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Approximation Properties of Modified q-Gamma Operators Preserving Linear Functions

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Abstract. In this paper, we introduce the q-analogue of modified Gamma operators preserving linear functions. We establish the moments of the operators using the q-Gamma functions. Next, some local approximation for the above operators are discussed. Also, the rate of convergence and weighted approximation by these operators in terms of modulus of continuity are studied. Furthermore, we obtain the Voronovskaja type theorem.

1. Introduction

It is well known that the modified Gamma operators are given by

$$G_n(f;x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-ux}u^n f\left(\frac{n}{u}\right)\mathrm{d}u \tag{1}$$

In 2007, Xu and Wang [6] introduced and estimated approximation properties for functions satisfying exponential growth condition of G_n defined above. They obtained the approximation properties to the locally bounded functions and absolutely continuous functions by using some inequalities and results of probability theory with the method of Bojanic-Cheng.

The q-calculus has attracted attention of many researchers because of its application in various fields such as numerical analysis, CAGD, differential equations, and so on. In the field of approximation theory, the application of q-calculus has been the area of many recent researches, it seems there are no papers mentioning the q-analogue of these operators defined in (1), which motivates us to introduce the q-analogue of this of modified Gamma operators.

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Before introducing the operators, we recall some concepts of q-calculus, details can be founded in ([1],[3],[4]). For $q > 0$ the q-integer $[n]_q$ is defined by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & \text{if } q \neq 1; \\ n, & \text{if } q = 1, \end{cases}$$

for $n \in \mathbb{N}$. Also, the q-factorial $[n]_q!$ is defined as

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q, & \text{if } n = 1, 2, \cdots; \\ 1, & \text{if } n = 0. \end{cases}$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x) \mathrm{d}_q x = (1-q) \sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, A > 0,$$

provided the sums converge absolutely.

The q-exponential function $E_q(x)$ is given as

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}, |q| < 1.$$

The q-Gamma integral is defined as

$$\Gamma_q(t) = \int_0^{\infty/A} x^{t-1} E_q(-qx) \mathrm{d}_q t, t > 0,$$

which satisfies the following functional equations: $\Gamma_q(n+1) = [n]_q!$.

Definition 1.1. *For $f \in C[0,\infty)$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce a kind of modified q-Gamma operators $G_{n,q}(f;x)$ as follows:*

$$G_n(f;x) = \frac{x^{n+1}}{\Gamma_q(n+1)} \int_0^{\infty/A} E_q(-qux) u^n f\left(\frac{[n]_q}{u}\right) \mathrm{d}_q u. \tag{2}$$

Obviously, $G_{n,q}(f;x)$ are positive linear operators. It is observed that for $q \to 1^-$, $G_{n,1^-}(f;x)$ become the modified Gamma operators defined in (1).

The paper is organized as follows: In the first section, we give the basic notations and the definition of modified q-Gamma operators. In the second section, we obtain the moments of these operators. In the third section and the fourth section, we study the local approximation and rate of convergence for these operators. In the fifth section and the sixth section, we establish weighted approximation and voronovskaja type theorem.

2. Auxiliary Results

In this section, we give some basic lemmas which will be useful to prove our main results.

Lemma 2.1. *For $q \in (0,1)$, $x \in [0,\infty)$ and $k = 0, 1, \cdots$, we have*

$$G_n(t^k;x) = \frac{[n]_q^k [n-k]_q!}{[n]_q!} x^k \tag{3}$$

Proof. Direct computation gives

$$\begin{aligned} G_n(t^k;x) &= \frac{x^{n+1}}{\Gamma_q(n+1)} \int_0^{\infty/A} E_q(-qux)u^n \left(\frac{[n]_q}{u}\right)^k \mathrm{d}_q u \\ &= \frac{[n]_q^k x^k}{\Gamma_q(n+1)} \int_0^{\infty/A} E_q(-qux)(ux)^{n-k} \mathrm{d}_q(ux) \\ &= \frac{[n]_q^k x^k}{\Gamma_q(n+1)} \Gamma_q(n-k+1) = \frac{[n]_q^k [n-k]_q!}{[n]_q!} x^k. \end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2. *For the operators $G_{n,q}(f;x)$ as defined in* (2), *the following equalities hold:*

1. $G_{n,q}(1;x) = 1$;
2. $G_{n,q}(t;x) = x$;
3. $G_{n,q}(t^2;x) = \frac{[n]_q}{[n-1]_q} x^2$, *for* $n > 1$;
4. $G_{n,q}(t^3;x) = \frac{[n]_q^2}{[n-1]_q[n-2]_q} x^3$, *for* $n > 2$;
5. $G_{n,q}(t^4;x) = \frac{[n]_q^3}{[n-1]_q[n-2]_q[n-3]_q} x^4$, *for* $n > 3$.

Proof. The proof of this Lemma is an immediate consequence of Lemma 2.1. Hence the details are omitted. □

Remark 2.3. *For every $q \in (0,1)$, we have*

$$G_{n,q}(t-x;x) = 0;$$

$$A(x) := G_{n,q}((t-x)^2;x) = \frac{q^{n-1}}{[n-1]_q} x^2;$$

$$G_{n,q}\left((t-x)^4;x\right) = \frac{3q^{n-3}(1-q)^2[n]_q^2 + 3q^{2n-5}(6-3q-2q^2)[n]_q + 18q^{3n-6}}{[n-1]_q[n-2]_q[n-3]_q} x^4.$$

Remark 2.4. *The sequences (q_n) satisfying $0 < q_n < 1$ such that $q_n \to 1$, $q_n^n \to a$ and $[n]_{q_n} \to \infty$ as $n \to \infty$ where $a \in [0,1]$, then*

1. $\lim_{n\to\infty} [n-1]_{q_n} G_{n,q_n}\left((t-x)^2;x\right) = ax^2$;
2. $\lim_{n\to\infty} [n-1]_{q_n} G_{n,q_n}\left((t-x)^4;x\right) = 0$.

Lemma 2.5. *For $f \in C_B[0,\infty)$ (space of all bounded and uniformly continuous functions on $[0,\infty)$ endowed with norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$, one has*

$$|G_{n,q}(f;x)\| \leq \|f\|.$$

Proof. In view of (2) and Lemma 2.2, the proof of this lemma easily follows. □

3. Local Approximation

In this section we establish direct local approximation theorem in connection with the operators $G_{n,q}(f;x)$. Let us consider the following K-functional:

$$K(f,\delta) = \inf_{g \in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [2],(p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K(f,\delta) \leq C\omega_2\left(f, \sqrt{\delta}\right) \tag{4}$$

where

$$\omega_2(f,\delta) = \sup_{0<|h|\leq\delta} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of f. By

$$\omega(f,\delta) = \sup_{0<|h|\leq\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

we denote the usual modulus of continuity of $\in C_B[0,\infty)$.

Our first result is a direct local approximation theorem for the operators $G_{n,q}(f;x)$:

Theorem 3.1. *Let $f \in C_B[0,+\infty)$, $q \in (0,1)$, then for every $x \in [0,\infty)$ and $n > 2$ we have*

$$|G_{n,q}(f;x) - f(x)| \leq C\omega_2\left(f, \sqrt{A(x)}\right),$$

where C is some positive constant.

Proof. Let for all $g \in C_B^2[0,\infty)$, using the Taylor's expansion for $x \in [0,\infty)$, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying the operators $G_{n,q}$ to both sides of above equality and using Remark 2.3, we get

$$\begin{aligned}\left|G_{n,q}(g;x) - g(x)\right| &= \left|G_{n,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u; x\right)\right| \\ &\leq G_{n,q}\left(\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right|; x\right) \\ &\leq G_{n,q}\left(\|g''\|(t-x)^2; x\right) \\ &\leq A(x)\|g''\|.\end{aligned}$$

By Lemma 2.5 , we have

$$\begin{aligned}\left|G_{n,q}(f;x) - f(x)\right| &\leq \left|G_{n,q}(f-g;x) - (f-g)(x)\right| + \left|G_{n,q}(g;x) - g(x)\right| \\ &\leq 2\left\|f-g\right\| + A(x)\|g''\|\end{aligned}$$

Lastly, taking infimum on both side of the above inequality over all $g \in C_B^2[0,\infty)$

$$\left|G_{n,q}(f;x) - f(x)\right| \leq 2K\left(f; A(x)\right)$$

for which we have the desired result by (4). This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $0 < \gamma \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. If $f \in C_B[0,\infty)$ is locally in* Lip(γ), i.e., *the condition*

$$|f(x) - f(t)| \leq L|x-t|^\gamma, t \in E \quad \text{and} \quad x \in [0,\infty)$$

holds, then, for each $x \in [0,\infty)$, we have

$$|G_{n,q}(f;x) - f(x)| \leq L\left\{(A(x))^{\frac{\gamma}{2}} + 2(\mathrm{d}(x;E)^\gamma)\right\},$$

where L is a constant depending on γ amd f; and d$(x;E)$ *is the distance between x and E defined by*

$$\mathrm{d}(x;E) = \inf\{|t-x| : t \in E\}.$$

Proof. From the properties of infinum, there is at least an point t_0 in the closure of E, that is $t_0 \in E$, such that

$$d(x;E) = |t_0 - x|.$$

Using the triangle inequality, we have

$$\begin{aligned}|G_{n,q}(f;x) - f(x)| &\le G_{n,q}(|f(t) - f(x)|;x)\\ &\le G_{n,q}(|f(t) - f(t_0)|;x) + G_{n,q}(|f(t_0) - f(x)|;x)\\ &\le L\left(G_{n,q}(|t - t_0|^{\gamma};x) + G_{n,q}(|t_0 - x|^{\gamma};x)\right)\\ &\le L\left(G_{n,q}(|t - x|^{\gamma};x) + 2|t_0 - x|^{\gamma}\right)\end{aligned}$$

Choosing $a_1 = \dfrac{2}{\gamma}$ and $a_2 = \dfrac{2}{2-\gamma}$ and using the well-known Hölder inequality

$$\begin{aligned}|G_{n,q}(f;x) - f(x)| &\le L\left\{\left(G_{n,q}(|t - x|^{\gamma a_1};x)\right)^{\frac{1}{a_1}}\left(G_{n,q}(1^{a_a};x)\right)^{\frac{1}{a_2}} + 2|t_0 - x|^{\gamma}\right\}\\ &\le L\left\{\left(G_{n,q}((t - x)^2;x)\right)^{\frac{\gamma}{2}} + 2|t_0 - x|^{\gamma}\right\}\\ &\le L\left\{A^{\frac{\gamma}{2}}(x) + 2(d(x;E))^{\gamma}\right\}\end{aligned}$$

This completes the proof. □

4. Rate of convergence

Let $C_\rho[0,\infty)$ be the set of all functions defined on $(0,\infty)$ satisfying the condition $|f(x)| \le C_f\rho(x)$, where $C_f > 0$ is a constant depending only on f and $\rho(x)$ is a weight function. Let $C^0_\rho[0,\infty)$ be the space of all continuous functions in $C_\rho[0,\infty)$ with the norm $\|f\|_\rho = \sup\limits_{x\in[0,\infty)} \dfrac{|f(x)|}{\rho(x)}$ and $C^0_\rho[0,\infty) = \left\{f \in C_\rho[0,\infty) : \lim\limits_{x\to\infty} \dfrac{|f(x)|}{\rho(x)} < \infty\right\}$. We consider $\rho(x) = 1 + x^2$ in the following two theorems. Meantime, we denote the modulus of continuity on f on the closed interval $[0,a]$, $a > 0$ by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]} |f(t) - f(x)|.$$

Obviously, for the function $f \in C_\rho[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero. Now we give a rate of convergence theorem for the operators $G_{n,q}(f;x)$.

Theorem 4.1. *Let $f \in C_\rho[0,\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then, for every $n > 2$,*

$$\|G_{n,q}(f;x) - f(x)\|_{C[0,a]} \le 4C_\rho(1 + a^2)A(x) + 2\omega_{a+1}\left(f, \sqrt{A(x)}\right)$$

Proof. For all $x \in [0,a]$ and $t > a+1$, we easily have $(t-x)^2 \ge (t-a)^2 \ge 1$, therefore,

$$\begin{aligned}|f(t) - f(x)| &\le |f(t)| + |f(x)| \le C_\rho(2 + x^2 + t^2)\\ &= C_\rho\left(2 + x^2 + (x - t - x)^2\right) \le C_\rho\left(2 + 3x^2 + 2(x - t)^2\right)\\ &\le C_\rho(4 + 3x^2)(t - x)^2 \le 4C_\rho(1 + a^2)(t - x)^2.\end{aligned} \tag{5}$$

and for all $x \in [0,a]$, $t \in [0,a+1]$ and $\delta > 0$, we have

$$|f(t) - f(x)| \le \omega_{a+1}(f,|t - x|) \le \left(1 + \frac{|t - x|}{\delta}\right)\omega_{a+1}(f,\delta) \tag{6}$$

From (5) and (6) , we get

$$|f(t) - f(x)| \leq 4C_\rho(1 + a^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{a+1}(f, \delta).$$

By Schwarz's inequality and Remark 2.3, we have

$$\begin{aligned} |G_{n,q}(f; x) - f(x)| &\leq G_{n,q}(|f(t) - f(x)|; x) \\ &\leq 4C_\rho(1 + a^2)G_{n,q}((t - x)^2; x) + G_{n,q}\left(\left(1 + \frac{|t - x|}{\delta}\right); x\right)\omega_{a+1}(f, \delta) \\ &\leq 4C_\rho(1 + a^2)G_{n,q}((t - x)^2; x) + \omega_{a+1}(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{G_{n,q}((t - x)^2; x)}\right) \\ &\leq 4C_\rho(1 + a^2)A(x) + \omega_{a+1}(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{A(x)}\right) \end{aligned}$$

by taking $\delta = \sqrt{A(x)}$, we get the proof of Theorem 4.1. □

As is known, if f is not uniformly continuous on the interval $(0, \infty)$, the usual first modulus of continuity $\omega(f; \delta)$ does not tend to zero as $\delta \to 0$. For every $f \in C^0_\rho[0, \infty)$, we would like to take a weighted modulus of continuity $\Omega(f; \delta)$ which tends to zero as $\delta \to 0$.

Let

$$\Omega(f; \delta) = \sup_{0<h\leq\delta, x\geq 0} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}, \quad \text{for every } f \in C^0_\rho[0, \infty).$$

The weighted modulus of continuity $\Omega(f; \delta)$ was defined by Yuksel and Ispir in [7]. It is known that $\Omega(f; \delta)$ has the following properties.

Lemma 4.2. *Let $f \in C^0_\rho[0, \infty)$, then:*
i) $\Omega(f; \delta)$ is a monotone increasing function of δ,
ii) For each $f \in C^0_\rho[0, \infty)$, $\lim_{\delta\to 0^+} \Omega(f; \delta) = 0$,
iii) For each $m \in \mathbb{N} \setminus \{0\}$, $\Omega(f; m\delta) \leq m\Omega(f; \delta)$,
iv) For each $\lambda \in \mathbb{R}^+$, $\Omega(f; \lambda\delta) \leq (1 + \lambda)\Omega(f; \delta)$.

Theorem 4.3. *Let $f \in C^0_\rho[0, \infty)$ and $q = q_n \in (0, 1)$ such that $q_n \to 1$ as $n \to \infty$, then there exists a positive constant K such that the inequality*

$$\sup_{x\in[0,\infty)} \frac{|G_{n,q_n}(f; x) - f(x)|}{(1 + x^2)^{\frac{5}{2}}} \leq K\Omega\left(f; \frac{1}{\sqrt{[n - 1]_{q_n}}}\right) \tag{7}$$

holds.

Proof. For $t > 0$, $x \in (0, \infty)$ and $\delta > 0$, by the definition of $\Omega(f; \delta)$ and Lemma 4.2, we get

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (x + |x - t|))^2\, \Omega\left(f; |t - x|\right) \\ &\leq 2(1 + x^2)\left(1 + (t - x)^2\right)\left(1 + \frac{|t - x|}{\delta}\right)\Omega(f; \delta). \end{aligned}$$

Since G_{n,q_n} is linear and positive, we have

$$\begin{aligned} \left|G_{n,q_n}(f; x) - f(x)\right| \leq 2(1 + x^2)\Omega(f; \delta)\Big\{1 + G_{n,q_n}\left((t - x)^2; x\right) + \\ G_{n,q_n}\left(\left(1 + (t - x)^2\right)\frac{|t - x|}{\delta}; x\right)\Big\}. \end{aligned} \tag{8}$$

Using Remark 2.3, we have

$$G_{n,q_n}\left((t-x)^2;x\right) \le K_1 \frac{1+x^2}{[n-1]_{q_n}}, \tag{9}$$

for some positive constant K_1. To estimate the second term of (8), applying the Cauchy-Schwartz inequality, we have

$$G_{n,q_n}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta};x\right) \le 2\left(G_{n,q_n}\left(1+(t-x)^4;x\right)\right)^{\frac{1}{2}}\left(G_{n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}}.$$

By Remark 2.3 and (9), there exist two positive constants K_2, K_3 such that

$$\left(G_{n,q_n}\left(1+(t-x)^4;x\right)\right)^{\frac{1}{2}} \le K_2(1+x^2)$$

and

$$\left(G_{n,q_n}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}} \le \frac{K_3}{\delta}\sqrt{\frac{1+x^2}{[n-1]_{q_n}}}.$$

Now we take $K = 2+2K_1+4K_2K_3$ and $\delta = \frac{1}{\sqrt{[n-1]_{q_n}}}$. Combining the above estimates, we obtain the inequality (7). The proof is completed. □

5. Weighted Approximation

Now, we obtaion the weighted approximation theorem as follows:

Theorem 5.1. *Let the sequence $q=\{q_n\}$ satisfy $q_n \in (0,1)$, $q_n \to 1$ and $[n]_{q_n} \to \infty$ as $n \to \infty$. Then for $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} \|G_{n,q_n}(f) - f\|_\rho = 0.$$

Proof. Using Korovkin's theorem (see[5]), it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \|G_{n,q_n}(t^k) - x^k\|_\rho = 0, k = 0,1,2. \tag{10}$$

Since $G_{n,q_n}(1;x) = 1$, $G_{n,q_n}(t;x) = x$, (10) holds for $k = 0,1$.

By Lemma 2.2, we have,

$$\begin{aligned}\left\|G_{n,q_n}(t^2;x) - x^2\right\|_\rho &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|G_{n,q_n}(t^2;x) - x^2\right| \\ &= \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2}\left|\frac{[n]_{q_n}}{[n-1]_{q_n}} - 1\right| \\ &= \frac{q_n^{n-1}}{[n-1]_{q_n}} \\ &= 0, n \to \infty.\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|G_{n,q_n}(t^2;x) - x^2\right\|_\rho = 0.$$

Thus the proof is completed. □

Now, we present a weighted approximation theorem for function in $C_\rho[0,\infty)$.

Theorem 5.2. *Let $q = q_n \in (0,1)$ satisfies $q_n \to 1$ as $n \to \infty$. For every $f \in C_\rho[0,\infty)$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|G_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0.$$

Proof. Let $x_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned} \sup_{x\in[0,\infty)} \frac{|G_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} &= \sup_{x\in[0,x_0]} \frac{|G_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x\in(x_0,\infty)} \frac{|G_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} \\ &\leq \|G_{n,q_n}(f;x) - f(x)\|_{C[0,x_0]} + \|f\|_\rho \sup_{x\in(x_0,\infty)} \frac{|G_{n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\alpha}} \\ &+ \sup_{x\in(x_0,\infty)} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \end{aligned} \tag{11}$$

Since $|f(x)| \leq \|f\|_\rho(1+x^2)$, we have $\sup\limits_{x\in(x_0,\infty)} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_\rho}{(1+x_0^2)^\rho}$. Let $\epsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\|f\|_\rho}{(1+x_0^2)^\rho} < \epsilon. \tag{12}$$

In view of Lemma 2.2, while $x \in (x_0,\infty)$, we obtain

$$\|f\|_\rho \lim_{n\to\infty} \frac{|G_{n,q_n}((1+t^2);x)|}{(1+x^2)^{1+\alpha}} = \frac{(1+x^2)}{(1+x^2)^{1+\alpha}}\|f\|_\rho = \frac{\|f\|_\rho}{(1+x^2)} \leq \frac{\|f\|_\rho}{(1+x_0^2)} < \epsilon.$$

Using Theorem 3.1, we can see that the first term of the inequality (11), implies that

$$\|G_{n,q_n}(f;x) - f(x)\|_{C[0,x_0]} < \epsilon, \text{ as } n \to \infty. \tag{13}$$

Combining (11)-(13), we get the desired result. □

6. Voronovskaja Type Theorem

In this section, we give a Voronovskaja-type asymptotic formula for $G_{n,q_n}(f;x)$ by means of the second and fourth central moments.

Theorem 6.1. *Let $q = q_n \in (0,1)$ satisfying $q_n \to 1$, $q_n^n \to a \in [0,1]$, $[n]_{q_n} \to \infty$. For $f \in C_B^2[0,\infty)$, the following equality holds*

$$\lim_{n\to\infty} [n-1]_{q_n} \left(G_{n,q_n}(f;x) - f(x)\right) = \frac{a}{2} f''(x)x^2 \tag{14}$$

for every $x \in [0,A]$, $A > 0$.

Proof. Let $x \in [0,\infty)$ be fixed. In order to prove this identity, we use Taylor's expansion

$$f(t) - f(x) = (t-x)f'(x) + (t-x)^2\left(\frac{f''(x)}{2} + \Phi_{q_n}(t,x)\right),$$

where $\Phi_{q_n}(x,t)$ is bounded and $\lim_{t\to x}\Phi_{q_n}(t,x)=0$. By applying the operator $G_{n,q_n}(f;x)$ to the above relation, we obtain

$$\begin{aligned} G_{n,q_n}(f;x)-f(x) &= f'(x)G_{n,q_n}((t-x);x)+\frac{1}{2}f''(x)G_{n,q_n}\left((t-x)^2;x\right) \\ &+ G_{n,q_n}\left(\Phi_{q_n}(t,x)(t-x)^2;x\right) \\ &= \frac{1}{2}f''(x)G_{n,q_n}\left((t-x)^2;x\right)+G_{n,q_n}\left(\Phi_{q_n}(t,x)(t-x)^2;x\right). \end{aligned}$$

Since $\lim_{t\to x}\Phi_{q_n}(t,x)=0$, then for all $\epsilon>0$, there exists $\delta>0$ such that $|t-x|<\delta$ implies $|\Phi_{q_n}(t,x)|<\epsilon$ for all fixed $x\in[0,\infty)$ where n large enough. While if $|t-x|\geq\delta$, then $|\Phi_{q_n}(t,x)|\leq\frac{C}{\delta^2}(t-x)^2$, where $C>0$ is a constant. Using Remark 2.3, we have

$$\lim_{n\to\infty}[n-1]_{q_n}G_{n,q_n}((t-x)^2;x)=ax^2$$

and

$$[n-1]_{q_n}\left|G_{n,q_n}\left(\Phi_{q_n}(t,x)(t-x)^2;x\right)\right|\leq\epsilon[n-1]_{q_n}G_{n,q_n}\left((t-x)^2;x\right)$$

$$+\frac{C}{\delta^2}[n-1]_{q_n}G_{n,q_n}\left((t-x)^4;x\right)\to 0\ (n\to\infty).$$

The proof is completed. □

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