{Eff}(G, f, \leq)$ if for every open set U containing $\bar{x}$ (*respectively*, $S(f, \leq, f(\bar{x}))$ there exists $\lambda > 0$ such that $L_f(\bar{x}, v, \lambda) \subseteq U$. Indeed, for $v' \in \mathcal{V}$ we have $v' \leq \gamma v$ for some $\gamma > 0$, so $L_f(\bar{x}, v', \lambda/\gamma) \subseteq L_f(\bar{x}, v, \lambda)$.

(iii) If X is a metric space, $(G, f, \leq)$ is DH-well-posed at $\bar{x}$ if and only if $\inf_{\lambda>0} \mathrm{diam}\, L_f(\bar{x}, v, \lambda) = 0$ for all $v \in \mathcal{V}$, where diam denotes the diameter of a set. If $(G, f, \leq)$ is DH-well-posed at $\bar{x}$ then $S(f, \leq, f(\bar{x})) = \{\bar{x}\}$. Also, $(G, f, \leq)$ is weakly DH-well-posed at $\bar{x}$ if and only if

$$\inf_{\lambda>0} \mathrm{diam}\, L_f(\bar{x}, v, \lambda) = \mathrm{diam}\, S(f, \leq, f(\bar{x})) \quad \text{for all } v \in \mathcal{V}.$$

According to [9, Ch I, 3.2], for a fixed element $v \in \mathcal{V}$, the *local preorder* $\leq_v$ is defined by $a \leq_v b$ for $a, b \in \mathcal{P}$ if and only if $a \leq b + \lambda v$ for all $\lambda > 0$. Clearly $a \leq b$ in the orginal preorder implies $a \leq_v b$. For $v \in \mathcal{V}$, it is easy to see that

$$\bigcap_{\lambda>0} L_f(\bar{x}, v, \lambda) = \{x \in G : f(x) \leq_v f(\bar{x})\} = S(f, \leq_v, f(\bar{x})).$$

Proposition 2.5. *Let $\bar{x} \in Eff(G, f, \leq)$. If $(G, f, \leq)$ is weakly DH-well-posed at $\bar{x}$ then for every $v \in \mathcal{V}$ we have $S(f, \leq, f(\bar{x})) = S(f, \leq_v, f(\bar{x}))$.*

Proof. For $v \in \mathcal{V}$, let $x \in S(f, \leq_v, f(\bar{x}))$ and $x \notin S(f, \leq, f(\bar{x}))$. Then there is an open set U such that $S(f, \leq, f(\bar{x})) \subseteq U$ and $x \notin U$. Since $(G, f, \leq)$ is weakly DH-well-posed at $\bar{x}$ and $x \in L_f(\bar{x}, v, \lambda)$ for all $\lambda > 0$, then $x \in U$ which is a contradiction. □

Proposition 2.6. *If $(G, f, \leq)$ is L-well-posed at $\bar{x} \in Min(G, f, \leq)$, then it is weakly L-well-posed at $\bar{x}$.*

Proof. If $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ is a net such that $f(x_\alpha) \uparrow \bar{y}$ and $v \in \mathcal{V}$ then, for every $n \in \mathbb{N}$, there exists $\alpha_n \in \mathcal{I}$ such that $f(x_{\alpha_n}) \leq \bar{x} + 1/n\, v$, i.e., $\{x_{\alpha_n}\}_{n \in \mathbb{N}}$ is an $\bar{x}$-minimizing net to problem $(G, f, \leq)$. By the L-well-posedness, $\{x_{\alpha_n}\}_{n \in \mathbb{N}}$ has a subnet that converges to an element of $S(f, \leq, \bar{x})$, hence $(G, f, \leq)$ is weakly L-well-posed at $\bar{x}$. □

In general, the converse of Proposition 2.6 does not hold. However, L-well-posedness is equivalent to weakly L-well-posedness when in problem $(G, f, \le)$ the elements of the abstract neighborhood system $\mathcal{V}$ are bounded. For a full locally convex cone $(\mathcal{P}, \mathcal{V})$, we denote by $Conv(\mathcal{P})$ the set of all non-empty convex subsets of $\mathcal{P}$ which is a cone with its usual addition and scalar multiplication. If we identify the elements of $\mathcal{V}$ with singleton sets $\overline{v} = \{v\}$ then $\overline{\mathcal{V}} = \{\overline{v} : v \in \mathcal{V}\}$ is a subset of $Conv(\mathcal{P})$, which can be preordered using the preorder of $\mathcal{P}$. For $A, B \in Conv(\mathcal{P})$, we define

$$A \le B \text{ if for each } a \in A \text{ there is some } b \in B \text{ such that } a \le b,$$

then $(Conv(\mathcal{P}), \overline{\mathcal{V}})$ becomes a full locally convex cone. For locally convex vector space E with neighborhood base $\mathcal{V}$, with set inclusion as order and abstract neighborhood system $\mathcal{V}$, $Conv(E)$ forms a full locally convex cone. For details see [9].

Example 2.7. (i) Let $\mathcal{P} = Conv(\overline{\mathbb{R}})$ endowed with neighborhood system $\overline{\varepsilon} = \{\overline{\epsilon} : \epsilon > 0\}$. If $X = \mathbb{R}$ and $G = [0, 1] \subset X$, then for the mappings $f, g : G \to \mathcal{P}$ defined by $f(x) = [x, 1]$ and

$$g(x) = \begin{cases} (-1, 0) & \text{if } \quad x = 0, \\ [0, 1 - x] & \text{if } \quad 0 < x \le 1, \end{cases}$$

we have $\mathrm{Eff}(G, f, \le) = [0, 1]$ and $\mathrm{Eff}(G, g, \le) = \{0\}$. For every $x \in G$, we have $F(f(x)) = S(f, \le, f(x)) = [0, 1]$; so F is upper semicontinuous on $f(G)$. This implies the B-well-posedness of $(G, f, \le)$ at $f(x)$ for all $x \in \mathrm{Eff}(G, f, \le)$. But, the problem $(G, g, \le)$ is not B-well-posed at $g(0)$. Actually, $F(g(0)) = \{0\}$ and $F(g(1)) = \{0, 1\}$. Since $g(1) \in \overline{\epsilon}(g(0))$ for all $\epsilon \in \varepsilon$ and for neighborhood $U = (-1, 1)$ of $F(g(0))$ we have $F(g(1)) \not\subseteq U$, i.e., the set-valued map F is not upper semicontinuous at $g(0)$.

(ii) Let $\mathcal{P} = Conv(\mathbb{R})$ endowed with neighborhood system $\mathcal{V} = \{(-\epsilon, \epsilon) : \epsilon > 0\}$. If $X = \mathbb{R}$ and for $G = [0, 1] \subseteq X$, the mapping $f : G \to (\mathcal{P}, \mathcal{V})$ is defined by $f(x) = [-x, x]$ for all $x \in G$, then $\mathrm{Eff}(G, f, \le) = \{0\}$. It is easy to see that problem $(G, f, \le)$ is weakly L-well-posed at $f(0)$ and since all elements of $\mathcal{V}$ are bounded, $(G, f, \le)$ is L-well-posed at $f(0)$.

Proposition 2.8. *If $\bar{y} \in Min(G,\ f,\ \le)$, then*

(a) *if $(G, f, \le)$ is L-well-posed at $\bar{y}$ then it is B-well-posed at $\bar{y}$,*
(b) *if $(G, f, \le)$ is B-well-posed at $\bar{y}$ and $S(f, \le, \bar{y})$ is a singleton, then $(G, f, \le)$ is weakly L-well-posed at $\bar{y}$.*

Proof. (a) By contradiction suppose that F is not upper semicontinuous at $\bar{y}$. There exists an open set $W \subseteq X$ containing $S(f, \le, \bar{y})$ such that $F(v(\bar{y}) \cap f(G)) \not\subseteq W$ for all $v \in \mathcal{V}$. Then for $v \in \mathcal{V}$, one can take a sequence $\{x_n\}_{n \in \mathbb{N}} \subset G$ with $x_n \in F(1/n\, v(\bar{y}) \cap f(G))$ and $x_n \notin W$. In other words, $\{x_n\}_{n \in \mathbb{N}} \subset X \backslash W$ and $f(x_n) \le \bar{y} + 1/n\, v$, i.e., $\{x_n\}_{n \in \mathbb{N}}$ is a $\bar{y}$-minimizing sequence. Since $(G, f, \le)$ is L-well-posed at $\bar{y}$, there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $\{x_{n_k}\}_{k \in \mathbb{N}}$ converges to an element of $S(f, \le, \bar{y})$. But $X \setminus W$ is closed, so the convergent point of $\{x_{n_k}\}_{k \in \mathbb{N}}$ belongs to $X \setminus W$, which is a contradiction. For (b), since $(G, f, \le)$ is B-well-posed at $\bar{y}$ and $S(G, \le, \bar{y})$ is a singleton, by Proposition 2.2, every net $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subseteq G$ with $f(x_\alpha) \uparrow \bar{y}$ has a subnet converging to $S(f, \le, \bar{y})$ and this proves the assertion. □

Proposition 2.9. *If $G \subseteq X$ and $\bar{x} \in Eff(G, f, \le)$, then*

(a) *if $(G, f, \le)$ is DH-well-posed at $\bar{x}$ then it is L-well-posed at $f(\bar{x})$,*
(b) *if $(G, f, \le)$ is L-well-posed at $f(\bar{x})$ then it is weakly DH-well-posed at $\bar{x}$,*
(c) *if $(G, f, \le)$ is DH-well-posed at $\bar{x}$, then it is H-well-posed at $\bar{x}$.*

Proof. (a) Let $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ be a $f(\bar{x})$-minimizing net. There exist $v \in \mathcal{V}$ and a real positive net $\{\lambda_\alpha\}_{\alpha \in \mathcal{I}}$ with $\lambda_\alpha \to 0$ such that $f(x_\alpha) \le f(\bar{x}) + \lambda_\alpha v$, i.e., $x_\alpha \in L_f(\bar{x}, v, \lambda_\alpha)$. If $(G, f, \le)$ is DH-well-posed at $\bar{x}$ then for every neighborhood U of $\bar{x}$ there exists $\lambda > 0$ such that $L_f(\bar{x}, v, \lambda) \subseteq U$. We may choose $\alpha_0 \in \mathcal{I}$ with $\lambda_{\alpha_0} \le \lambda$ such that $L_f(\bar{x}, v, \lambda_\alpha) \subseteq U$ for all $\alpha \ge \alpha_0$, which yields $x_\alpha \in U$ for all $\alpha \ge \alpha_0$, so $x_\alpha \to \bar{x}$. For (b), suppose that $(G, f, \le)$ is L-well-posed at $f(\bar{x})$ but it is not weakly DH-well-posed at $\bar{x}$. There exist $v \in \mathcal{V}$ and an open set

U containing $S(f, \leq, f(\bar{x}))$ such that $L_f(\bar{x}, v, \lambda) \nsubseteq U$ for all $\lambda > 0$. Then for a positive sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \to 0$, we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset G$ such that $x_n \in L_f(\bar{x}, v, \lambda_n)$ and $x_n \notin U$. Since $x_n \in L_f(\bar{x}, v, \lambda_n)$, $f(x_n) \leq f(\bar{x}) + \lambda_n v$, i.e., $\{x_n\}_{n\in\mathbb{N}}$ is a $f(\bar{x})$-minimizing sequence, so it has a subsequence convergent to an element of $S(f, \leq, f(\bar{x}))$, which is a contradiction.

For (c), let $\{x_\alpha\}_{\alpha\in\mathcal{I}} \subset G$ be a net such that $f(x_\alpha) \to f(\bar{x})$ and U be a neighborhood of $\bar{x}$ in X. For $v \in \mathcal{V}$, since $(G, f, \leq)$ is DH-well-posed at $\bar{x}$, there exists $\lambda > 0$ such that $L_f(\bar{x}, v, \lambda) \subset U$. Since $f(x_\alpha) \to f(\bar{x})$, we have $f(x_\alpha) \uparrow f(\bar{x})$, so there exists $\alpha_0 \in \mathcal{I}$ such that $f(x_\alpha) \in (\lambda v)(f(\bar{x}))$ for all $\alpha \geq \alpha_0$. Since $L_f(\bar{x}, v, \lambda) = \{x \in G : f(x) \in (\lambda v)(f(\bar{x}))\}$, we have $x_\alpha \in L_f(\bar{x}, v, \lambda) \subset U$ for all $\alpha \geq \alpha_0$. This means that $x_\alpha \to \bar{x}$. □

Remark 2.10. Let E be a real locally convex topological vector space and let $K \subset E$ be a closed *pointed* cone, i.e., $K \cap -K = \{0\}$. The orders $\leq$ and $\ll$ on E for all $a, b \in E$ defined as $x \leq y$ if $y - x \in K$ and $x \ll y$ if $y - x \in \text{int}K$, where int$K$ is the topological interior of K. We have:

(i) $\text{int}K + \text{int}K \subseteq \text{int}K$, K+int$K\subseteq$ int$K$ and $\lambda intK \subseteq intK$ for all $\lambda > 0$.

(ii) For every $c_1, c_2 \in \text{int}K$, there exists $c \in \text{int}K$ such that $c \ll c_1$ and $c \ll c_2$. Indeed, since int$K$ is open, there are symmetric neighborhoods u_1 and u_2 of zero such that $c_1 + u_1 \subseteq \text{int}K$ and $c_2 + u_2 \subseteq \text{int}K$. Thus, we have $c_1 - c \in intK$ and $c_2 - c \in intK$ for $c \in u_1 \cap u_2$, that is, $c \ll c_1$ and $c \ll c_2$.

(iii) $(E, \text{int}K)$ is a full locally convex cone; for, clearly every element of int$K$ is strictly positive, so (v_1) holds. The condition (v_2) comes from (ii). For (v_3), if $c_1, c_2 \in \text{int}K$ and $\lambda > 0$, then by (i), we have $\lambda c_1 \in \text{int}K$ and $c_1 + c_2 \in \text{int}K$. Thus int$K$ is an abstract neighborhood system on E. On the other hand, for every $x \in E$ and $c \in \text{int}K$ there exists a convex neighborhood u of zero and a $\lambda > 0$ such that $c + u \subseteq \text{int}K$ and $x \in \lambda u$. Thus, by (i), $\lambda c + \lambda u \subseteq \text{int}K$, i.e., $0 \ll \lambda c + x$. Thus every $x \in E$ is bounded below and $(E, \text{int}K)$ is a full locally convex cone.

(iv) The *order interval* $[-c, c] = \{z \in E : -c \leq_K z \leq_K c\}$ is a convex neighborhood of zero in E for all $c \in \text{int}K$ [1]. If E has a base at zero of order intervals $[-c, c]$ for all $c \in \text{int}K$, then one can see that the symmetric topology on E coincides with the original one. The elements of E are bounded with respect to *intK*; indeed, if $a \in E$, then for every $c \in \text{int}K$ there exists $\lambda > 0$ such that $0 \leq_K -a + \lambda c$, i.e., $a \leq_K \lambda c$. If K is closed, then for every $e \in \text{int}K$ the local preorder $\leq_e$ coincides with the original one.

Remark 2.11. If $(\mathcal{P}, \mathcal{V})$ is indeed a locally convex ordered topological vector space then our kinds of pointwise well-posedness is reduced to the vectorial kinds considered by Miglierina et al. in [13]. In fact, in problem $(G, f, \leq_K)$, if $(\mathcal{P}, \mathcal{V}) = (E, \text{int}K)$, then

(i) If $\bar{y} \in \text{Min}(f(G), \leq_K)$ and the set-valued map $\mathcal{Q}_{\bar{y}} : K \to 2^G$ is defined by

$$\mathcal{Q}_{\bar{y}}(k) = \{x \in G : f(x) \leq_K \bar{y} + k\}$$

for all $k \in K$, then $(G, f, \leq_K)$ is B-well-posed at $\bar{y}$ if and only if $\mathcal{Q}_{\bar{y}}$ is upper semicontinuous at $k = 0$. In fact, suppose that $(G, f, \leq_K)$ is B-well-posed at $\bar{y}$ and W is an open set such that $\mathcal{Q}_{\bar{y}}(0) = S(f, \leq_K, \bar{y}) \subset W$. There exists $e \in \text{int}K$ such that $F(e(\bar{y}) \cap f(G)) \subset W$. If we choose a symmetric zero neighborhood $U \subset E$ with $e + U \subset \text{int}K$, then $k \leq_K e$ for all $k \in U \cap K$ which yields

$$\mathcal{Q}_{\bar{y}}(k) \subseteq \mathcal{Q}_{\bar{y}}(e) = F(e(\bar{y}) \cap f(G)) \subset W,$$

i.e., $\mathcal{Q}_{\bar{y}}$ is upper semicontinuous at zero. Conversely, suppose that for every open set W containing $\mathcal{Q}_{\bar{y}}(0)$, there exists a zero neighborhood U such that $\mathcal{Q}_{\bar{y}}(k) \subset W$ for all $k \in U \cap K$. Taking $e \in U \cap \text{int}K$, we have $F(e(\bar{y}) \cap f(G)) = \mathcal{Q}_{\bar{y}}(e) \subset W$. Therefore $(G, f, \leq_K)$ is B-well-posed at $\bar{y}$.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in G is $\bar{y}$-minimizing to problem $(G, f, \leq_K)$ if and only if there exists a sequence $\{k_n\}_{n\in\mathbb{N}} \subseteq K \setminus \{0\}$ with $k_n \to 0$ such that $f(x_n) \leq_K \bar{y} + k_n$ (see Proposition 5.2 in [7]).

(iii) If $K \subseteq E$ is pointed, then the preorder $\leq_K$ is antisymmetric, hence for every $\bar{y} \in \text{Min}(f(G), \leq_K)$ we have $f^{-1}(\bar{y}) = S(f, \leq_K, \bar{y})$. Thus the all kinds of well posedness for locally convex cone-valued functions are extension of the corresponding well-posedness for convex spaces.

(iv) From Remarks 2.10 (iv) and 2.3 (i), if E has a zero neighborhood base consisting of order intervals $[-c, c], c \in \text{int}K$, then the definition of H-well-posedness in sequential sense reduces to the definition 3.4 in [13].

3. Scalarization functions

In this section, we introduce a non-linear scalarization function for locally convex cones and discuss some of its properties. Then, we investigate the relationships between pointwise well-posedness of optimization problems and well-posedness of the associated scalar problems for locally convex cone-valued functions. For $v \in \mathcal{V}$ and $b \in \mathcal{P}$, we define the *scalarization function* $\varphi_{v,b} : (\mathcal{P}, \mathcal{V}) \to (\overline{\mathbb{R}}_+, \varepsilon)$ for all $a \in \mathcal{P}$ by

$$\varphi_{v,b}(a) = \inf\{t > 0 \ : \ a \leq b + tv\},$$

where $\overline{\mathbb{R}}_+ = [0, +\infty]$ and $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$.

Remark 3.1. (i) If $a \in \mathcal{P}$ is v-bounded then $\varphi_{v,b}(a) < +\infty$; indeed, there is $\lambda > 0$ such that $a \leq \lambda v$. Since every element of $\mathcal{P}$ is bounded below, there is $\gamma > 0$ such that $0 \leq b + \gamma v$. Thus $a \leq b + (\lambda + \gamma)v$, hence $\{t > 0 : a \leq b + tv\} \neq \emptyset$, i.e., $\varphi_{v,b}(a) < +\infty$.

(ii) If b is v-bounded and $\varphi_{v,b}(a) < +\infty$, then a is v-bounded.

Theorem 3.2. *If $v \in \mathcal{V}$ and $b \in \mathcal{P}$, then for the scalarazation function $\varphi_{v,b}$, the following hold:*

(a) *For every $\epsilon > 0$ and $x, y \in \mathcal{P}$, if $x \leq y + \epsilon v$ then $\varphi_{v,b}(x) \leq \varphi_{v,b}(y) + \epsilon$.*

(b) *$\varphi_{v,b}$ is monotone even with respect to the local preorder $\preceq_v$ of $\mathcal{P}$.*

(c) *$\varphi_{v,b}(b) = 0$.*

(d) *$x \preceq_v b$ if and only if $\varphi_{v,b}(x) = 0$.*

Proof. (a) If $t > 0$ such that $y \leq b + tv$ then, by assumption, $x \leq b + (t + \epsilon)v$, so $\varphi_{v,b}(x) \leq t + \epsilon$, i.e., $\varphi_{v,b}(x) \leq \varphi_{v,b}(y) + \epsilon$. For (b), if $x \preceq_v y$ and $x \leq y + \epsilon v$ for all $\epsilon > 0$, then by (a), $\varphi_{v,b}(x) \leq \varphi_{v,b}(y) + \epsilon$ for all $\epsilon > 0$, which implies that $\varphi_{v,b}(x) \leq \varphi_{v,b}(y)$. Part (c) is obvious. For (d), from (b) and (c), if $x \preceq_v b$ then $\varphi_{v,b}(x) \leq 0$, so $\varphi_{v,b}(x) = 0$. Conversely, if $\varphi_{v,b}(x) = 0$ then $x \leq b + \lambda v$ for all $\lambda > 0$, i.e., $x \preceq_v b$. □

Let $h : X \to (\overline{\mathbb{R}}_+, \varepsilon)$ and consider the scalar optimization problem (G, h) Min $h(G)$, where $h(G) = \{h(x) : x \in G\}$. We note that, if $\inf_G h$ denotes the infimum of h on G and $\mathrm{Eff}(G, h) \neq \emptyset$, then $\mathrm{Min}(G, h) = \{\inf_G h\}$.

Lemma 3.3. *If $\bar{x} \in Eff(G, f, \leq)$, then $Eff(G, \varphi_{v,f(\bar{x})} \circ f) = S(f, \preceq_v, f(\bar{x}))$.*

Proof. Since $\varphi_{v,f(\bar{x})}(f(\bar{x})) = 0$, we have $\inf_G \varphi_{v,f(\bar{x})} \circ f = 0$ by Theorem 3.2 (d), so $\mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f) = S(f, \preceq_v, f(\bar{x}))$. □

Theorem 3.4. *If $\bar{x} \in Eff(G, f, \leq)$ such that $S(f, \preceq_v, f(\bar{x})) = S(f, \leq, f(\bar{x}))$ for some $v \in \mathcal{V}$ and the problem $(G, \varphi_{v,f(\bar{x})} \circ f)$ is B-well-posed at zero, then $(G, f, \leq)$ is B-well-posed at $f(\bar{x})$.*

Proof. Consider the set-valued map $F : f(G) \to 2^G$ by $F(y) = S(f, \leq, y)$ and $\psi : \varphi_{v,f(\bar{x})} \circ f(G) \to 2^G$ as $\psi(t) = S(\varphi_{v,f(\bar{x})} \circ f, \leq, t)$. Suppose that $W \subseteq X$ is an open set such that $F(f(\bar{x})) = S(f, \leq, f(\bar{x})) \subset W$. By Lemma 3.3, we have

$$\begin{aligned} \psi(0) &= S(\varphi_{v,f(\bar{x})} \circ f, \leq, 0) = \mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f) \\ &= S(f, \preceq_v, f(\bar{x})) = S(f, \leq, f(\bar{x})) \subseteq W. \end{aligned}$$

For cone-valued function $\varphi_{v,f(\bar{x})} \circ f : G \to \overline{\mathbb{R}}_+$, since $(G, \varphi_{v,f(\bar{x})} \circ f)$ is B-well-posed at zero, there exists $\epsilon > 0$ such that $\psi(\epsilon(0) \cap \varphi_{v,f(\bar{x})} \circ f(G)) \subseteq W$, i.e., $\{x \in G : \varphi_{v,f(\bar{x})}(f(x)) \leq \epsilon\} \subseteq W$. Thus

$$\begin{aligned} F((\epsilon v)(f(\bar{x})) \cap f(G)) &= \{x \in G : f(x) \leq f(\bar{x}) + \epsilon v\} \\ &\subseteq \{x \in G : \varphi_{v,f(\bar{x})}(f(x)) \leq \epsilon\} \subseteq W, \end{aligned}$$

i.e., F is upper semicontinuous at $f(\bar{x})$, so $(S, f, \leq)$ is B-well-posed at $f(\bar{x})$. □

Let S be a set and let $\mathcal{F}(S,\overline{\mathbb{R}})$ be the cone of all $(\overline{\mathbb{R}},\varepsilon)$-valued functions on S endowed with the pointwise operations and order. We may identify the elements $\epsilon\in\varepsilon$ with the constant functions $\hat{\epsilon}$ on S, that is $t\longmapsto\epsilon$ for all $t\in S$. Hence $\widehat{\varepsilon}=\{\hat{\epsilon}:\epsilon\in\varepsilon\}$ is a subset and a neighborhood system for $\mathcal{F}(S,\overline{\mathbb{R}})$. The neighborhoods $\hat{\epsilon}\in\widehat{\varepsilon}$ are defined for functions $f,g\in\mathcal{F}(S,\overline{\mathbb{R}})$ as

$$f\leq g+\hat{\epsilon}\quad\text{if}\quad f(x)\leq g(x)+\hat{\epsilon}(x)\quad\text{for all}\quad x\in S.$$

We consider the subcone $\mathcal{F}_{\widehat{\varepsilon}_b}(S,\overline{\mathbb{R}})$ of all functions in $\mathcal{F}(S,\overline{\mathbb{R}})$ that are bounded below relative to the functions in $\widehat{\varepsilon}$, that is $f\in\mathcal{F}_{\widehat{\varepsilon}_b}(S,\overline{\mathbb{R}})$ if for every $\epsilon\in\varepsilon$ there is $\lambda>0$ such that $0\leq f+\lambda\hat{\epsilon}$. In this way $(\mathcal{F}_{\widehat{\varepsilon}_b}(S,\overline{\mathbb{R}}),\widehat{\varepsilon})$ forms a full locally convex cone. For details see Example 4.1 (e) in [21].

Example 3.5. Let $X=\mathbb{R}$, $G=[0,1]$ and $S=(0,+\infty)$. If the mapping $f:G\to(\mathcal{F}_{\widehat{\varepsilon}_b}(S,\overline{\mathbb{R}}),\widehat{\varepsilon})$ is defined by

$$f(x)=\begin{cases}0, & \text{if } \ x=0,\\ xt, & \text{if } \ 0<x\leq 1,\end{cases}$$

then $\mathrm{Eff}(G,f,\leq)=\{0\}$. We show that the problem $(G,f,\leq)$ is B-well-posed at $f(0)$. Obviously, for every $\varepsilon\in\mathcal{V}$ we have $S(f,\leq_{\hat{\varepsilon}},f(0))=S(f,\leq,f(0))$. Let $\epsilon=1$. We have

$$\varphi_{1,f(0)}\circ f(x)=\begin{cases}0, & \text{if } \ x=0,\\ +\infty, & \text{if } \ 0<x\leq 1.\end{cases}$$

Thus, for $\psi:\varphi_{1,f(0)}\circ f(G)\to 2^G$ defined by $\psi(t)=S(\varphi_{1,f(0)}\circ f,\leq,t)$, we have

$$\psi(t)=\begin{cases}\{0\}, & \text{if } \ t=0,\\ [0,1], & \text{if } \ t=+\infty.\end{cases}$$

We can easily check that $\psi(t)$ is upper semicontinuous at 0, so the problem $(G,f,\leq)$ is B-well-posed at $f(0)$.

The notions of well-posedness in the scalar optimization problems are formally stronger than classical notions of well-posedness in [4]. We recall that the problem (G,h) is called *Tykhonov well-posed* if $\mathrm{Eff}(G,h)$ is a singleton and for every minimizing net $\{x_\alpha\}_{\alpha\in\mathcal{I}}\subseteq G$, $h(x_\alpha)$ converges to $\mathrm{Eff}(G,h)$, i.e., $h(x_\alpha)\to\inf_G h$. Also, (G,h) is *Tykhonov well-posed in the generalized sense* if $\mathrm{Eff}(G,h)\neq\emptyset$ and every minimizing net in G clusters to an element of $\mathrm{Eff}(G,h)$.

Remark 3.6. The nets Tykhonov well-posedness is equivalent to the sequential notion in [4] by Remark 2.3 (i). If $\mathcal{P}=\overline{\mathbb{R}}_+$ endowed with $\varepsilon=\{\epsilon:\epsilon>0\}$ and $G\subseteq\mathcal{P}$, then the definition of H-well-posedness of $(G,f,\leq)$ reduces to the notion of Tykhonov well-posedness; indeed, the symmetric topology of locally convex cone $\overline{\mathbb{R}}_+$ is the usual topology on $\overline{\mathbb{R}}_+$ with $+\infty$ as an isolated point, so if $(G,f,\leq)$ is H-well-posed at $\bar{x}$ then $\mathrm{Eff}(G,f)=\{\bar{x}\}$ by Remark 2.3 (ii). Also, the definition of weakly L-well-posedness at $\inf_G f$ reduces to the generalized Tykhonov well-posedness; in fact, $f(x_\alpha)\uparrow\inf_G f$ in $\overline{\mathbb{R}}_+$ implies that $f(x_\alpha)\to\inf_G f$ in symmetric topology of $\overline{\mathbb{R}}_+$.

Lemma 3.7. *The problem (G,h) is DH-well-posed at $\bar{x}\in\mathrm{Eff}(G,h)$ if and only if it is Tykhonov well-posed.*

Proof. Assume that (G,h) is DH-well-posed at $\bar{x}$. By Proposition 2.9 (c), (G,h) is H-well-posed at $\bar{x}$ and so, by Remark 3.6, (G,h) is Tykhonov well-posed. Conversely suppose that (G,h) is Tykhonov well-posed but there exist $\varepsilon>0$ and a neighborhood U of $\bar{x}$ such that $L_h(\bar{x},\varepsilon,\lambda)\nsubseteq U$ for all $\lambda>0$. Then, for a positive sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n\to 0$ one can find a sequence $\{x_n\}_{n\in\mathbb{N}}\subset G$ such that $x_n\in L_h(\bar{x},\varepsilon,\lambda_n)$ but $x_n\notin U$. Since $h(x_n)\leq\inf_G h+\lambda_n\varepsilon$, $\{x_n\}_{n\in\mathbb{N}}$ is a minimizing sequence and $x_n\to\bar{x}$, which is a contradiction. □

We next establish some equivalences between pointwise well-posedness of the optimization problem $(G,f,\leq)$ and well-posedness of the associated scalar problems.

Theorem 3.8. *The problem $(G, f, \leq)$ is DH-well-posed at $\bar{x} \in \mathit{Eff}(G, f, \leq)$ if and only if the scalarized problem $(G, \varphi_{v,f(\bar{x})} \circ f)$ is Tykhonov well-posed for all $v \in \mathcal{V}$.*

Proof. From Lemma 3.3, $\bar{x} \in \mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f)$ for all $v \in \mathcal{V}$, so by Lemma 3.7, it is enough to prove that $(G, f, \leq)$ is DH-well-posed at $\bar{x}$ if and only if the scalarized problem $(G, \varphi_{v,f(\bar{x})} \circ f, \leq)$ is DH-well-posed at $\bar{x}$ for all $v \in \mathcal{V}$. For $v \in \mathcal{V}$, $\epsilon = 1$ and $\lambda > 0$ we have

$$\begin{aligned} L_f(\bar{x}, v, \lambda) &= \{x \in G : f(x) \leq f(\bar{x}) + \lambda v\} \\ &\subseteq \{x \in G : \varphi_{v,f(\bar{x})}(f(x)) \leq \lambda\} \\ &= L_{\varphi_{v,f(\bar{x})} \circ f}(\bar{x}, 1, \lambda). \end{aligned}$$

Therefore if $(G, \varphi_{v,f(\bar{x})} \circ f)$ is DH-well-posed for all $v \in \mathcal{V}$ at $\bar{x}$, then $(G, f, \leq)$ is also DH-well-posed at $\bar{x}$. On the other hand, for every $v \in \mathcal{V}$ and $\lambda > \beta > 0$ we have

$$L_{\varphi_{v,f(\bar{x})} \circ f}(\bar{x}, 1, \beta) = \{x \in G : \varphi_{v,f(\bar{x})}(f(x)) \leq \beta\} \subseteq L_f(\bar{x}, v, \lambda),$$

so if $(G, f, \leq)$ is DH-well-posed at $\bar{x}$ then $(G, \varphi_{v,f(\bar{x})} \circ f)$ is DH-well-posed at $\bar{x}$ for all $v \in \mathcal{V}$. □

Corollary 3.9. *If $\bar{x} \in \mathit{Eff}(G, f, \leq)$ and $S(f, \leq, f(\bar{x})) = S(f, \leq_v, f(\bar{x}))$ for all $v \in \mathcal{V}$, then the problem $(G, f, \leq)$ is L-well-posed at $f(\bar{x})$ if and only if the scalarized problem $(G, \varphi_{v,f(\bar{x})} \circ f)$ is Tykhonov well-posed in generalized sense for all $v \in \mathcal{V}$.*

Proof. Suppose that $(G, \varphi_{v,f(\bar{x})} \circ f)$ is Tykhonov well-posed in the generalized sense for all $v \in \mathcal{V}$. If $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ is a $f(\bar{x})$-minimizing net, there exists $v \in \mathcal{V}$ and a positive real net $\{\lambda_\alpha\}_{\alpha \in \mathcal{I}}$ with $\lambda_\alpha \to 0$ such that $f(x_\alpha) \leq f(\bar{x}) + \lambda_\alpha v$. By parts (a) and (c) of Theorem 3.2, we have

$$0 = \inf_G(\varphi_{v,f(\bar{x})} \circ f) \leq \varphi_{v,f(\bar{x})}(f(x_\alpha)) \leq \varphi_{v,f(\bar{x})}(f(\bar{x})) + \lambda_\alpha = \lambda_\alpha,$$

so $\varphi_{v,f(\bar{x})}(f(x_\alpha)) \to 0$ which means that $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ is a minimizing net to problem $(G, \varphi_{v,f(\bar{x})} \circ f)$. Thus, $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ clusters to an element of $\mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f)$. By assumption and Lemma 3.3, we have $\mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f) = S(f, \leq_v, f(\bar{x})) = S(f, \leq, f(\bar{x}))$, so $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ clusters to an element of $S(f, \leq, f(\bar{x}))$,. i.e., $(G, f, \leq)$ is L-well-posed at $f(\bar{x})$. Conversely, let $v \in \mathcal{V}$. If $\{x_\alpha\}_{\alpha \in \mathcal{I}}$ is a minimizing net to problem $(S, \varphi_{v,f(\bar{x})} \circ f)$, then $\varphi_{v,f(\bar{x})}(f(x_\alpha)) \to 0$. Let $\alpha_0 \in \mathcal{I}$ such that $\varphi_{v,f(\bar{x})}(f(x_\alpha)) < +\infty$ for all $\alpha \geq \alpha_0$. For an arbitrary positive real net $\{\lambda_\alpha\}_{\alpha \in \mathcal{I}}$ with $\lambda_\alpha \to 0$, we have $f(x_\alpha) \leq f(\bar{x}) + (\varphi_{v,f(\bar{x})}(f(x_\alpha)) + \lambda_\alpha v$ for all $\alpha \geq \alpha_0$. Since $\varphi_{v,f(\bar{x})}(f(x_\alpha)) + \lambda_\alpha \to 0$, $\{x_\alpha\}_{\alpha \geq \alpha_0}$ is a $f(\bar{x})$-minimizing net, hence it clusters to an element of $S(f, \leq, f(\bar{x})) = S(f, \leq v, f(\bar{x})) = \mathrm{Eff}(G, \varphi_{v,f(\bar{x})} \circ f)$. Therefore, problem $(G, \varphi_{v,f(\bar{x})} \circ f)$ is Tykhonov well-posed in generalized sense. □

References

[1] C.D. Aliprantis, R. Tourky, *Cones and Duality. Graduate Studies in Mathematics*, vol. 84, American Mathematical Society, Providence, (2007)

[2] E.M. Bednarczuck, *An approach to well-posedness in vector optimization: consequences to stability and parametric optimization*, Control Cybern. **23**, 107-122 (1994)

[3] D. Dentcheva, S. Helbig, *On variational principles, level sets, well-posedness and -solutions in vector optimization*, J. Optim. Theory Appl. **89**, 325-349 (1996)

[4] A.L. Dontchev, T. Zolezzi, *Well-Posed Optimization Problems*, in: Lecture Notes in Math., vol. 1543, Springer-Verlag, Berlin, (1993)

[5] M. Durea, *Scalarization for pointwise well-posedness vectorial problems*, Math. Methods Oper. Res. **66**, 409-418 (2007)

[6] M. Furi, A. Vignoli, *About well-posed minimization problems for functionals in metric spaces*, J. Optim. Theory Appl. **5**, 225-229 (1970)

[7] C. Gutiérrez, E. Miglierina, E. Molho, V. Novo, *Pointwise well-posedness in set optimization with cone proper sets*, Nonlinear Anal. **75** 1822-1833 (2012)

[8] X.X. Huang, *Pointwise well-posedness of perturbed vector optimization problems in a vector-valued variational principle*, J. Optim. Theory Appl. **108**, 671-687 (2001)

[9] K. Keimel, W. Roth, *Ordered cones and approximation*, Lecture notes in math, vol. 1517, Springer-Verlag, Heidelberg, (1992)

[10] S. Khoshkhabar-amiranloo, E. Khorram, *Pointwise well-posedness and scalarization in set optimization*, Math. Meth. Oper. Res. **82**,195-210 (2015)

[11] X.J. Long, J.W. Peng, Z.Y. Peng, *Scalarization and pointwise well-posedness for set optimization problems*, J Global Optim. **62**, 763-773 (2015)

[12] P. Loridan, *Well-posedness in vector optimization*, In: Lucchetti, R., Recalski, J. (eds.) Recent Developments in CariationalWell-posedness Probems. Mathematics and Its Applications, vol. 331, pp. 171-192. Kluwer, Dordrecht (1995)
[13] E. Miglierina, E. Molho, M. Rocca, *Well-posedness and scalarization in vector optimization*, J. Optim. Theory Appl. **126**, 391-409 (2005)
[14] M.R. Motallebi, *Completeness on locally convex cones*, C. R. Math. Acad. Sci. Paris **352** 785-789 (2014)
[15] M.R. Motallebi, *On weak completeness of products and direct sums in locally convex cones.* Period. Math. Hung. 75(2), 322-329 (2017)
[16] M.R. Motallebi, H. Saiflu, *Products and direct sums in locally convex cones*, Can. Math. Bull. **55**(4), 783-798 (2012)
[17] W. Roth, *A uniform boundedness theorem for locally convex cones*, Proc. Amer. Math. Soc. **126**(7), 1973–1982 (1998)
[18] W. Roth, *Hahn-Banach type theorems for locally convex cones*, J. Aust. Math. Soc. (Series A). **68**, 104-125 (2000)
[19] W. Roth, *Separation properties for locally convex cones*, J. Conv. Anal. **9**, 301-307 (2002)
[20] W. Roth, *Boundedness and connectedness components for locally convex cones*. N. Z. J. Math. **34**, 143-158 (2005).
[21] W. Roth, *Operator-valued measures and integrals for cone-valued functions*, Lecture Notes in Mathematics, vol. 1964, Springer, Heidelberg, (2009)
[22] A.N. Tykhonov, *On the stability of the functional optimization problems*, USSR Comput. Math. Phys. **6**, 28-33 (1966)
[23] W.Y. Zhang, S.J. Li, K.L. Teo, *Well-posedness for set optimization problems*, Nonlinear Anal. **71**, 3768-3788 (2008)