Filomat 34:5 (2020), 1611–1620
https://doi.org/10.2298/FIL2005611V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximate Controllability of Random Impulsive Quasilinear Evolution Equation

A. Vinodkumar[a], C. Loganathan[b], S.Vijay[c]

[a]*Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Coimbatore-641 112, INDIA.*
[b]*Department of Mathematics, Maharaja Arts and Science college, Coimbatore- 641 407, Tamil Nadu, INDIA.*
[c]*Department of Mathematics, Government Arts college, Udhagamandalam - 643002, Tamil Nadu, INDIA.*

Abstract.
In this paper, we study the approximate controllability of quasilinear evolution equation with random impulsive moments with less restriction and sufficient condition. The results are obtained by the theory of C_0 semigroup of bounded linear operators on evolution equations.

1. Introduction

Approximate controllability plays an important role in science and engineering. There exists different types of controllability like null control, exact control, approximate control, feedback control, relative control ect., which were studied in literature. The exact controllability is not always realizable in infinite dimensional space, to overcome this situation we look for approximate controllability. Further, the quasilinear evolution equation is one form of an evolution equation. The time dependent problems in physics can represent by such evolution equation. For further study on quasilinear evolution equation and approximate controllability see [1, 6–8, 13–18] and the reference therein.

The processes of short-term perturbations can be modelled by an impulsive systems. Many authors studied different type of impulsive systems which have existed in the history see [12, 19]. Most of the papers deal the problem with fixed time impulses, but in real time situation it need not be at fixed times may be at random time. When the impulses exist at random times, then the solutions of the differential equations are a stochastic process. It is very different from deterministic impulsive control systems and also it is different from impulsive stochastic control systems. Thus the random impulsive system gives more realistic than deterministic impulsive system.

There are few publications in this field, In [3], the author studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [4] through Banach fixed point method for the system of differential equations with random impulsive effect. In [5, 20, 21] the author studied the existence results for the random impulsive neutral functional differential

2010 *Mathematics Subject Classification*. Primary 93B05, 35R60; Secondary 93E99.
Keywords. Quasilinear control system, random impulses, approximate controllability.
Received: 10 August 2018; Revised: 12 February 2020; Accepted: 22 March 2020
Communicated by Adrian Petrusel
Email addresses: vinod026@gmail.com, a_vinodkumar@cb.amrita.edu (A. Vinodkumar), clogu@rediffmail.com (C. Loganathan), vijay_ms_oty@yahoo.co.in (S.Vijay)

equations and differential inclusions with delays. In [2], the author detailed the stabilization nature for random impulsive moments in differential equations with exponential distribution. Recently, in [24, 25], the authors investigated the unstable continuous time delay systems controlled by the random impulses and the effects of random impulses shown by the simulation results. Further, the inaccuracy in finding expectation for the solution $x(t)$ of the random impulsive differential systems [3, 4, 20–23] is rectified in this paper.

Only a few papers study the existence result and controllability result of quasilinear evolution equation sees [9–11] and the reference therein. Motivated by the above mentioned works, the main purpose of this paper is to study the approximate controllability of random impulsive quasilinear control systems. We extend the results with and without fixed impulsive term in quasilinear control systems to fill the gap in the approximate controllability of abstract control systems. To the best of our knowledge, there is no paper which study the random impulsive quasilinear control systems. We utilize the technique developed in [12, 15, 17, 19].

The paper will be organized as follows: In section 2, we recall briefly the notations, definitions, preliminary facts which are used throughout this paper. In section 3, we study the approximate controllability of random impulsive quasilinear control systems. Further, we extend our results to fixed impulsive quasilinear control systems and non-impulsive quasilinear control systems.

2. Preliminaries

Let X and Y be two real separable Hilbert spaces such that $Y \hookrightarrow X$ is dense and continuous and Ω a nonempty set. Assume that $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent exponentially distributed random variable with parameter λ, and each random variable τ_k is defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Let $\tau, T \in \Re$ be two constants satisfying $\tau < T$. Let us denote $\Re_\tau = [\tau, T]$.

We consider quasi-linear control system with random impulses of the form

$$\begin{aligned} x' - A(t,x)x &= (Bu)(t) + f(t, x_t), \quad \xi_k < t < \xi_{k+1}, \quad t \in [t_0, T], && (1) \\ x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \ldots, && (2) \\ x_{t_0} &= \varphi, && (3) \end{aligned}$$

where $A(t, x)$ is a linear operator in X for each x in an open subset $\bar{B}$ of X; $f : [t_0, T] \times C \to X, C = C([-r, 0], X)$ is the set of piecewise continuous functions mapping $[-r, 0]$ in to X with some given $r > 0$; u belongs to $L_2[t_0, T; U]$ be the control function. Let $\mathcal{Z} = L_2[t_0, T; X]$ and $\mathcal{Y} = L_2[t_0, T; U]$ be function spaces, $B : \mathcal{Y} \to \mathcal{Z}$ is a bounded linear operator, x_t is a function when t is fixed, defined by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$; $\xi_0 = t_0 \geq 0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \cdots$, here $t_0 \in \Re_\tau$ is arbitrary given real number. The impulse moments $\{\xi_k\}$ form a strictly increasing sequence, ie., $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$; $b_k : D_k \to X$ for each $k = 1, 2, \cdots$; $x(\xi_k^-) = \lim_{t\uparrow\xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{t-r\leq s\leq t} |x(s)|$ for each t satisfying $t \geq t_0$, $\|\cdot\|$ is any given norm in X; φ is a function defined from $[-r, 0]$ to X.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \mathcal{F}_t, X)$ denote the Hilbert space of all $\mathcal{F}_t$ - measurable square integrable random variables with values in X.

Assume that $T > t_0$ is any fixed time to be determined later and let $\mathcal{B}$ denote the Banach space $\mathcal{B}\Big([t_0 - r, T], L_2\Big)$, the family of all $\mathcal{F}_t$-measurable, C-valued random variables ψ with the norm

$$\|\psi\|_{\mathcal{B}} = \left(\sup_{t_0\leq t\leq T} E\|\psi\|_t^2\right)^{1/2}.$$

Let $L_2^0(\Omega, \mathcal{B})$ denote the family of all $\mathcal{F}_0$ - measurable, $\mathcal{B}$ - valued random variable φ.

Remark 2.1. *For any given time $t > 0$ and integer $n \geq 1$, ξ_n and B_t are related by $\{B_t \geq n\} = \{\xi_n \leq t\}$. It is understood that, $\{\xi_n \leq t\}$ is the event that the n^{th} arrival occurs by time t, implies that B_t, the number of arrivals by time t, must be at least n. Similarly, $\{B_t \geq n\}$ implies $\{\xi_n \leq t\}$, yielding the equality. The counting process $\{B_t, t \geq 0\}$ is a stochastic process in time. The given filtration $\mathcal{F}_t$ represents the evolution of knowledge about the random system through time. The information at time t carried by filtration $\mathcal{F}_t$ determines the value of the random variable ξ_k.*

lemma 2.2. *[2] The probability that there will be exactly k impulses until the time $t, t \geq t_0$,where impulse moments $\xi_k, k = 1, 2, \cdots$ follow exponential distribution with parameter λ, is given by the equality $P(I_{[\xi_k,\xi_{k+1})}(t)) = \frac{\lambda^k (t-t_0)^k}{k!} e^{-\lambda(t-t_0)}$. where the events $I_{[\xi_k,\xi_{k+1})}(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_{k+1}(\omega)\}, k = 1, 2, \cdots$.*

Remark 2.3. *From [2], expected value of the solution $x(t)$ for the random impulsive differential equations given as*

$$E[\|x(t)\|] = \sum_{k=0}^{\infty} E[\|x(t)\| \| I_{[\xi_k,\xi_{k+1})}(t)] P(I_{[\xi_k,\xi_{k+1})}(t)),$$

where the impulse moments $\xi_k, k = 1, 2, \cdots$ follow exponential distribution with parameter λ.

The following definition for the evolution family of operators. For further read on quasilinear operator and evolution operator see monograph [17] and [1, 9] and the references therein.

Definition 2.4. *A two parameter family of bounded linear operators $S(t, s), t \geq s \geq 0$, on X is called an evolution system if*

(i) $S(s, s) = I$ and $S(t, r)S(r, s) = S(t, s), t \geq r \geq s \geq 0$;
(ii) $(t, s) \to S(t, s)$ is strongly continuous for $t \geq s \geq 0$.

If $u \in \mathcal{B}$ and the family $\{A(t, w),\ (t, w) \in [t_0, T] \times X\}$, then there exists an evolution system $S_h(t, s)$ in X satisfying for any fixed $h \in \mathcal{B}$:

(iii) $\|S_h(t, s)\|_{B(X)} \leq M$, for $t \geq s \geq 0$, where M is a constant,
(iv) $S_h(t, t) = I,\ S_h(t, s)S_h(s, r) = S_h(t, r),\ (t, s, r) \in [t_0, T] \times [t_0, T] \times [t_0, T]$, and moreover,
(v) $\frac{\partial S_h(t,s)}{\partial t} = A_h(t)S_h(t, s)$, for almost all $t \in [t_0, T]$, for all $s \in [t_0, T]$.

Moreover, there exist a constant $\mu > 0$ such that for every $u, v \in \mathcal{B}$ with values in $\bar{B}$ and every $y \in Y$ we have

$$\|S_u(t, s)y - S_v(t, s)y\|^2 \leq \mu \|y\|_Y^2 \int_s^t \|u(\tau) - v(\tau)\|^2 d\tau.$$

3. Main result

We consider quasi-linear differential equations with random impulses of the form

$$x' - A(t, x)x = f(t, x_t),\ \ \xi_k < t < \xi_{k+1},\ \ t \in [t_0, T], \tag{4}$$
$$x(\xi_k) = b_k(\tau_k)x(\xi_k^-),\ \ k = 1, 2, \ldots, \tag{5}$$
$$x_{t_0} = \varphi, \tag{6}$$

Definition 3.1. *For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B},\ t_0 - r \leq t \leq T\}$ is said to be a mild solutions to equation (4) -(6) in $(\Omega, P, \{\mathcal{F}_t\})$, if*

(i) *$x(t) \in \mathcal{B}$ is piecewise continuous and $\mathcal{F}_t$−adapted for $t \in [t_0, T)$;*
(ii) *$x(t_0 + s) = \varphi(s) \in L_2^0(\Omega, \mathcal{B})$, when $s \in [-r, 0]$,*
and

$$\begin{aligned} x(t) = \sum_{k=0}^{+\infty} \Bigg[&\prod_{i=1}^{k} b_i(\tau_i) S_x(t, t_0)\varphi + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_x(t, s) f(s, x_s) ds \\ &+ \int_{\xi_k}^{t} S_x(t, s) f(s, x_s) ds \Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [t_0, T], \end{aligned} \tag{7}$$

where $\prod_{j=m}^{n}(\cdot) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i)$, *and* $I_A(\cdot)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

Now we introduce following hypotheses used in our discussion:
(H_1): The function $f : [t_0, T] \times C \to X$ satisfy the Lipschitz conditions: that is, there exit constants $L_1 > 0$ for $\zeta, \varsigma \in X$ and for every $t_0 \leq t \leq T$ such that

$$\begin{aligned} \left\| f(t,\zeta) - f(t,\varsigma) \right\|^2 &\leq L_1 \, \|\zeta - \varsigma\|_t^2 , \\ \text{and} \quad \left\| f(t,0) \right\|^2 &\leq \kappa_1, \quad \kappa_1 > 0 \end{aligned}$$

(H_2): $\sup\{\|U_x(t,s)\|^2_{B(X)} = M, \quad (t,s) \in [t_0,T] \times [t_0,T]\}$
(H_3): $E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\left\|b_j(\tau_j)\right\|\right\}\right\}$ is uniformly bounded, that is, there is $C > 0$ such that

$$E\left\{\max_{i,k}\left\{\prod_{j=i}^{k}\left\|b_j(\tau_j)\right\|\right\}\right\} \leq C \quad \text{for all } \tau_j \in D_j, \;\; j = 1, 2, \cdots .$$

lemma 3.2. $E\|\psi(t)\|^2 \leq \kappa$, *where* $\psi(t) = \int_{t_0}^{t} f(s, y_s)ds$.

Proof.

$$\begin{aligned} E\|\psi(t)\|^2 &\leq 2E\int_{t_0}^{t}\left[\|f(s,y_s) - f(s,0)\|^2 + \|f(s,0)\|^2\right]ds \\ &\leq 2L_1\int_{t_0}^{t} E\|y\|_s^2 ds + 2\kappa_1(T - t_0)ds. \end{aligned}$$

Hence the result. □

Theorem 3.3. *Assume that hypotheses* (H_1)-(H_3) *be hold. Then, the system* (4) -(6) *has unique mild solution on* $[-r, T]$.

Proof. Let T be an arbitrary number $T > t_0$ such that

$$\Lambda(T) = \left[3(T - t_0)\{e^{-\lambda(1-C)(t-t_0)}\mu\|\varphi\|^2 + e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T - t_0)(M^2L_1 + \mu\kappa)\}\right] < 1. \tag{8}$$

In order to apply the contraction principle, we define the nonlinear operator $\Phi : \mathcal{B} \to \mathcal{B}$ as follows

$$(\Phi x)(t) = \varphi(t - t_0), \;\; \text{for } t \in [-r, t_0]$$

and for $t \in [t_0, T]$

$$(\Phi x)(t) = \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(\tau_i)S_x(t,t_0)\varphi + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S_x(t,s)f(s,x_s)ds + \int_{\xi_k}^{t} S_x(t,s)f(s,x_s)ds\right]I_{[\xi_k,\xi_{k+1})}(t),$$

It is easy to prove the continuity of Φ. Now, we have to show Φ is a contraction mapping. For any $x, y \in \mathcal{B}$, we have

$$\begin{aligned}
E\|(\Phi x)(t)-(\Phi y)(t)\|^2 &\leq 3E\|S_x(t,t_0)\varphi - S_y(t,t_0)\varphi\|^2\\
&\times \sum_{k=0}^{+\infty}\Big(\prod_{i=1}^{k} E\|b_i(\tau_i)\|^2 I_{[\xi_k,\xi_{k+1})}(t)\Big)P(I_{[\xi_k,\xi_{k+1})}(t))\\
&+ 3\sum_{k=0}^{+\infty} E\Big[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\tau_j)\right\|^2\right\}\Big]\\
&\times \Big(\int_{t_0}^{t}\{\|S_x(t,s)\|\|f(s,x_s)-f(s,y_s)\|\}ds I_{[\xi_k,\xi_{k+1})}(t)\Big)^2 P(I_{[\xi_k,\xi_{k+1})}(t))\\
&+ 3\sum_{k=0}^{+\infty} E\Big[\max_{i,k}\left\{1,\prod_{j=i}^{k}\left\|b_j(\tau_j)\right\|^2\right\}\Big]\\
&\times \Big(\int_{t_0}^{t}\|S_x(t,s)f(s,y_s)-S_y(t,s)f(s,y_s)\|\}ds I_{[\xi_k,\xi_{k+1})}(t)\Big)^2 P(I_{[\xi_k,\xi_{k+1})}(t)).
\end{aligned}$$

By (H_3), lemma 2.2 and lemma 3.2,

$$\begin{aligned}
E\|(\Phi x)-(\Phi y)\|_t^2 &\leq 3\mu\|\varphi\|^2(T-t_0)E\|x-y\|_t^2 e^{-\lambda(1-C)(t-t_0)}\\
&+ 3M^2L_1(T-t_0)^2E\|x-y\|_t^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}\\
&+ 3\mu\kappa(T-t_0)^2E\|x-y\|_t^2 e^{-\lambda(1-\max\{1,C\})(t-t_0)}.\\
E\|(\Phi x)-(\Phi y)\|_t^2 &\leq 3(T-t_0)\{e^{-\lambda(1-C)(t-t_0)}\mu\|\varphi\|^2\\
&+ e^{-\lambda(1-\max\{1,C\})(t-t_0)}(T-t_0)(M^2L_1+\mu\kappa)\}E\|x-y\|_t^2
\end{aligned}$$

Hence,

$$\|(\Phi x)-(\Phi y)\|_{\mathcal{B}}^2 \leq \Lambda(T)\,\|x-y\|_{\mathcal{B}}^2.$$

By (8), we get $\Lambda(T)<1$.

Then we can take a suitable $0<T_1<T$ sufficient small such that $\Lambda(T_1)<1$, and hence Φ is a contraction on $\mathcal{B}_{T_1}$ ($\mathcal{B}_{T_1}$ denotes $\mathcal{B}$ with T substituted by T_1). Thus, by the well-known Banach fixed point theorem we obtain a unique fixed point $x\in\mathcal{B}_{T_1}$ for operator Φ, and hence $\Phi x = x$ is a mild solution of (4) − (6). This procedure can be repeated to extend the solution to the entire interval $[-r,T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $[-r,T]$.

□

From the above result, a mild solution for the control system (1)-(3) can be written as follows

Definition 3.4. *For a given $T\in(t_0,+\infty)$, a stochastic process $\{x(t)\in\mathcal{B},\ t_0-r\leq t\leq T\}$ is said to be a mild solution to equation* (1)-(3) *in* $(\Omega,P,\{\mathcal{F}_t\})$, *if*

(i) $x(t)\in\mathcal{B}$ *is piecewise continuous and $\mathcal{F}_t$–adapted for $t\in[t_0,T)$;*

(ii) $x(t_0+s)=\varphi(s)\in L_2^0(\Omega,\mathcal{B})$, *when $s\in[-r,0]$;*

and

$$\begin{aligned}
x(t) &= \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)S_x(t,t_0)\varphi + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S_x(t,s)\Big[Bu(s)+f(s,x_s)\Big]ds\\
&+\int_{\xi_k}^{t} S_x(t,s)\Big[Bu(s)+f(s,x_s)\Big]ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t\in[t_0,T],
\end{aligned} \tag{9}$$

where $\prod_{j=m}^{n}(\cdot) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i)$, *and* $I_A(\cdot)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & \text{if } \ t \in A, \\ 0, & \text{if } \ t \notin A. \end{cases}$$

Definition 3.5. *The control system* (1)-(3) *is said to be approximately controllable on* $[t_0, T]$ *if for any* $\epsilon > 0$*, the initial function* $\varphi \in C$*, there exist a control* $u \in \mathcal{Y}$ *such that the mild solution* $x(\cdot)$ *of* (1)-(3) *satisfies*

$$E\|x(T) - x_1\|^2 \leq \epsilon.$$

Let $x_t(\varphi(t_0), u)$ denotes state value of the system (1)-(3) at time t corresponding to the control $u \in \mathcal{Y}$ and the initial value $\varphi(t_0)$. The set of all possible trajectories, denoted by

$$K_\alpha(f) = \{x_\alpha(\varphi(t_0), u) \in C([\alpha, T], X) : u \in \mathcal{Y}, 0 < \alpha \leq T\} \tag{10}$$

is called the trajectory reachable set of system (1). In particular, the reachable set of the system (1)-(3) at terminal time T is defined by

$$K_T(f) = \{x_T(\varphi(t_0), u) : u \in \mathcal{Y}\}. \tag{11}$$

A control system is said to be approximately controllable on $[t_0, T]$, if $\overline{K_T(f)} = X$.

Now we define
(i) the solution $\mathcal{W}$ from $\mathcal{Z}$ to $C([t_0, T], X)$ can be defined by $(\mathcal{W}u)(t) = x(\varphi(t_0), u)(\cdot)$, $u \in \mathcal{Z}$.
(ii) the continuous operator $\mathcal{Q}$ from $\mathcal{Z}$ to $C([t_0, T], X)$ is

$$(\mathcal{Q}p)(t) = \int_{t_0}^{t} S_x(t, s)p(s)ds, \ \ p \in \mathcal{Z}, \ t \in [t_0, T].$$

(iii) the functions, $F : L_2[t_0, T; C] \to \mathcal{Z}$ as $(Fx)(t) = f(t, x_t(\cdot)); x \in L_2[t_0, T; C]$ and $\hat{B} : \mathcal{Y} \to L_2[t_0, T; X]$, as $(\hat{B}u)(t) = Bu(t)$.

Now we introduce additional hypotheses used in our following discussion:
(H_4): For any given $\epsilon > 0$, $p(\cdot) \in \mathcal{Z}$, there exists some control $u(\cdot) \in \mathcal{Y}$ such that

$$E\|\mathcal{Q}p - \mathcal{Q}Bu\|^2 < \epsilon.$$

(H_5): $R(F) \subseteq \overline{R(\hat{B})}$.

In the following lemma, we prove the approximate controllability of the following linear system of the form

$$\begin{aligned} x' - A(t, x)x &= (Bu)(t), \ \ \xi_k < t < \xi_{k+1}, \quad t \in [t_0, T], && (12)\\ x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \ \ k = 1, 2, \ldots, && (13)\\ x_{t_0} &= \varphi, && (14) \end{aligned}$$

lemma 3.6. *Under hypothesis* (H_4), $\overline{K_\alpha(0)} = X$.

Proof. Since, $D(A)$ is dense in X, it is sufficient to prove that $D(A) \subseteq \overline{K_\alpha(0)}$, that is, for given $\epsilon > 0$, and $\mathcal{O} \in D(A)$ there exists a control function $u(\cdot) \in \mathcal{Y}$ such that

$$E\|\mathcal{O} - \hat{\mathcal{O}} - \mathcal{Q}Bu\|^2 < \epsilon,$$

where $\hat{\mathcal{O}} = \sum_{k=0}^{+\infty} \left[\prod_{i=1}^{k} b_i(\tau_i) S_x(t, t_0)\varphi \right] I_{[\xi_k, \xi_{k+1})}(t), \ t \in [\alpha, T].$

Let us take $O \in D(A)$, then $O - \hat{O} \in D(A)$. It can be seen that there exists some function $p(\cdot) \in \mathcal{Z}$ such that $O - \hat{O} = Qp$.

By the assumption (H_4), for any given $\epsilon > 0$, there exists some control $u(\cdot) \in \mathcal{Y}$ such that $E\|Qp - QBu\|^2 < \epsilon$.

Since ϵ is arbitrary, we infer that $K_\alpha(0) \subseteq D(A)$. The denseness domain $D(A)$ in X implies the approximate controllability of the linear system (12) − (14). □

In the following theorem, we prove the approximate controllability of the quasi-linear control system (1)−(3)

Theorem 3.7. *Let the hypotheses* $(H_1) - (H_5)$ *be hold. Then we have* $K_\alpha(0) \subseteq \overline{K_\alpha(f)}$.

Proof. Let $x(\cdot) \in K_\alpha(0)$, there exists a $u \in \mathcal{Y}$, which can be written as

$$x(t) = \varphi(t - t_0), \quad \text{for } t \in [-r, t_0],$$

for $t \in [t_0, T]$,

$$\begin{aligned} x(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S_x(t, t_0)\varphi \\ &+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_x(t, s)Bu(s)ds + \int_{\xi_k}^{t} S_x(t, s)Bu(s)ds \Big] I_{[\xi_k, \xi_{k+1})}(t), \end{aligned}$$

Since $Fx \in \overline{R(B)}$ for a given $\epsilon > 0$ there exists a $w \in \mathcal{Y}$ such that

$$E\|Fx - \hat{B}w\|^2_{\mathcal{B}} \leq \epsilon. \tag{15}$$

Now, let $y(t)$ be mild solution of (1)-(3) corresponding to the control $u - w$, then

$$\begin{aligned} x(t) - y(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S_x(t, t_0)\varphi \\ &+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_x(t, s)Bu(s)ds + \int_{\xi_k}^{t} S_x(t, s)Bu(s)ds \Big] I_{[\xi_k, \xi_{k+1})}(t) \\ &- \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S_y(t, t_0)\varphi \\ &+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_y(t, t_0)[Fy](s)ds + \int_{\xi_k}^{t} S_y(t, t_0)[Fy](s)ds \Big] I_{[\xi_k, \xi_{k+1})}(t) \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i)[S_x(t, s)\varphi - S_y(t, s)\varphi] + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_x(t, s)[\hat{B}w - Fx](s)ds \\ &+ \int_{\xi_k}^{t} S_x(t, s)[\hat{B}w - Fx](s)ds \Big] I_{[\xi_k, \xi_{k+1})}(t) \\ &+ \sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S_x(t, s)[Fx - Fy](s)ds + \int_{\xi_k}^{t} S_x(t, s)[Fx - Fy](s)ds \Big] I_{[\xi_k, \xi_{k+1})}(t) \\ &+ \sum_{k=0}^{+\infty} \left[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [S_x(t, s)Fy(s) - S_y(t, s)Fy(s)]ds + \int_{\xi_k}^{t} [S_x(t, s)Fy(s) - S_y(t, s)Fy(s)]ds \right] I_{[\xi_k, \xi_{k+1})}(t) \end{aligned}$$

$$
\begin{aligned}
\|x(t)-y(t)\|^2 \leq\;& 4\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\tau_i)\|\|S_x(t,s)\varphi - S_y(t,s)\varphi\|\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 P(I_{[\xi_k,\xi_{k+1})}(t)) \\
&+ 4\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S_x(t,s)\|\|[\hat{B}w - Fx](s)\|ds \\
&+ \int_{\xi_k}^{t}\|S_x(t,s)\|\|[\hat{B}w - Fx](s)\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 P(I_{[\xi_k,\xi_{k+1})}(t)) \\
&+ 4\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S_x(t,s)\|\|[Fx - Fy](s)\|ds \\
&+ \int_{\xi_k}^{t}\|S_x(t,s)\|\|[Fx - Fy](s)\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 P(I_{[\xi_k,\xi_{k+1})}(t)) \\
&+ 4\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S_x(t,s)Fy(s) - S_y(t,s)Fy(s)\|ds \\
&+ \int_{\xi_k}^{t}\|S_x(t,s)Fy(s) - S_y(t,s)Fy(s)\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^2 P(I_{[\xi_k,\xi_{k+1})}(t)) \\
E\|x-y\|_t^2 \leq\;& 4\mu\|\varphi\|^2(T-t_0)E\|x-y\|_t^2 e^{-\lambda(1-C)(t-t_0)} \\
&+4e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)\int_{t_0}^{t}E\|[\hat{B}w - Fx](s)\|^2 ds \\
&+4e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)\int_{t_0}^{t}E\|[Fx - Fy](s)\|^2 ds \\
&+4e^{-\lambda(1-\max\{1,C\})(t-t_0)}\mu\kappa(T-t_0)E\|x-y\|_t^2 \\
\leq\;& 4\mathcal{J}e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)^2\epsilon \\
&+4\mathcal{J}e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)L_1\int_{t_0}^{t}E\|x-y\|_s^2 ds
\end{aligned}
$$

Taking supremum over t, and by Grownwall's inequality we get,

$$
\|x-y\|_{\mathcal{B}}^2 \leq 4\mathcal{J}e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)^2\epsilon\exp(4\mathcal{J}e^{-\lambda(1-\max\{1,C\})(t-t_0)}M^2(T-t_0)^2L_1),
$$

where $\mathcal{J} = \frac{1}{1-4\mu(T-t_0)\{e^{-\lambda(1-C)(t-t_0)}\|\varphi\|^2+\kappa e^{-\lambda(1-\max\{1,C\})(t-t_0)}\}} > 0$. From the above inequality it is clear that $\|x-y\|_{\mathcal{B}}^2$ can be made arbitrarily small by choosing suitable w. Hence the theorem is proved. □

Theorem 3.8. *Under assumption of the above theorem, system* (1) − (3) *is approximately controllable if its corresponding linear system is approximately controllable.*

Proof. The proof is a particular case of Theorem 3.7 at $\alpha = T$. □

The following remark generalize the system (1) − (3) with fixed impulsive system under sufficient condition.

Remark 3.9. *If the impulses are exist at fixed times in the system* (1) − (3), *then by the similar argument as in the Theorem* 3.8, *the system* (1) − (3) *is approximately controllable.*

When there is no impulse condition, then the problem becomes abstract quasilinear differential equations with delay.

Remark 3.10. *If there is no impulses, then the system* (1) − (3) *becomes,*

$$x' - A(t,x)x = (Bu)(t) + f(t,x_t), \quad t \in [t_0, T], \tag{16}$$

$$x_{t_0} = \varphi, \tag{17}$$

Theorem 3.11. *Let the hypotheses* $(H_1)-(H_2)$ *and* $(H_4)-(H_5)$ *be hold. Then the system* (16) − (17) *is approximately controllable.*

Proof. The proof is a particular case of Theorem 3.7 at $\alpha = T$. □

4. An application

Example 4.1. *Consider the following partial differential control systems with random impulses of the form*

$$\begin{cases} \dfrac{\partial z(x,t)}{\partial t} + \dfrac{\partial^3 z(x,t)}{\partial x^3} + z(x,t)\dfrac{\partial z(x,t)}{\partial x} = \boldsymbol{B}u(x,t) + H(z(x,tsint),t), \ \xi_k < t < \xi_{k+1}, \\ z(x,\xi_k) = q(k)\tau_k z(x,\xi_k^-), \qquad t = \xi_k, \\ z(0,t) = z(\pi,t) = 0 \\ z(x,t) = \varphi(x,t) \qquad 0 \le x \le \pi, \quad -r < t \le 0, \ \ t \ge 0. \end{cases} \tag{18}$$

where $\mu > 0$ *and* q *is a function of* k; $\xi_0 = t_0$; $\xi_k = \xi_{k-1} + \tau_k$ *for* $k = 1, 2, \ldots$; τ_i *and* τ_j *are independent with each other as* $i \neq j$ *for* $i, j = 1, 2, \ldots$ *and follow exponentially distributed random variable with parameter* λ.

For every reals s *we introduce a Hilbert space* $H^s(R)$ *as follows [17]. Let the linear space functions* $z \in L^2(R)$ *and* $\hat{z}$ *is the Fourier transform of* z, *then*

$$\|z\|_s = \Big(\int_R (1+\gamma^2)^s |\hat{z}(\gamma)|^2 d\gamma\Big)^{\frac{1}{2}}.$$

Let the linear space functions $z \in L^2(R)$ *with the inner product is defined as*

$$< z, y >_s = \Big(\int_R (1+\gamma^2)^s |\hat{z}(\gamma)| \overline{\hat{y}(\gamma)} d\gamma\Big)^{\frac{1}{2}}.$$

Denote Hilbert space $H^s(R)$ *with respect to the norm* $\|\cdot\|_s$. *From this, it is clear that* $H^0(R) = L^2(R)$.

Let us consider $X = U = H^0(R) = L^2(R)$ *and* $Y = H^s(R), s \ge 3$. *Define an operator* A_0 *by* $D(A_0) = H^3(R)$ *and* $A_0 z = D^3 z$ *for* $z \in D(A_0)$ *where* $D = d/dx$. *Then* A_0 *is the infinitesimal generator of a* C_0 *group of isometries on* X. *Now we define for every* $v \in Y$ *an operator* $A_1(v)$ *by* $D(A_1(v)) = H^1(R)$ *and* $z \in D(A_1(v)), A_1(v)z = vDz$. *Then for every* $v \in Y$ *the operator* $A(v) = A_0 + A_1(v)$ *is the infintesimal generator of* C_0 *semigroup* $S_x(t,0)$ *on* X *satisfying* $\|S_x(t,0)\| \le e^{\kappa t}$ *for every* $\kappa \ge c\|v\|_s$, *where* c *is a constant independent of* $v \in Y$. *Let* B_r *be the ball of radius* $r > 0$ *in* Y *and it is proved that the family of operators* $A(v)$, $v \in B_r$, *satisfies the conditions of definition* (2.4) *and* H_2 *(see [17]).*

Now we define an infinite dimensional control space U *by* $U = \sum_{n=2}^{\infty} u_n \kappa_n$ *with* $\sum_{n=2}^{\infty} u_n^2 < \infty$ *with the norm* $|u|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. *Define a continuous linear map from* U *to* X *as follows* $Bu = 2u_2\kappa_1 + \sum_{n=2}^{\infty} u_n \kappa_n$, *for* $u = \sum_{n=2}^{\infty} u_n \kappa_n \in U$.

We assume the following conditions hold:

(i) $E\Big[\|q(j)(\tau_j)\|^2\Big] < \infty$.

Assuming that condition (*i*) *is verified. Further* (H_1) *holds, then the problem* (18) *can be modeled as the abstract form of the equations* (1) *by defining*

$f(t,x_t) = H(z(x,tsint),t), \quad b_k(\tau_k) = q(k)\tau_k. \quad$ *and* $R(F) \subseteq \overline{R(\hat{B})}$.

Proposition 4.2. *Let the hypotheses* (H_1)-(H_5) *be hold. Then the mild solution* z *of the system* (18) *is approximately controllable.*

Proof. Condition (*i*) *implies that* (H_3) *holds.* □

References

[1] S. Abbas, D. Bahuguna, *Existence of solutions to quasilinear functional differential equations*, Electron. J. Differ. Equ. 2009 (164) (2009), 1-8.

[2] R. Agarwal, S. Hristova and Donal O'Regan, *Exponential stability for differntial equations with random impulses at random times*, Advances in Difference Equations 2013, 2013:372.

[3] A.Anguraj, S. Wu and A. Vinodkumar, *Existence and Exponential Stability of Semilinear Functional Differential Equations with Random Impulses under Non-uniqueness*, Nonlinear Anal. TMA, 74(2011)331-342.

[4] A. Anguraj, A.Vinodkumar, *Existence, Uniqueness and Stability Results of Random Impulsive Semilinear Differential Systems*, Nonlinear Anal. Hybrid Syst. Vol.4,3(2010), 475-483.

[5] A. Anguraj, A.Vinodkumar, *Existence and Uniqueness of Neutral Functional Differential Equations with Random Impulses*, I. J. Nonlinear Sci., 8:4(2009), 412-418.

[6] K. Balachandran, J.Y. Park and S. H. Park, *Controllability of nonlocal impulsive quasilinear integrodifferential systems in Banach spaces*, Reports on Math. Phys., Vol. 65, No. 2, pp. 247-257, 2010.

[7] P. Balasubramaniam, and C. Loganathan, *Controllability of neutral functional differential equations with unbounded delay in Banach spaces*, J. Indian Math. Soc., 68(2001), 191-203.

[8] Bouzahir Hassane, Fu Xianlong, *Controllability of neutral functional differential equations with infinite delay*, Acta Mathematica Scientia, 31B (1), pp. 73 -80.

[9] A. Debbouchea, D.Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comp. Math. Appl., 62 (2011), 1442–1450

[10] S. Kato, *Nonhomogeneous quasi-linear evolution equations in Banach spaces*, Nonlinear Anal., 9 (1985), 1061-1071.

[11] T. Kato, *Nonlinear semigroups and evolution equations in Banach spaces*, J. Math. Soc. Japan, 19 (1967), 508-520.

[12] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[13] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAMJ. Control Optim. 25 (1987), no. 3, 715–722.

[14] K. Naito, and J. Park, *Approximate Controllability for Trajectories of a Delay Volterra Control System*, J. Optim. ther. Appli., 61(1989), 271-279.

[15] N. K. Tomar and N. Sukavanam, *Approximate controllability of non-densely defined semilinear delayed control systems*, Nonlinear studies, 18(2) (2011), 229-234.

[16] N. K. Tomar and S. Kumar, *Approximate controllability of nonlocal semilinear time-varying delay control systems*, Nonlinear Dynamics and System Theory 12 (3) (2012), 303- 310.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Newyork, 1983.

[18] J.W. Ryu, J. Y. Park, and Y. C. Kwun, *Approximate controllability of delay Volterra control system*, Bull. Korean Math. Soc. 30 (1993), no. 2, 277–284.

[19] A.M. Samoilenko, N.A Perestyuk., *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[20] A.Vinodkumar, A. Anguraj, *Existence of random impulsive abstract neutral non-autonomous differential inclusions with delays*, Nonlinear Anal. Hybrid Syst. 5 (2011), 413–426.

[21] A.Vinodkumar, *Existence results on random impulsive semilinear functional differential inclusions with delays*, Ann. Funct. Anal. 3 (2012), no. 2, 89-106.

[22] Vinodkumar, A., K. Malar, M. Gowrisankar, and P. Mohankumar., *Existence, uniqueness and stability of random impulsive fractional differential equations*, Acta Mathematica Scientia, 36B(2) (2016) 428-442.

[23] S.Vijay, C. Loganathan and A. Vinodkumar, *Approximate controllability of random impulsive semilinear control systems*, Nonlinear Studies, 23(2)(2016), 273-280.

[24] A. Vinodkumar, T. Senthilkumar, X. Li, *Robust exponential stability results for uncertain infinite delay differential systems with random impulsive moments*, Advances in Difference Equations 2018 (1) (2018) 39.

[25] A. Vinodkumar, T. Senthilkumar, X. Li, *Exponential stability results on random and fixed time impulsive differential systems with infinite delay*, Mathematics 7 (9) (2019) 843.