Connecting Quantum Calculus and Harmonic Starlike Functions

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Abstract. Quantum calculus or $q-$calculus plays an important role in hypergemetric series, quantum physics, operator theory, approximation theory, sobolev spaces, geometric functions theory and others. But role of $q-$calculus in the theory of harmonic univalent functions is quite new. In this paper, we make an attempt to connect quantum calculus and harmonic univalent starlike functions. In particular, we introduce and investigate the properties of $q-$harmonic functions and $q-$harmonic starlike functions of order α.

1. Introduction

Quantum calculus is the traditional calculus without the use of limits. Quantum calculus or $q-$calculus dates back to Leonhard Euler (1707-1783) and Carl Gustav Jakobi (1804-1851). But $q-$calculus became popular only after its usefulness in quantum mechanics after 1905 paper by Albert Einstein. In 1909 and 1910, Jackson initiated in-depth study of $q-$calculus (see [13–15]).

Throughout this paper, we shall assume that q satisfies the condition $q \in (0, 1)$. The power series converges for $|z| < 1$ when $q \in (0, 1)$ and this guarentees the analyticity of the power series in the open unit disc; see for details [10].

Definition 1.1. *Let $q \in (0, 1)$ and $\lambda \in \mathbb{C}$. The $q-$number, denoted by $[\lambda]_q$, is defined by*

$$[\lambda]_q = \frac{1 - q^\lambda}{1 - q}.$$

When $\lambda = n \in \mathbb{N}$, we obtain $[n]_q = 1 + q + q^2 + ... + q^{n-1}$, and when $q \to 1^-$, then $[n]_q = n$.

Applying the above $q-$number and motivated by Jackson [13], $q-$derivative is defined below.

Definition 1.2. *The $q-$derivative (or $q-$difference operator) of a function f, defined on a subset of $\mathbb{C}$, is given by*

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0. \end{cases} \tag{1}$$

We note that $\lim_{q \to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z.

Note that such an operator plays an important role in the theory of hypergeometric series, quantum physics, sobolev spaces, geometric functions theory; see for instance [6, 9, 10, 17].

For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}.$$

It is obvious that q−derivative of a function f is a linear operator. That is, for any constants a and b, we have

$$D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z).$$

It is also straight forward to verify that

$$D_q(f(z).g(z)) = g(z)D_q f(z) + f(qz)D_q g(z),$$

and

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_q f(z) - f(qz)D_q g(z)}{g(z)g(qz)}, \quad g(z)g(qz) \neq 0.$$

Jackson [14] also introduced the q−integral of any function f by

$$\int_0^z f(t)d_q t = z(1 - q) \sum_{n=0}^{\infty} q^n f(zq^n),$$

provided that the series on right hand side converges.

Let $\mathcal{A}$ denote the class of normalized functions of the form

$$F(z) = \sum_{n=1}^{\infty} a_n z^n, \ a_1 = 1 \tag{2}$$

which are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. In view of (1) and (2), it follows that for any $F \in \mathcal{A}$, we have

$$D_q F(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \quad \text{and} \quad D_q(zD_q F(z)) = \sum_{n=1}^{\infty} [n]_q^2 a_n z^{n-1}, \tag{3}$$

where $q \in (0, 1)$.

A q−analog of the class of analytic starlike functions, denoted by PS_q, was introduced and studied by Ismail, Merkes and Styer in 1990 [12]. In fact, these authors defined the class

$$PS_q = \left\{F \in \mathcal{A} : \left|\frac{z(D_q F)(z)}{F(z)} - \frac{1}{1 - q}\right| \leq \frac{1}{1 - q}, \ z \in \mathbb{D}\right\}.$$

In [20], the researchers studied the following three types of classes of q−starlike functions of order α, $\alpha \in [0, 1)$.

$$S_{q,1}^*(\alpha) = \left\{F \in \mathcal{A} : Re\left(\frac{z(D_q F)(z)}{F(z)}\right) > \alpha, \ z \in \mathbb{D}\right\},$$

$$S_{q,2}^*(\alpha) = \left\{F \in \mathcal{A} : \left|\frac{\frac{z(D_q F)(z)}{F(z)} - \alpha}{1 - \alpha} - \frac{1}{1 - q}\right| \leq \frac{1}{1 - q}, \ z \in \mathbb{D}\right\},$$

$$S_{q,3}^*(\alpha) = \left\{F \in \mathcal{A} : \left|\frac{z(D_q F)(z)}{F(z)} - 1\right| \leq 1 - \alpha, \ z \in \mathbb{D}\right\}.$$

In [12], the researchers showed that the class PS_q satisfies the property $S^* = \bigcap_{0<q<1} PS_q$, where S^* is the well-known traditional class of starlike functions.

In order to connect $q-$calculus and harmonic univalent functions, we first need some notations and terminology of harmonic univalent functions.

Let $\mathcal{H}$ denote the family of continuous complex-valued sense-preserving functions $f = h + \overline{g}$ in the unit disc $\mathbb{D}$ that are harmonic and where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{4}$$

Note that $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)/h'(z)| < 1$ in $\mathbb{D}$. We also let $\mathcal{S}_\mathcal{H}$ be a subclass of functions f in $\mathcal{H}$ that are univalent in $\mathbb{D}$. Clunie and Sheil-Small [7] observed that $\mathcal{S}_\mathcal{H}$, when $b_1 = 0$, is a compact family with respect to the topology of locally uniform convergence. We observe that for $g(z) \equiv 0$ in $\mathbb{D}$, the class $\mathcal{S}_\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions in $\mathbb{D}$. For history of famous family $\mathcal{S}$, one may refer to the survey article by the first author [1].

We also recall that convolution of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} z^n} \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{2n} z^n}$$

is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b_{1n} b_{2n} z^n}, \quad (z \in \mathbb{D}).$$

A comprehensive study for the theory of harmonic univalent functions may be found in Duren [8]. One may also refer to the survey articles by the first author [2, 3].

In [5], Ahuja *et al.* defined the class $\mathcal{H}_q$ consisting of $q-$harmonic functions in $\mathbb{D}$.

Definition 1.3. *A harmonic function $f = h + \overline{g}$ defined by (4) is said to be $q-$harmonic, locally univalent and sense-preserving in $\mathbb{D}$ denoted by $\mathcal{H}_q$, if and only if the second dilatation w_q satisfies the condition*

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1, \tag{5}$$

where $q \in (0, 1)$ and $z \in \mathbb{D}$. Note that as $q \to 1^-$, $\mathcal{H}_q$ reduces to the family $\mathcal{H}$.

We now define a new subclass $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ of the class $\mathcal{H}_q$.

Definition 1.4. *A function $f = h + \overline{g}$ in $\mathcal{H}_q$ is said to be in $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ if*

$$Re\left(\frac{z D_q h(z) - \overline{z D_q g(z)}}{h(z) + \overline{g(z)}} \right) > \alpha, \quad |z| = r < 1, \tag{6}$$

where $q \in (0, 1)$ and $\alpha \in [0, 1)$. A function f in $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ is called $q-$harmonic starlike function of order α.

It is easy to verify that

$$\bigcap_{0 < q < 1} \mathcal{S}^*_{\mathcal{H}_q}(\alpha) \subset \mathcal{S}^*_{\mathcal{H}}(\alpha).$$

Remark 1.5. *When $q \to 1^-$, then the class $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ reduces to the traditional class $\mathcal{S}^*_{\mathcal{H}}(\alpha)$. This class is called harmonic starlike functions of order α (see for example [2, 3, 8, 16]).*

Remark 1.6. *When $q \to 1^-$ and $g(z) \equiv 0$ in $\mathbb{D}$, then the class $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ reduces to the traditional class $\mathcal{S}^*(\alpha)$. This class is called starlike functions of order α which was studied by Robertson in 1936 [18].*

Remark 1.7. *Suppose $g(z) \equiv 0$ in $\mathbb{D}$. Then*

(i) $\mathcal{S}_{q,3}^*(\alpha) \subset \mathcal{S}_{q,2}^*(\alpha) \subset \mathcal{S}_{q,1}^*(\alpha)$ *for* $\alpha \in [0, 1)$, *[20].*

(ii) $\mathcal{S}_{\mathcal{H}_q}^*(\alpha) \equiv \mathcal{S}_q^*(\alpha)$, *[19].*

(iii) $\bigcap_{0<q<1} \mathcal{S}_{q,1}^*(\alpha) = \bigcap_{0<q<1} \mathcal{S}_{q,2}^*(\alpha) = \mathcal{S}^*(\alpha)$, *[20].*

(iv) $\bigcap_{0<q<1} \mathcal{S}_{q,1}^*(\alpha) = \bigcap_{0<q<1} \mathcal{S}_{q,3}^*(\alpha) \subset \mathcal{S}^*(\alpha)$, *[20].*

In this paper, we make an attempt to investigate some applications of q–calculus in the theory of harmonic univalent functions. In particular, we obtain convolution characterization and sufficient coefficient condition for the class $\mathcal{S}_{\mathcal{H}_q}^*(\alpha)$. We also determine coefficient characterization, coefficient bounds, extreme points, convex combinations, distortion and covering theorems for a subclass of $\mathcal{S}_{\mathcal{H}_q}^*(\alpha)$ of functions with negative coefficients.

2. Analytic Criteria

We first obtain necessary and sufficient convolution conditions for q–harmonic starlike functions of order α.

Theorem 2.1. *Let $f = h + \overline{g}$ with h and g of the form (4) and suppose that f belongs to $\mathcal{H}_q$. Then $f \in \mathcal{S}_{\mathcal{H}_q}^*(\alpha)$ if and only if*

$$h(z) * \frac{\left(2 - 2\alpha\right)z + \left(2\alpha + \zeta - 1\right)qz^2}{(1 - z)(1 - qz)} - \overline{g(z)} * \frac{\left(2\zeta + 2\alpha\right)\overline{z} - \left(2\alpha + \zeta - 1\right)q\overline{z}^2}{(1 - \overline{z})(1 - q\overline{z})} \neq 0, \tag{7}$$

where $|\zeta| = 1$, $\zeta \neq -1$, $\alpha \in [0, 1)$, $q \in (0, 1)$ *and* $0 < |z| < 1$.

Proof. Since at $z = 0$ we have

$$\frac{zD_qh(z) - \overline{zD_qg(z)}}{h(z) + \overline{g(z)}} = 1, \tag{8}$$

it follows that the required condition (6) is equivalent to

$$\frac{\frac{zD_qh(z) - \overline{zD_qg(z)}}{h(z) + \overline{g(z)}} - \alpha}{1 - \alpha} \neq \frac{\zeta - 1}{\zeta + 1}. \tag{9}$$

It is straight forward to verify that

$$f(z) * \frac{z}{1 - z} = f(z) \quad \text{and} \quad f(z) * \frac{z}{(1 - z)(1 - qz)} = zD_qf(z). \tag{10}$$

Using (10) and algebraic manipulations, (9) is equivalent to

$$0 \neq (\zeta + 1)\left[zD_qh(z) - \overline{zD_qg(z)} - \alpha h(z) - \alpha\overline{g(z)}\right] - (\zeta - 1)\left[h(z) + \overline{g(z)} - \alpha h(z) - \alpha\overline{g(z)}\right]$$

$$= h(z) * \left(\frac{(\zeta + 1)z}{(1 - z)(1 - qz)} - \frac{\alpha(\zeta + 1)z}{1 - z} - \frac{(\zeta - 1)z}{1 - z} + \frac{\alpha(\zeta - 1)z}{1 - z}\right)$$

$$- \overline{g(z)} * \overline{\left(\frac{(\overline{\zeta} + 1)z}{(1 - z)(1 - qz)} + \frac{\alpha(\overline{\zeta} + 1)z}{1 - z} + \frac{(\overline{\zeta} - 1)z}{1 - z} - \frac{\alpha(\overline{\zeta} - 1)z}{1 - z}\right)}$$

$$= h(z) * \left(\frac{\left(2 - 2\alpha\right)z + \left(2\alpha + \zeta - 1\right)qz^2}{(1 - z)(1 - qz)}\right) - \overline{g(z)} * \overline{\left(\frac{\left(2\overline{\zeta} + 2\alpha\right)z - \left(2\alpha + \overline{\zeta} - 1\right)qz^2}{(1 - z)(1 - qz)}\right)}.$$

This proves the required convolution condition given by (7). $\quad\square$

Remark 2.2. *When $q \to 1^-$ and $\alpha = 0$, the condition (7) reduces to the corresponding results obtained in [4].*

Theorem 2.1 helps us to establish the following sufficient condition.

Theorem 2.3. *Let $f = h + \overline{g}$ with h and g of the form (4). Suppose*

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q - \alpha}{1 - \alpha} |a_n| + \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right) \leq 2, \tag{11}$$

where $a_1 = 1, q \in (0,1)$ and $\alpha \in [0,1)$. Then

(a) $f \in \mathcal{H}_q$,
(b) f univalent in $\mathbb{D}$, and
(c) $f \in \mathcal{S}_{\mathcal{H}_q}^*(\alpha)$, where f belongs to $\mathcal{H}_q$.

Proof. (a) In view of Definition 1.3, we only need to establish $|D_q h(z)| > |D_q g(z)|$. This condition follows from the following statements

$$|D_q h(z)| \geq 1 - \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1}$$

$$> 1 - \sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} |a_n|$$

$$\geq \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} |b_n|$$

$$> \sum_{n=1}^{\infty} [n]_q |b_n| |z|^{n-1} \geq |D_q g(z)|.$$

This also proves that f in $\mathcal{H}_q$ is locally univalent and sense-preserving in $\mathbb{D}$.

(b) In order to prove that f is univalent, we will show that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$. Suppose $z_1, z_2 \in \mathbb{D}$ so that $z_1 \neq z_2$. Since $\mathbb{D}$ is simply connected and convex, we have $z(t) = (1 - t)z_1 + tz_2 \in \mathbb{D}$, where $0 \leq t \leq 1$. Then we can write

$$f(z_2) - f(z_1) = \int_0^1 \left[(z_2 - z_1) D_q h(z(t)) + \overline{(z_2 - z_1) D_q g(z(t))} \right] d_q t.$$

Dividing the above equation by $z_2 - z_1 \neq 0$ and taking the real parts, we obtain

$$Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 Re \left[D_q h(z(t)) + \frac{\overline{(z_2 - z_1)}}{z_2 - z_1} \overline{D_q g(z(t))} \right] d_q t$$

$$> \int_0^1 \left[Re D_q h(z(t)) - |D_q g(z(t))| \right] d_q t$$

$$\geq \int_0^1 \left[Re D_q h(z(t)) - \sum_{n=1}^{\infty} [n]_q |b_n| \right] d_q t$$

$$\geq \int_0^1 \left[1 - \sum_{n=2}^{\infty} [n]_q |a_n| - \sum_{n=1}^{\infty} [n]_q |b_n| \right] d_q t$$

$$\geq \int_0^1 \left[1 - \sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} |a_n| - \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right] d_q t.$$

The last inequality is non-negative by the condition (11). This proves that $f(z_1) \neq f(z_2)$. This leads to the univalence of f.

(c) For $f = h + \overline{g}$ belonging to $\mathcal{H}_q$, $|\zeta| = 1$, $\zeta \neq -1$ and using (10), we have

$$\left| h(z) * \left(\frac{(2 - 2\alpha)z + (2\alpha + \zeta - 1)qz^2}{(1 - z)(1 - qz)} \right) - \overline{g(z)} * \overline{\left(\frac{(2\overline{\zeta} + 2\alpha)z - (2\alpha + \overline{\zeta} - 1)qz^2}{(1 - z)(1 - qz)} \right)} \right|$$

$$= \left| (2 - 2\alpha)zD_q h(z) + (2\alpha + \zeta - 1)(zD_q h(z) - h(z)) \right.$$

$$\left. + \overline{(2\overline{\zeta} + 2\alpha)zD_q \overline{g(z)} - (2\alpha + \overline{\zeta} - 1)(zD_q \overline{g(z)} - \overline{g(z)})} \right|$$

$$= \left| (2 - 2\alpha)z + \sum_{n=2}^{\infty} \left((\zeta + 1)[n]_q - 2\alpha - (\zeta - 1) \right)a_n z^n \right.$$

$$\left. + \sum_{n=1}^{\infty} \overline{\left((\overline{\zeta} + 1)[n]_q + 2\alpha + (\overline{\zeta} - 1) \right)b_n z^n} \right|$$

$$\geq (2 - 2\alpha)|z| \left[1 - \sum_{n=2}^{\infty} \left| \frac{(\zeta + 1)[n]_q - 2\alpha - (\zeta - 1)}{2 - 2\alpha} \right| |a_n||z|^{n-1} \right.$$

$$\left. - \sum_{n=1}^{\infty} \left| \frac{(\overline{\zeta} + 1)[n]_q + 2\alpha + (\overline{\zeta} - 1)}{2 - 2\alpha} \right| |b_n||z|^{n-1} \right]$$

$$> (2 - 2\alpha)|z| \left[1 - \sum_{n=2}^{\infty} \left| \frac{(\zeta + 1)[n]_q - 2\alpha - (\zeta - 1)}{2 - 2\alpha} \right| |a_n| \right.$$

$$\left. - \sum_{n=1}^{\infty} \left| \frac{(\overline{\zeta} + 1)[n]_q + 2\alpha + (\overline{\zeta} - 1)}{2 - 2\alpha} \right| |b_n| \right]$$

$$= 2(1 - \alpha)|z| \left[1 - \sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} |a_n| - \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \right].$$

This last expression is non-negative because of the condition (11). In view of Theorem 2.1, it follows that $f \in \mathcal{S}^*_{\mathcal{H}_q}(\alpha)$.

The q−harmonic starlike mappings

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q - \alpha} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{[n]_q + \alpha} \overline{y_n z^n}, \tag{12}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by (11) is sharp. □

Remark 2.4. *If $g(z) \equiv 0$ in $\mathbb{D}$, then we obtain the corresponding result for q−starlike functions of order α obtained in [20].*

Remark 2.5. *When $q \to 1^-$, (11) reduces to the corresponding sufficient coefficient condition for harmonic starlike functions of order α obtained in [16]. Moreover, when $q \to 1^-$, $\alpha = 0$ and $g(z) \equiv 0$ in $\mathbb{D}$, Theorem 2.3 reduces to the corresponding results for analytic starlike functions discovered by Goodman [11].*

Suppose $\mathcal{T}^0 \mathcal{H}_q$ is a subclass of $\mathcal{H}_q$ which consists of functions $f = h + \overline{g}$, where h and g are of the following form.

$$h(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \quad \text{and} \quad g(z) = - \sum_{n=2}^{\infty} |b_n|z^n. \tag{13}$$

Theorem 2.6. *A function $f \in \mathcal{T}^0\mathcal{H}_q$ if and only if*

$$\sum_{n=2}^{\infty}[n]_q(|a_n| + |b_n|) \le 1, \tag{14}$$

where $q \in (0, 1)$.

Proof. We first note that for $\alpha = 0$, $a_1 = 1$ and $b_1 = 0$, (11) reduces to (14). It therefore follows from Theorem 2.3 that $f \in \mathcal{T}^0\mathcal{H}_q$. In order to prove "only if" part, suppose $f \in \mathcal{T}^0\mathcal{H}_q$ and $\sum_{n=2}^{\infty}[n]_q(|a_n|+|b_n|) = 1+\epsilon, (\epsilon > 0)$. Then there exists an integer N such that

$$\sum_{n=2}^{N}[n]_q(|a_n| + |b_n|) > 1 + \epsilon.$$

Suppose z is real in the open interval $(\frac{1}{1+\epsilon})^{\frac{1}{N-1}} < z < 1$. Then for any function $f = h + \overline{g}$, where h and g are given by (13), we have

$$\begin{aligned}
D_q f(z) &= 1 - \sum_{n=2}^{\infty}[n]_q(|a_n| + |b_n|)z^{n-1} \\
&\le 1 - \sum_{n=2}^{N}[n]_q(|a_n| + |b_n|)z^{n-1} \\
&\le 1 - z^{N-1}\sum_{n=2}^{N}[n]_q(|a_n| + |b_n|) \\
&\le 1 - (1 + \epsilon)z^{N-1} < 0.
\end{aligned}$$

Since $D_q f(0) = f'(0) > 0$, there exists a real number z_0 in $(0, 1)$ for which $D_q f(z_0) = 0$. Hence $f \notin \mathcal{T}^0\mathcal{H}_q$. Since this contradicts our assumption, the proof is completed. $\square$

In order to establish that (11) is also a necessary condition for $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$, we need to define a class $\mathcal{T}\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$. The class $\mathcal{T}\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ is a subclass of $\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ which consists of functions $f = h + \overline{g}$, where h and g of the form

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty}|b_n|z^n, |b_1| < 1. \tag{15}$$

In the following result, it is shown that the condition (11) is also necessary for functions $f = h + \overline{g}$, where h and g are of the form (15).

Theorem 2.7. *Let $f = h + \overline{g}$ be given by (15), where f belongs to $\mathcal{H}_q$. Then $f \in \mathcal{T}\mathcal{S}^*_{\mathcal{H}_q}(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{[n]_q - \alpha}{1 - \alpha}|a_n| + \frac{[n]_q + \alpha}{1 - \alpha}|b_n|\right) \le 2, \tag{16}$$

where $a_1 = 1, q \in (0, 1)$ and $\alpha \in [0, 1)$.

Proof. Since $\mathcal{T}\mathcal{S}^*_{\mathcal{H}_q}(\alpha) \subset \mathcal{S}^*_{\mathcal{H}_q}(\alpha)$, we only need to prove the "only if" part of this theorem. To this end, for functions $f = h + \overline{g}$ of the form (15), we notice that the condition (6) is equivalent to

$$Re\left\{\frac{(1 - \alpha)z - \sum_{n=2}^{\infty}([n]_q - \alpha)a_n z^n - \sum_{n=1}^{\infty}([n]_q + \alpha)b_n\overline{z}^n}{z - \sum_{n=2}^{\infty}a_n z^n + \sum_{n=1}^{\infty}b_n\overline{z}^n}\right\} \ge 0. \tag{17}$$

The above required condition (17) must hold for all values of $z \in \mathbb{D}$, $|z| = r < 1$. By choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{(1 - \alpha) - \sum\limits_{n=2}^{\infty} ([n]_q - \alpha)a_n r^{n-1} - \sum\limits_{n=1}^{\infty} ([n]_q + \alpha)b_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n r^{n-1} + \sum\limits_{n=1}^{\infty} b_n r^{n-1}} \ge 0. \tag{18}$$

If the condition (16) does not hold, then the numerator in (18) is negative for r sufficiently close to 1. Thus there exists a point $z_0 = r_0$ in $(0, 1)$ for which the quotient in (18) is negative. This contradicts the required condition for $f \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ and so the proof is completed. $\square$

In order to explore relationship between $\mathcal{T}^0 \mathcal{H}_q$ and $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, we need the following.

$$\mathcal{T}^0 \mathcal{S}^*_{\mathcal{H}_q}(\alpha) := \{f : f \in \mathcal{T}^0 \mathcal{H}_q \quad \text{and} \quad f \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)\}.$$

Setting $\alpha = 0$ and $b_1 = 0$ in Theorem 2.7, we obtain the following nice result.

Corollary 2.8. $\mathcal{T}^0 \mathcal{H}_q = \mathcal{T}^0 \mathcal{S}^*_{\mathcal{H}_q}(0).$

3. Extreme Points, Convolution and Convex Combinations

In this section, we first determine the extreme points of the closed convex hulls of $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, denoted by $clco\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$.

Theorem 3.1. *Let f be given by (15). Then $f \in clco\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ if and only if $f(z) = \sum\limits_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$, where $h_1(z) = z$, $h_n(z) = z - \frac{1-\alpha}{[n]_q - \alpha} z^n$, $(n \ge 2)$, $g_n(z) = z + \frac{1-\alpha}{[n]_q + \alpha} \bar{z}^n$, $(n \ge 1)$ and $\sum\limits_{n=1}^{\infty} (x_n + y_n) = 1$ where $x_n \ge 0$ and $y_n \ge 0$. In particular, the extreme points of $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ are $\{h_n\}$ and $\{g_n\}$.*

Proof. For a function f of the form $f(z) = \sum\limits_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$, where $\sum\limits_{n=1}^{\infty} (x_n + y_n) = 1$, we have

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q - \alpha} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{[n]_q + \alpha} y_n \bar{z}^n.$$

Then $f \in clco\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ because

$$\sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} \left(\frac{1 - \alpha}{[n]_q - \alpha} x_n \right) + \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} \left(\frac{1 - \alpha}{[n]_q + \alpha} y_n \right)$$

$$= \sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1 - x_1 \le 1.$$

Conversely, suppose $f \in clco\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$. Then $|a_n| \le \frac{1-\alpha}{[n]_q - \alpha}$ and $|b_n| \le \frac{1-\alpha}{[n]_q + \alpha}$. Set

$$x_n = \frac{[n]_q - \alpha}{1 - \alpha} |a_n|, (n \ge 2) \quad \text{and} \quad y_n = \frac{[n]_q + \alpha}{1 - \alpha} |b_n|, (n \ge 1).$$

By Theorem 2.7, $\sum\limits_{n=2}^{\infty} x_n + \sum\limits_{n=1}^{\infty} y_n \le 1$. Therefore we define $x_1 = 1 - \sum\limits_{n=2}^{\infty} x_n - \sum\limits_{n=1}^{\infty} y_n \ge 0$. Consequently, we obtain $f(z) = \sum\limits_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$ as required. $\square$

Using definition of convolution, we show that the class $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ is closed under convolution.

Theorem 3.2. *For $0 \leq \beta \leq \alpha < 1$, let $f \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ and $F \in \mathcal{TS}^*_{\mathcal{H}_q}(\beta)$. Then $f * F \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha) \subset \mathcal{TS}^*_{\mathcal{H}_q}(\beta)$.*

Proof. Let $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| z^n + \sum\limits_{n=1}^{\infty} |b_n| \bar{z}^n$ be in $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ and $F(z) = z - \sum\limits_{n=2}^{\infty} |A_n| z^n + \sum\limits_{n=1}^{\infty} |B_n| \bar{z}^n$ be in $\mathcal{TS}^*_{\mathcal{H}_q}(\beta)$. Due to definition of convolution, we get

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_n||B_n| \bar{z}^n.$$

We need to show that the coefficients of $f * F$ satisfy the required condition given in Theorem 2.7. For $F \in \mathcal{TS}^*_{\mathcal{H}_q}(\beta)$, we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Therefore, we have

$$\sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} |a_n||A_n| + \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} |b_n||B_n|$$

$$\leq \sum_{n=2}^{\infty} \frac{[n]_q - \alpha}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{[n]_q + \alpha}{1 - \alpha} |b_n| \leq 1.$$

In view of Theorem 2.7, it follows that $f * F \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha) \subset \mathcal{TS}^*_{\mathcal{H}_q}(\beta)$. $\square$

We now prove that $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ is closed under convex combination of its members.

Theorem 3.3. *The class $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$ is closed under convex combination.*

Proof. For $j = 1, 2, 3, ..$, let $f_j \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, where f_j is given by

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n_j}| z^n + \sum_{n=1}^{\infty} |b_{n_j}| \bar{z}^n.$$

Then, by Theorem 2.7 we have

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q - \alpha}{1 - \alpha} |a_{n_j}| + \frac{[n]_q + \alpha}{1 - \alpha} |b_{n_j}| \right) \leq 2. \tag{19}$$

For $\sum\limits_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{n=2}^{\infty} \sum_{j=1}^{\infty} t_j |a_{n_j}| z^n + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} t_j |b_{n_j}| \bar{z}^n.$$

Using (19), we have

$$\sum_{n=1}^{\infty} \left(\frac{[n]_q - \alpha}{1 - \alpha} \sum_{j=1}^{\infty} t_j |a_{n_j}| + \frac{[n]_q + \alpha}{1 - \alpha} \sum_{j=1}^{\infty} t_j |b_{n_j}| \right)$$

$$= \sum_{j=1}^{\infty} t_j \sum_{n=1}^{\infty} \left(\frac{[n]_q - \alpha}{1 - \alpha} |a_{n_j}| + \frac{[n]_q + \alpha}{1 - \alpha} |b_{n_j}| \right)$$

$$\leq 2 \sum_{j=1}^{\infty} t_j = 2,$$

and so by Theorem 2.7, we have $\sum\limits_{j=1}^{\infty} t_j f_j(z) \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$. $\square$

4. Distortion Bounds and Covering Theorem

The coefficient bounds in Theorem 2.7 enables us to prove distortion bounds for functions in $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, which yields covering result for this class.

Theorem 4.1. *If $f \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, then for $|z| = r < 1$ we have*

$$|f(z)| \le (1 + |b_1|)r + \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha}|b_1|\right)r^2 \tag{20}$$

and

$$|f(z)| \ge (1 - |b_1|)r - \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha}|b_1|\right)r^2. \tag{21}$$

These inequalities are sharp.

Proof. Let $\mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$. Taking the absolute value of f, we obtain

$$
\begin{aligned}
|f(z)| &\le (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n \\
&\le (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^2 \\
&= (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q - \alpha}\sum_{n=2}^{\infty}\left(\frac{[2]_q - \alpha}{1 - \alpha}|a_n| + \frac{[2]_q - \alpha}{1 - \alpha}|b_n|\right)r^2 \\
&\le (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q - \alpha}\sum_{n=2}^{\infty}\left(\frac{[n]_q - \alpha}{1 - \alpha}|a_n| + \frac{[n]_q + \alpha}{1 - \alpha}|b_n|\right)r^2 \\
&\le (1 + |b_1|)r + \frac{1 - \alpha}{[2]_q - \alpha}\left(1 - \frac{1 + \alpha}{1 - \alpha}|b_1|\right)r^2 \\
&= (1 + |b_1|)r + \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha}|b_1|\right)r^2.
\end{aligned}
$$

The proof of the inequality (21) is similar and is omitted. The bounds (20) and (21) are sharp for the functions

$$f(z) = z + |b_1|\bar{z} + \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1 - |b_1|)z - \left(\frac{1 - \alpha}{[2]_q - \alpha} - \frac{1 + \alpha}{[2]_q - \alpha}|b_1|\right)z^2,$$

where $|b_1| \le (1 - \alpha)/(1 + \alpha)$. $\square$

The following covering result follows from the inequality (21) by letting r approaches to 1.

Corollary 4.2. *If $f \in \mathcal{TS}^*_{\mathcal{H}_q}(\alpha)$, then*

$$\left\{w : |w| < \frac{[2]_q - 1 + \left(2\alpha - [2]_q + 1\right)|b_1|}{[2]_q - \alpha}\right\} \subset f(\mathbb{D}).$$

Remark 4.3. *For $q \to 1^-$, the covering theorem in Corollary 4.2 yields the corresponding traditional result for harmonic starlike functions of order α obtained in [16].*

We conclude this paper by a remark that the corresponding definition of q−harmonic convex function lead to several interesting results; see [5].

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