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Bounded Pseudo-Amenability and Contractibility of Certain Banach Algebras

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Abstract. The notion of bounded pseudo-amenability was introduced by Y. Choi and et al. [CGZ]. In this paper, similarly, we define bounded pseudo-contractibility and then investigate bounded pseudo-amenability and contractibility of various classes of Banach algebras including ones related to locally compact groups and discrete semigroups. We also introduce a multiplier bounded version of approximate biprojectivity for Banach algebras and determine its relation to bounded pseudo-amenability and contractibility.

1. Introduction

Let A be a Banach algebra and X a Banach A-bimodule. A bounded linear map $D : A \to X$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A),$$

and it is termed *inner* if there is $x \in X$ such that

$$D(a) = a \cdot x - x \cdot a \qquad (a \in A).$$

The notion of amenability of Banach algebras was established by B. E. Johnson in 1972 ([Joh2]). If every bounded derivation from A into the dual Banach A-bimodule X^* is inner for all Banach A-bimodules X, then A is said to be *amenable*. A Banach algebra A is called *contractible*, if every bounded derivation from A into any Banach A-bimodule is inner. In 2004, Ghahramani and Loy developed these concepts and introduced new notions of amenability and contractibility ([GhL]). The basic definition of their notions is referred to be approximately inner derivation. For an A-bimodule X, a derivation $D : A \to X$ is called *approximately inner* if there is a net of inner derivations $\{D_\alpha : A \to X\}_\alpha$ such that $D(a) = \lim_\alpha D_\alpha(a)$ for any $a \in A$. The Banach algebra A is said to be *(boundedly) approximately amenable* if for any A-bimodule X, every derivation $D : A \to X^*$ is the pointwise limit of a (bounded) net of inner derivations from A into X^*. In a similar manner (boundedly) approximate contractibility was defined. All notions of amenability are characterized in terms of approximate diagonals. We recall definitions needed in this article.

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Definition 1.1. *Let A be a Banach algbera. A net $\{m_i\} \subset A\hat{\otimes}A$ satisfying*

$$am_i - m_ia \to 0, \qquad a\pi(m_i) \to a,$$

is called an approximate diagonal, where $\pi : A\hat{\otimes}A \to A$ is the diagonal map determined by $\pi(a \otimes b) = ab$. According to [CGZ], we say that the diagonal $\{m_i\}$ is multiplier-bounded if there exists a constant $K > 0$ such that for all $a \in A$ and all i,

$$\|am_i - m_ia\| \leq K\|a\|, \quad \|a\pi(m_i) - a\| \leq K\|a\|, \quad \|\pi(m_i)a - a\| \leq K\|a\|.$$

Johnson proved in [Joh1] that a Banach algebra A is amenable if and only if there exists a bounded approximate diagonal, i.e. an approximate diagonal $\{m_i\}$ satisfying $\sup_\alpha \|m_i\| < \infty$.

According to [GhZh] a Banach algebra A is called *pseudo-amenable* if it has an approximate diagonal, and it is pseudo-contractible if it possesses a central approximate diagonal $\{m_i\}$, i.e. $am_i = m_ia$ for all $a \in A$ and all i.

Definition 1.2. *A Banach algebra A is called boundedly pseudo-amenable if it has a multiplier-bounded approximate diagonal. The term "K-pseudo-amenable" refers to bounded pseudo-amenability with multiplier bound $K > 0$.*

Like Definition 1.2 we introduce the concept of bounded pseudo-contractibility.

Definition 1.3. *A Banach algebra A is called boundedly pseudo-contractible if it has a central multiplier-bounded approximate diagonal, that is to say there are a central approximate diagonal $\{m_i\}$ and a constant $K > 0$ such that*

$$\|a\pi(m_i) - a\| \leq K\|a\| \qquad (a \in A).$$

Similarly, the term "K-pseudo-contractible" refers to bounded pseudo-contractibility with multiplier bound $K > 0$.

It is needless to say that every boundedly pseudo-contractible Banach algebra is boundedly pseudo-amenable.

Motivated by the earlier investigations, in this paper, we verify bounded pseudo-amenability and contractibility of some important Banach algebras in harmonic analysis such as group and measure algebras of a locally compact group, Fourier algebra of a discrete group and some algebras constructed on discrete semigroups. We also introduce a multiplier-bounded approximate biprojectivity for Banach algebras and verify its relation with bounded pseudo-amenability and contractibility.

2. Bounded pseudo-amenability and contractibility

In this section we give some general properties of bounded pseudo-amenable and contractible Banach algebras including hereditary properties.

Let A be a Banach algebra. We say that a net (e_α) is an *approximate identity* for A, if $\|ae_\alpha - a\| \to 0$ and $\|e_\alpha a - a\| \to 0$ for all $a \in A$. It is called *central* if $ae_\alpha = e_\alpha a$ for each $a \in A$. We call (e_α) a *bounded approximate identity* for A, if it is also bounded. The net (e_α) is termed a *multiplier-bounded approximate identity* for A if there exists a constant $k > 0$ such that $\|ae_\alpha\| \leq k\|a\|$ and $\|e_\alpha a\| \leq k\|a\|$ for all $a \in A$ and all α. It is clear that boundedly pseudo-amenable Banach algebras possess a multiplier-bounded approximate identity and pseudo-contractible Banach algebras have a multiplier-bounded central approximate identity.

The unitization of a Banach algebra A is denoted by $A^\#$ which is $\mathcal{A} \oplus \mathbb{C}$ with the following product:

$$(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu) \qquad (a, b \in A, \lambda, \mu \in \mathbb{C}).$$

It is obvious that with l^1-norm $A^\#$ is a Banach algebra as well.

Proposition 2.1. *([CGZ, Proposition 2.2]) A Banach algebra A is boundedly approximately contractible if and only if its unitization $A^\#$ is boundedly pseudo-amenable.*

The next proposition provide an example of a pseudo-amenable Banach algebra which is not boundedly pseudo-amenable.

Proposition 2.2. *There is a unital Banach algebra which is pseudo-amenable but not boundedly pseudo-amenable.*

Proof. Consider the Banach algebra A constructed in [GhR] which is boundedly approximately amenable but not boundedly approximately contractible. Then it follows from [CGZ, Proposition 2.4] that $A^\#$ is boundedly approximately amenable and so $A^\#$ is pseudo-amenable by [Pou1, Corollary 3.7]. Using Proposition 2.1 and the fact that A is not boundedly approximately contractible we conclude that $A^\#$ is not boundedly pseudo-amenable. □

Theorem 2.3. *Let A be a K-pseudo-amenable (-contractible) Banach algebra, B a Banach algebra and $\theta : A \to B$ a continuous epimorphism. Then B is boundedly pseudo-amenable (-contractible) with bound $K' = \max\{K\|\theta\|^2, K\|\theta\|\}$.*

Proof. By the assumption there is a net $\{m_i\}$ in $A\hat{\otimes}A$ such that

$$\begin{array}{lll} am_i - m_i a \to 0, & a\pi(m_i) \to a, & \\ \|am_i - m_i a\| \leq K\|a\|, & \|a\pi(m_i) - a\| \leq K\|a\|, & \|\pi(m_i)a - a\| \leq K\|a\|. \end{array}$$

For each $i \in \mathbb{N}$ let $\{a_n^i\}_{n=1}^\infty, \{b_n^i\}_{n=1}^\infty \subset A$ be sequences such that $m_i = \sum_{n=1}^\infty a_n^i \otimes b_n^i$ and $\sum_{n=1}^\infty \|a_n^i\|\|b_n^i\| < \infty$. Set $C = \|\theta\|$ and define

$$M_i = (\theta \otimes \theta)(m_i) = \sum_{n=1}^{\infty} \theta(a_n^i) \otimes \theta(b_n^i).$$

Then $\|M_i\| \leq C^2\|m_i\|$ and for each $a \in A$,

$$\begin{aligned} \|\theta(a)M_i - M_i\theta(a)\| &= \|(\theta \otimes \theta)(am_i - m_i a)\| \leq C^2\|am_i - m_i a\| \leq C^2K\|a\|, \\ \|\theta(a)\pi(M_i) - \theta(a)\| &= \|\theta(a)\pi(\theta \otimes \theta(m_i)) - \theta(a)\| = \|\theta(a)\theta(\pi(m_i)) - \theta(a)\| \\ &= \|\theta(a\pi(m_i) - a)\| \leq C\|a\pi(m_i) - a\| \leq CK\|a\|, \end{aligned}$$

and similarly

$$\|\pi(M_i)\theta(a) - \theta(a)\| \quad \leq CK\|a\|.$$

Therefore, $\{M_i\}$ is a multiplier-bounded approximate diagonal for B, with bound $K' = \max\{KC^2, KC\}$. □

Corollary 2.4. *Let A be a K-pseudo-amenable (contractible) Banach algebra and I be a closed two-sided ideal of A. Then A/I is K-pseudo-amenable (contractible).*

Corollary 2.5. *Let A and B be two Banach algebras such that $A\hat{\otimes}B$ is boundedly pseudo-amenable (contractible) and B has a non-zero character. Then A is boundedly pseudo-amenable (contractible).*

Proof. Suppose that $A\hat{\otimes}B$ is K-pseudo amenable, φ is a non-zero character of B and consider the epimorphism $\theta(A\hat{\otimes}B) \to A$ by $\theta(a \otimes b) = \varphi(b)a$. Now Theorem 2.3 implies that A is K-pseudo-amenable. □

Theorem 2.6. *Suppose that A is a boundedly pseudo-amenable Banach algebra and J is a two-sided closed ideal of A. Suppose also $\{e_\alpha\} \subseteq A$ is a central approximate identity for J that is multiplier-bounded in A. Then J is also boundedly pseudo-amenable.*

Proof. By the assumption there is a constant $M \geq 1$ such that for all α and $a \in A$,

$$\|ae_\alpha\| \leq M\|a\|, \quad \|e_\alpha a\| \leq M\|a\|.$$

So for each α and $m \in A\hat{\otimes}A$ we infer that

$$\|me_\alpha\| \le M\|m\|, \quad \|e_\alpha m\| \le M\|m\|.$$

Let $\{m_i\} \subset A\hat{\otimes}A$ be a net satisfying conditions of Definition 1.2 with bound $K > 0$. For any $\varepsilon > 0$ and finite set $F \subset J$, there are i and α such that

$$\|am_i - m_ia\|M^2 \le \varepsilon/2, \quad \|\pi(m_i)a - a\|M \le \varepsilon/2 \qquad (a \in F),$$

and

$$\|e_\alpha a - a\| \le \varepsilon/4, \quad \|\pi(m_i)(e_\alpha a - a)\|M \le \varepsilon/4 \quad (a \in F).$$

Similar to the proof of [GhZh, Proposition 2.6], we obtain

$$\|ae_\alpha m_ie_\alpha - e_\alpha m_ie_\alpha a\| \le \varepsilon, \qquad \|\pi(e_\alpha m_ie_\alpha)a - a\| < \varepsilon \qquad (a \in F).$$

Passing to a subnet we may suppose that $\{e_\alpha m_ie_\alpha\} \subset J\hat{\otimes}J$ constitutes an approximate diagonal for J. Since $\{e_\alpha\}$ is central, for each i and $a \in J$ we have

$$\begin{aligned}\|ae_\alpha m_ie_\alpha - e_\alpha m_ie_\alpha a\| &= \|e_\alpha am_ie_\alpha - e_im_iae_\alpha\| = \|e_\alpha(am_i - m_ia)e_\alpha\|\\ &\le M^2\|am_i - m_ia\| \le M^2K\|a\|,\end{aligned}$$

and

$$\begin{aligned}\|\pi(e_\alpha m_ie_\alpha)a - a\| &= \|e_\alpha\pi(m_i)e_\alpha a - a\|\\ &= \|e_\alpha\pi(m_i)e_\alpha a - e_\alpha e_\alpha a + e_\alpha e_\alpha a - a\|\\ &\le \|e_\alpha(\pi(m_i)e_\alpha a - e_\alpha a)\| + \|e_\alpha e_\alpha a - a\|\\ &\le M\|\pi(m_i)e_\alpha a - e_\alpha a\| + \|e_\alpha e_\alpha a\| + \|a\|\\ &\le MK\|e_\alpha a\| + M\|e_\alpha a\| + \|a\|\\ &\le M^2K\|a\| + M^2\|a\| + \|a\|\\ &= (M^2K + M^2 + 1)\|a\|.\end{aligned}$$

Likewise, $\|a\pi(e_\alpha m_ie_\alpha) - a\| \le (M^2K + M^2 + 1)\|a\|$. These imply that J is $(M^2K + M^2 + 1)$-pseudo-amenable. □

Corollary 2.7. *Suppose that A is a boundedly pseudo-amenable Banach algebra, J a closed two-sided ideal of A with a bounded central approximate identity. Then J is boundedly pseudo-amenable.*

The proof of the next proposition is the same as that of [GhZh, Proposition 3.3] and is omitted.

Proposition 2.8. *Let A be a M-boundedly approximately contractible Banach algebra. If A has a bounded central approximate identity $\{e_\alpha\}$ with bound K, then A is $(2K^2 + M)$-pseudo-amenable.*

Corollary 2.9. *Let A be a boundedly approximately contractible commutative Banach algebra. Then A is boundedly pseudo-amenable.*

Proof. Every boundedly approximately contractible Banach algebra has a bounded approximate identity. □

Theorem 2.10. *Suppose that A is a boundedly pseudo-amenable Banach algebra and X is a Banach A-bimodule for which each multiplier bounded left (right) approximate identity of A is a mutiplier bounded left (right) approximate identity for X. Then*

1. *Every derivation $D : A \to X$ is boundedly approximately inner.*

2. *Every derivation $D : A \to X^*$ is boundedly weak* approximately inner.*

Proof. (1): Let $\Phi : A\hat{\otimes}A \to X$ be defined by $\Phi(a \otimes b) = D(a) \cdot b$ and let $\{m_i\}$ be a net satisfying conditions of Definition 1.2 with corresponding bound $K > 0$. If we set $\psi_i = -\Phi(m_i)$, then as in [GhZh, Proposition 3.5] for each $a \in A$ we obtain

$$D(a) = \lim_i (a\psi_i - \psi_i a),$$

and also we get

$$\begin{aligned}\|a \cdot \psi_i - \psi_i \cdot a\| - \|D(a)\pi(m_i)\| &\le \|a \cdot \psi_i - \psi_i \cdot a - D(a)\pi(m_i)\| = \|\Phi(a \cdot m_i - m_i \cdot a)\| \\ &\le \|\Phi\|\|a \cdot m_i - m_i \cdot a\| \le K\|\Phi\|\|a\| \le K\|D\|\|a\|,\end{aligned}$$

and so

$$\|a \cdot \psi_i - \psi_i \cdot a\| \le K\|D\|\|a\| + \|D(a)\pi(m_i)\| \le K\|D\|\|a\| + (K' + 1)\|D(a)\| \le K''\|D(a)\|.$$

Whence D is boundedly approximately inner.

(2) can be proven similarly. □

Obviously, every contractible Banach algebra is boundedly pseudo-contractible. We end this section by presenting an example of a boundedly pseudo-contractible Banach algebra which is not amenable and consequently not contractible.

Example 2.11. *For $1 \le p < \infty$ let ℓ^p be the usual Banach sequence algebra with pointwise multiplication. Since ℓ^p does not have a bounded approximate identity, it is not amenable. Now for each $i \in \mathbb{N}$ let δ_i be the characteristic function of the singleton $\{i\}$. Then every $f \in \ell^p$ is of the form $\sum_{i=1}^{\infty} f(i)\delta_i$. For each $n \in \mathbb{N}$ put $u_n := \sum_{i=1}^{n} \delta_i \otimes \delta_i$. It is seen that*

$$f \cdot u_n = \sum_{i=1}^{n} f(i)\delta_i \otimes \delta_i = \sum_{i=1}^{n} \delta_i \otimes \delta_i f(i) = u_n \cdot f,$$

and

$$\|f\pi(u_n) - f\|_p = \left\| \sum_{i=1}^{n} f(i)\delta_i - \sum_{i=1}^{\infty} f(i)\delta_i \right\|_p \to 0, \qquad \|f\pi(u_n)\| \le \|f\|.$$

Hence, ℓ^p is 1-pseudo-contractible. We also remark that ℓ^p is not approximately amenable[DLZh]. Therefore $(\ell^p)^\#$ is not approximately amenable and thus $(\ell^p)^\#$ is not pseudo-amenable by [GhZh, Proposition 3.2]. Therefore, bounded pseudo-contractibility of a Banach algebra A does not imply not only bounded pseudo-contractibility but also bounded pseudo-amenability of $A^\#$.

3. Banach algebras on locally compact groups

In this section we will verify Bounded pseudo-amenability and contractibility of some important Banach algebras on locally compact groups. We commence with the convolution group and measure algebras $L^1(G)$ and $M(G)$ and their second duals.

Proposition 3.1. *For a locally compact group G, $L^1(G)$ is boundedly pseudo-amenable if and only if G is amenable.*

Proof. If G is amenable then $L^1(G)$ is amenable and so it is boundedly pseudo-amenable. If $L^1(G)$ is boundedly pseudo-amenable, then it is pseudo-amenable. Thus G is amenable by [GhZh, Proposition 4.1]. □

The next proposition is a consequence of [GhZh, Proposition 4.2].

Proposition 3.2. *Let G be a locally compact group. Then*

1. *the convolution measure algebra $M(G)$ is boundedly pseudo-amenable if and only if G is discrete and amenable.*
2. *$L^1(G)^{**}$ is boundedly pseudo-amenable if and only if G is finite.*

The following proposition determines the bounded pseudo-amenability and contractibility of the Fourier algebra $A(G)$ of a discrete group G which provides an example of a non-amenable, boundedly pseudo-contractible Banach algebra.

Proposition 3.3. *Let G be a discrete group and $A(G)$ be its Fourier algebra. Then the following are equivalent.*

1. *$A(G)$ has a multiplier-bounded approximate identity.*
2. *$A(G)$ is boundedly pseudo-contractible.*
3. *$A(G)$ is boundedly pseudo-amenable.*

Proof. (1) $\Longrightarrow$ (2): Let $\{e_\alpha\}$ be a multiplier-bounded approximate identity of $A(G)$ with bound M. As it is mentioned in Remark 3.4 of [GhS], we may suppose that every e_α has finite support, say S_α. Now let

$$m_\alpha = \sum_{x\in S_\alpha} e_\alpha(x)\delta_x \otimes \delta_x,$$

where δ_x is the evaluational function at x. For each $f \in A(G)$ and $x \in G$ we have

$$\begin{aligned} f\cdot(\delta_x\otimes\delta_x) - (\delta_x\otimes\delta_x)\cdot f &= (f\delta_x)\otimes\delta_x - \delta_x\otimes(\delta_x f)\\ &= (f(x)\delta_x)\otimes\delta_x - \delta_x\otimes(\delta_x f(x))\\ &= f(x)(\delta_x\otimes\delta_x - \delta_x\otimes\delta_x) = 0. \end{aligned}$$

Therefore, $f\cdot m_\alpha = m_\alpha\cdot f$. Since $\pi(m_\alpha) = e_\alpha$, for all $f \in A(G)$ we have $\pi(m_\alpha)f - f \to 0$. Hence $\{m_\alpha\}$ is central approximate diagonal for $A(G)$. Furthermore, for any $f \in A(G)$ we have

$$\|f\pi(m_\alpha) - f\| = \|fe_\alpha - f\| \leq (M+1)\|f\|.$$

Hence, $A(G)$ is $(M+1)$-pseudo-contractible.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (1): This is immediate inasmuch as every boundedly pseudo-amenable Banach algebras has a multiplier-bounded approximate identity. □

The following example shows that bounded pseudo-contractibility does not imply amenability.

Example 3.4. *Let G be a free group. It is shown in [Haa, Theorem 2.1] that $A(G)$ has a multiplier-bounded approximate identity consisting of functions with finite support. Thus the Fourier algebra of a free group is boundedly pseudo-contractible. Nonetheless, free groups with at least 2 generators are not amenable and so, by Leptin's theorem, their Fourier algebras lack a bounded approximate identity; consequently they are not amenable.*

For a locally compact group G, let $PF_p(G)$ denote the Banach algebra of p-pseudofunctions on G which is the norm closure of the image of $L^1(G)$ in $B(L^p(G))$, the space of bounded operators on $L^p(G)$, under the left regular representation. It is shown in [CGZ, Theorem 7.1] that for a discrete group G, amenability and pseudo amenability of $PF_p(G)$ is equivalent to the amenability of G. We therefore have the following proposition.

Proposition 3.5. *Let G be a discrete group and $p \in (1,\infty)$. Then $PF_p(G)$ is boundedly pseudo-amenable if and only if G is amenable.*

4. Banach algebras on discrete semigroups

This section is devoted to the Bounded pseudo-amenability and contractibility of many significant Banach algebras constructed on semigroups.

Like Example 3.4, the following is an example of a boundedly pseudo-contractible Banach algebra which is not amenable and consequently is not contractible.

Example 4.1. *Let Λ be non-empty, totally ordered set which is a semigroup if the product of two elements is defined to be their maximum. In fact it is a semilattice and is denoted by $\Lambda_\vee$. Proposition 6.2 of [CGZ] shows that the semigroup algebra $\ell^1(\Lambda_\vee)$ is boundedly pseudo-amenable.*

Let $\{A_i\}_{i\in I}$ be a family of Banach algebras and $1 \le q < \infty$. Then their ℓ^q-direct sum

$$A = \ell^q - \bigoplus_{i\in I} A_i = \left\{ a = (a_i)_{i\in I} \,\middle|\, a_i \in A_i,\ \|a\|_A = \left(\sum_{i\in I} \|a_i\|^q_{A_i} \right)^{1/q} < \infty \right\},$$

is a Banach algebra under componentwise product.

Theorem 4.2. *Let $\{A_i\}_{i\in I}$ be a family of K-pseudo-amenable (contractible) Banach algebras, $1 \le q < \infty$ and $A = \ell^q - \oplus_{i\in I} A_i$. Then A is $(K+1)$-pseudo-amenable (contractible).*

Proof. We follow the proof of Proposition 2.1 of [GhZh]. For arbitrary $\varepsilon > 0$ and a finite set $F \subset A$, there is a finite set $J \subset I$ such that $\|P_J(a) - a\|_A < \frac{\varepsilon}{2}$ for $a \in A$, where $P_J : A \to \ell^q - \oplus_{i\in J} A_i$ is the natural projection and P_i is defined to be $P_{\{i\}}$. Since A_i is K-pseudo-amenable, there are $i \in J$ and $u_i \in A_i \hat{\otimes} A_i$ such that

$$\|P_i(a)u_i - u_i P_i(a)\| < \frac{\varepsilon}{|J|^{\frac{1}{q}}}, \quad \|\pi_i(u_i)P_i(a) - P_i(a)\| < \frac{\varepsilon}{2|J|^{\frac{1}{q}}} \qquad (a \in F),$$

and for all $b \in A$,

$$\|P_i(b)u_i - u_i P_i(b)\| < K\|P_i(b)\|, \quad \|\pi_i(u_i)P_i(b) - P_i(b)\| < K\|P_i(b)\|, \quad \|P_i(b)\pi_i(u_i) - P_i(b)\| < K\|P_i(b)\|,$$

where $\pi_i : A_i \hat{\otimes} A_i \to A_i$ is also the diagonal map. Setting $u = \{x_i\}_{i\in I}$ where $x_i = u_i$ for $i \in J$ and $x_i = 0$ for $i \in I\backslash J$ implies that $ua = uP_J(a)$ and $au = P_J(a)u$. Hence for each $a \in F$,

$$\|au - ua\|_A = \|P_J(a)u - uP_J(a)\|_A = (\sum_{i\in J} \|P_i(a)u_i - u_i P_i(a)\|^q)^{\frac{1}{q}} < \varepsilon;$$

and

$$\begin{aligned}
\|a\pi(u) - a\|_A &= \|P_J(a)\pi(u) - P_J(a) + P_J(a) - a\|_A \\
&\le \|P_J(a)\pi(u) - P_J(a)\|_A + \|P_J(a) - a\|_A \\
&= \sum_{i\in J} (\|P_i(a)\pi_i(u) - P_i(a)\|^q)^{\frac{1}{q}} + \|P_J(a) - a\|_A \le \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Also for each $b \in A$ we have

$$\begin{aligned}
\|bu - ub\|_A &= \|P_J(b)u - uP_J(b)\|_A = (\sum_{i\in J} \|P_i(b)u - uP_i(b)\|^q_{A_i})^{\frac{1}{q}} \\
&\le (\sum_{i\in J} K^q \|P_i(b)\|^q_{A_i})^{\frac{1}{q}} = K\|P_J(b)\|_A \le K\|b\|_A, \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\|b\pi(u) - b\|_A &\le \|P_J(b)\pi(u) - P_J(b)\|_A + \|P_J(b) - b\|_A \\
&\le (\sum_{i\in J} \|P_i(b)\pi_i(u) - P_i(b)\|_{A_i}^q)^{\frac{1}{q}} + \|b\|_A \\
&\le (\sum_{i\in J} K^q \|P_i(b)\|_{A_i}^q)^{\frac{1}{q}} + \|b\|_A = K\|P_J(b)\|_A + \|b\|_A \\
&\le (k+1)\|b\|_A, \qquad (2)
\end{aligned}$$

and similarly

$$\|\pi(u)b - b\|_A \le (K+1)\|b\|_A, \qquad (b \in A). \qquad (3)$$

□

So Theorem 4.2 shows that there are a large class of bounded pseudo-amenable(contractible) Banach algebras that are not amenable. We remark that $A = \ell^q - \oplus_{i\in I} A_i$ is amenable if and only if $|I| < \infty$ and each A_i is amenable.

Example 4.3. *Since $\ell^p = \ell^p - \bigoplus_1^\infty \mathbb{C}$, it is 2-pseudo-amenable invoking Theorem 4.2. Notice that, it is in fact ℓ^p is 1-pseudo-contractible by Example 2.11.*

Proposition 4.4. *Let A be a Banach algebra and $\mathbb{M}_n(A)$ be its ℓ^1-Munn algebra ($n \in \mathbb{N}$). Then $\mathbb{M}_n(A)$ is K-pseudo-amenable if and only if A is K-pseudo-amenable.*

Proof. Suppose that $\{\Psi_\alpha\}$ is an approximate diagonal of $M_n(A)$ with bound K. Keeping $\mathbb{M}_n(A)\hat{\otimes}\mathbb{M}_n(A) \cong \mathbb{M}_{n^2}(A\hat{\otimes}A)$ in mind, we may assume that

$$\Psi_\alpha = \begin{bmatrix} m_{11}^\alpha & m_{12}^\alpha & \dots & m_{1n^2}^\alpha \\ m_{21}^\alpha & m_{22}^\alpha & \dots & m_{2n^2}^\alpha \\ \vdots & \vdots & \vdots & \vdots \\ m_{n^21}^\alpha & m_{n^22}^\alpha & \dots & m_{n^2n^2}^\alpha \end{bmatrix},$$

where $m_{ij}^\alpha \in A\hat{\otimes}A$. For each $a \in A$ we have

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix} \Psi_\alpha - \Psi_\alpha \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix} =$$

$$\begin{bmatrix} am_{11}^\alpha & am_{12}^\alpha & \dots & am_{1n^2}^\alpha \\ am_{21}^\alpha & am_{22}^\alpha & \dots & am_{2n^2}^\alpha \\ \vdots & \vdots & \vdots & \vdots \\ am_{n^21}^\alpha & am_{n^22}^\alpha & \dots & am_{n^2n^2}^\alpha \end{bmatrix} - \begin{bmatrix} m_{11}^\alpha a & m_{12}^\alpha a & \dots & m_{1n^2}^\alpha a \\ m_{21}^\alpha a & m_{22}^\alpha a & \dots & m_{2n^2}^\alpha a \\ \vdots & \vdots & \vdots & \vdots \\ m_{n^21}^\alpha a & m_{n^22}^\alpha a & \dots & m_{n^2n^2}^\alpha a \end{bmatrix}.$$

Hence $am_{11}^\alpha - m_{11}^\alpha a \to 0$ and $\|am_{11}^\alpha - m_{11}^\alpha a\| \le K\|a\|$. With a similar fashion we can get $a\pi(m_{11}^\alpha) \to a$, $\pi(m_{11}^\alpha)a \to a$, $\|a\pi(m_{11}^\alpha) - a\| \le K\|a\|$ and $\|\pi(m_{11}^\alpha)a - a\| \le K\|a\|$.

Conversely, suppose that A is K-pseudo-amenable and $\{m_\alpha\}$ is an approximate diagonal for it, and set

$$\Psi_\alpha = \begin{bmatrix} m_\alpha & 0 & \dots & 0 \\ 0 & m_\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & m_\alpha \end{bmatrix}.$$

Obviously $\{\Psi_\alpha\}$ is an approximate diagonal for $\mathbb{M}_n(A)$ and for any $M \in \mathbb{M}_n(A)$ we have

$$\|\Psi_\alpha M - M\Psi_\alpha\| \le K\|M\|, \quad \|\bar{\pi}(\Psi_\alpha)M - M\| \le K\|M\|, \quad \|M\bar{\pi}(\Psi_\alpha) - M\| \le K\|M\|,$$

where $\bar{\pi} : \mathbb{M}_n(A)\hat{\otimes}\mathbb{M}_n(A) \to \mathbb{M}_n(A)$ is the diagonal map. □

Definition 4.5. *A (discrete) semigroup S is called an inverse semigroup if for any $s \in S$ there exists a unique $s^* \in S$ such that $s^*ss^* = s^*$ and $ss^*s = s$. The set of idempotent elements of S is denoted by $E(S)$, that is $E(S) = \{ss^* : s \in S\}$.*

Let S be a inverse semigroup. For $e \in E(S)$, $G_e = \{s \in S : ss^* = s^*s = e\}$ constitutes a group called *maximal subgroup of G at e*.

For all $s, t \in S$ the relation $\mathcal{D}$ defined on an inverse semigroup S by $s\mathcal{D}t$ if and only if there exists $x \in S$ with

$$Ss \cup \{s\} = Sx \cup \{x\}, \quad tS \cup \{t\} = xS \cup \{x\},$$

is an equivalence relation. There is also a natural partial order on S given by $s \le t \Leftrightarrow s = ss^*t$. For $p \in S$ we set $(p] = \{q \in S : q \le p\}$.

Definition 4.6. *An inverse semigroup S is called locally finite whenever $|(p]| < \infty$ for all $p \in S$, and it is called uniformly locally finite (ULF) if $\sup_{p\in S} |(p]| < \infty$.*

We recall that a Banach algebra A is called *biflat* if there exists a Banach A-bimodule morphim $\rho : (A\hat{\otimes}A)^* \to A^*$ such that $\rho \circ \pi^*(\gamma) = \gamma$ for all $\gamma \in A^*$, where $\pi^* : A^* \to (A\hat{\otimes}A)^*$ is adjoint of the diagonal map π.

Proposition 4.7. *Let S be a ULF inverse semigroup and $\{D_\lambda : \lambda \in \Lambda\}$ be the family of its $\mathcal{D}$-classes such that for all $\lambda \in \Lambda$, $|E(D_\lambda)| < \infty$. For each $\lambda \in \Lambda$ let $p_\lambda \in E(D_\lambda)$. Then the following statements are equivalent.*

1. *For each $\lambda \in \Lambda$ the maximal subgroup G_{p_λ} is amenable.*
2. *$\ell^1(S)$ is pseudo-amenable.*
3. *$\ell^1(S)$ is boundedly pseudo-amenable.*

Moreover, in this case $\ell^1(S)$ is biflat.

Proof. From [Ram, Theorem 2.18] we have the following isometric isomorphism

$$\ell^1(S) \cong \ell^1 - \bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}.$$

The proposition now follows from Propositions 3.1, 4.4, Theorem 4.2, and [Ram, Therem 3.7]. □

Definition 4.8. *An inverse semigroup S is called a Clifford semigroup if for all $s \in S$, $ss^* = s^*s$.*

Theorem 4.9. *Let S be a Clifford semigroup and $A(S)$ be its Fourier algebra introduced in [MP]. Then the following statements are equivalent.*

1. *$A(S)$ has a multiplier-bounded approximate identity.*
2. *$A(S)$ is boundedly pseudo-contractible.*
3. *$A(S)$ is boundedly pseudo-amenable.*

Proof. (1) $\Longrightarrow$ (2): *Suppose that $A(S)$ has a multiplier-bounded approximate identity with bound M. By [MP] we have the following useful decomposition*

$$A(S) = \ell^1 - \bigoplus_{e\in E(S)} A(G_e).$$

Thus it can be readily seen that for each $e \in E(S)$, $A(G_e)$ has a mutiplier-bounded approximate identity with bound M. From Proposition 3.3 we conclude that $A(G_e)$ is $(M + 1)$-pseudo-contractible for all $e \in E(S)$. Now Theorem 4.2 implies that $A(S)$ is $(M + 2)$-pseudo-contractible. The other parts of proof are obvious. □

Applying the above decomposition, as it is done in [MP], for a Clifford semigroup S with abelian maximal subgroups G_e, we obtain $A(S) \cong \ell^1 - \bigoplus_{e\in E(S)} L^1(\hat{G}_e)$, where $\hat{G}_e$ is the Pontrjagin dual of G_e. Since $\hat{G}_e$ is compact, it is amenable and so $L^1(\hat{G}_e)$ is 1-amenable. Hence $L^1(\hat{G}_e)$ is 1-pseudo-amenable for all $e \in E(S)$. From Theorem 4.2 it can be inferred that $A(S)$ is 2-pseudo-amenable.

Let $\{A_i\}_{i\in I}$ be a family of Banach algebras. Their c_0-direct sum

$$A = c_0 - \bigoplus_{i\in I} A_i = \left\{ a = (a_i)_{i\in I} \,\middle|\, a_i \in A_i,\ \|a_i\|_{A_i} \to 0,\ \|a\|_A = \sup_{i\in I} \|a_i\|_{A_i} \right\},$$

is a Banach algebra under componentwise product.
The next theorem gives the c_0-analogue of Theorem 4.2. Since the proof is similar, we omit it.

Theorem 4.10. *Let $\{A_i\}_{i\in I}$ be a family of K-pseudo-amenable (contractible) Banach algebras and $A = c_0 - \oplus_{i\in I} A_i$. Then A is $(K+1)$-pseudo-amenable (contractible).*

Corollary 4.11. *Let S be a Clifford semigroup and consider the Banach algebra $PF_P(S)$ of p-pseudofunctions on S introduced in [Pou2]. Then $PF_p(S)$ is boundedly pseudo-amenable if and only if every maximal subgroup G_e of S is amenable.*

Proof. By [Pou2] we have the following decomposition

$$PF_p(S) \cong c_0 - \bigoplus_{e\in E(S)} PF_p(G_e).$$

Combining Theorem 4.10 and Proposition 3.5 the corollary follows. □

5. Multiplier-bounded approximate biprojectivity

In this section we introduce an approximate version of biprojectivity and then investigate its relation with (bounded) pseudo-amenability.

Definition 5.1. *([Pou1]) A Banach algebra A is said to be approximately biprojective if there is a net $\{\rho_\alpha\} \subset \mathcal{B}(A\hat{\otimes}A, A)$ such that for each $a, b \in A$:*

$$\pi \circ \rho_\alpha(a) \to a, \quad \rho_\alpha(ab) - a\rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a)b \to 0.$$

We say that, A is called boundedly approximately biprojective when $\sup_\alpha \|\rho_\alpha\| < \infty$.

Definition 5.2. *An approximately biprojective Banach algebra A is termed multiplier-boundedly approximately biprojective if there is a $K > 0$ such that for each $a, b \in A$:*

$$\|\pi \circ \rho_\alpha(a) - a\| \le K\|a\|, \quad \|\rho_\alpha(ab) - a\rho_\alpha(b)\| \le K\|a\|\|b\|, \quad \|\rho_\alpha(ab) - \rho_\alpha(a)b\| \le K\|a\|\|b\|,$$

where $\{\rho_\alpha\}$ satisfies condition of Definition 5.1.

Obviously, every boundedly approximately biprojective Banach algebra is multiplier-boundedly approximately biprojective.

Corollary 5.3. *Let A be a boundedly pseudo-amenable Banach algebra. Then A is multiplier-boundedly approximately biprojective.*

Proof. Let $\{m_\alpha\}$ be an approximate diagonal of A with multiplier bound $K > 0$. Define $\rho_\alpha : A \to A\hat{\otimes}A$ by $\rho_\alpha(a) = a \cdot m_\alpha$. By [Pou1, Proposition 3.4], we have

$$\pi \circ \rho_\alpha(a) \to a, \quad \rho_\alpha(ab) - a \cdot \rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a) \cdot b \to 0, \qquad (a, b \in A).$$

Moreover, for each $a \in A$ and for each α we have

$$\|\pi \circ \rho_\alpha(a) - a\| = \|\pi(a \cdot m_\alpha) - a\| = \|a\pi(m_\alpha) - a\| \leq K\|a\|$$

On the other hand, for all α and every $a, b \in A$, $\rho_\alpha(ab) - a \cdot \rho_\alpha(b) = 0$ and

$$\|\rho_\alpha(ab) - \rho_\alpha(a) \cdot b\| = \|ab \cdot m_\alpha - (a \cdot m_\alpha) \cdot b\| \leq \|a\|\|b \cdot m_\alpha - m_\alpha \cdot b\| \leq K\|a\|\|b\|$$

Therefore A is multiplier-boundedly approximately biprojective. □

Proposition 5.4. *Let A be a multiplier-boundedly approximately biprojective Banach algebra with a central bounded approximate identity $\{e_\beta\}$. Then A is boundedly pseudo-amenable.*

Proof. Let $\{\rho_\alpha\}$ be a net satisfying Definition 5.2. As in Proposition 3.5 of [Pou1], there are subnets $\{e_{\beta_i}\}$ of $\{e_\beta\}$ and $\{\rho_{\alpha_i}\}$ of $\{\rho_\alpha\}$ such that $m_i := \rho_{\alpha_i}(e_{\beta_i})$ is an approximate diagonal for A. We show that $\{m_i\}$ is a multiplier-bounded approximate diagonal. Let $\{e_\beta\}$ be bounded by K_0. Then for each $a \in A$ we have

$$\begin{aligned}
\|a \cdot m_i - m_i \cdot a\| &= \|a \cdot \rho_{\alpha_i}(e_{\beta_i}) - \rho_{\alpha_i}(e_{\beta_i}) \cdot a\| \\
&= \|a \cdot \rho_{\alpha_i}(e_{\beta_i}) - \rho_{\alpha_i}(ae_{\beta_i}) + \rho_{\alpha_i}(e_{\beta_i}a) - \rho_{\alpha_i}(e_{\beta_i}) \cdot a\| \\
&\leq \|a \cdot \rho_{\alpha_i}(e_{\beta_i}) - \rho_{\alpha_i}(ae_{\beta_i})\| + \|\rho_{\alpha_i}(e_{\beta_i}a) - \rho_{\alpha_i}(e_{\beta_i}) \cdot a\| \\
&\leq K\|a\|\|e_{\beta_i}\| + K\|e_{\beta_i}\|\|a\| \\
&\leq 2KK_0\|a\|,
\end{aligned}$$

and

$$\begin{aligned}
\|\pi(m_i)a - a\| &= \|\pi \circ \rho_{\alpha_i}(e_{\beta_i})a - a\| \leq \|\pi \circ \rho_{\alpha_i}(e_{\beta_i})a - e_i a\| + \|e_{\beta_i}a - a\| \\
&\leq K\|a\|\|e_{\beta_i}\| + \|a\|\|e_{\beta_i}\| + \|a\| = (KK_0 + K_0 + 1)\|a\|;
\end{aligned}$$

Hence A is boundedly pseudo-amenable. □

The following example gives an approximately biprojective Banach algebra that is not multiplier-boundedly approximately biprojective.

Example 5.5. *Suppose that A is the algebra introduced in Proposition 2.2. Approximate amenability of $A^\#$ implies its approximate biprojectivity [Pou1, Proposition 3.4]. On the other hand, $A^\#$ is not boundedly pseudo-amenable and so by Proposition 5.4 is not multiplier-boundedly approximately biprojective.*

Here we give an example of multiplier-boundedly approximately biprojective Banach algebra which is not boundedly approximately biprojective.

Example 5.6. *Suppose that S is an infinite non-empty set and consider the Banach algebra $\ell^2(S)$ with pointwise multiplication. Let $\{e_i\}_{i\in S}$ be the canonical basis for $\ell^2(S)$ and let Λ be the set of finite subsets of S, which is an ordered set with respect to inclusion. For any $F \in \Lambda$ define $m_F = \sum_{i\in F} e_i \otimes e_i$. Then $\{m_F\}_{F\in\Lambda}$ is a central approximate diagonal for $\ell^2(S)$ satisfying conditions of Definition 1.2. Therefore it is boundedly pseudo contractible and consequently, by Proposition 5.3, multiplier-boundedly approximately biprojective. However, it is known that $\ell^2(S)$ is not boundedly approximately biprojective (see [Pou1, Example 4.1]).*

Corollary 5.7. *If G is an infinite Abelian compact group, then $L^2(G)$ is a multiplier-boundedly approximately biprojective Banach algebra.*

Proof. Suppose Γ is the dual group of G. From Plancherel Theorem we have $L^2(G) \cong \ell^2(\Gamma)$ and so Example 5.6 gives the desired result. □

The last example provides a boundedly pseudo-amenable Banach algebra which is not boundedly approximately biprojective.

Example 5.8. *Consider the inverse semigroup $S = (\mathbb{N}, *)$ whith $s * t = \min\{s, t\}$ for all $s, t \in N$. By [GLZ, Example 4.6], the convolution semigroup algebra $\ell^1(S)$ is sequentially approximately contractible. So the uniform boundedness principle implies that $\ell^1(S)$ is boundedly approximately contractible. Hence by Proposition* 2.1, *$(\ell^1(S))^\#$ is boundedly pseudo amenable. Nevertheless, since S is a locally finite, non-uniformly locally finite inverse semigroup, by [Ram, Theorem 3.7], $\ell^1(S)$ is not biflat and consequently its unitization $(\ell^1(S))^\#$ is not biflat. It now follows from [Ari, Theorem 3.6(A)] that $(\ell^1(S))^\#$ is not boundedly approximately biprojective.*

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