Filomat 34:5 (2020), 1713–1719
https://doi.org/10.2298/FIL2005713A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$PGL_2(q)$ cannot be determined by its cs

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Abstract. For a finite group G, let $Z(G)$ denote the center of G and $cs^*(G)$ be the set of non-trivial conjugacy class sizes of G. In this paper, we show that if G is a finite group such that for some odd prime power $q \geq 4$, $cs^*(G) = cs^*(PGL_2(q))$, then either $G \cong PGL_2(q) \times Z(G)$ or G contains a normal subgroup N and a non-trivial element $t \in G$ such that $N \cong PSL_2(q) \times Z(G)$, $t^2 \in N$ and $G = N.\langle t \rangle$. This shows that the almost simple groups cannot be determined by their set of conjugacy class sizes (up to an abelian direct factor).

1. Introduction

Throughout this paper, G is a finite group, $Z(G)$ is the center of G and for $a \in G$, $cl_G(a)$ is the conjugacy class in G containing a and $C_G(a)$ denotes the centralizer of the element a in G. We denote by $cs^*(G)$, the set of non-trivial conjugacy class sizes of G. Studying the interplay between the structure of a group and the set of its conjugacy class sizes is one of the interesting concepts in group theory. For instance, J. Thompson in 1988 conjectured that:

Thompson's conjecture. Let S be a simple group. If G is a finite centerless group with $cs^*(G) = cs^*(S)$, then $G \cong S$.

In a series of papers, it has been proved that Thompson's conjecture is true for many families of finite simple groups (see [1]-[6], [9], [11], [13], [16]).

G is named an almost simple group when there exists a simple group S such that $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$.

In [14] and [17], it has been shown that Thompson's conjecture is true for some almost simple groups.

Inspired by Thompson's conjecture, A. Camina and R. Camina come up with the following problem [10]:

Problem. If S is a simple group and G is a finite group with $cs^*(G) = cs^*(S)$, then is it true that $G \cong S \times Z(G)$?

In 2015, it has been investigated that the above problem is true when $S \cong PSL_2(q)$ [8]. Then, in [7], it has been proven that the answer of the above problem is true for many families of finite simple groups. Naturally, one can ask what happens for G in the above problem when S is an almost simple group. So, in this paper, we prove that:

Main theorem. Let $q > 4$ be an odd prime power. If G is a finite group with $cs^*(G) = cs^*(PGL_2(q))$, then either $G \cong PGL_2(q) \times Z(G)$ or G contains a normal subgroup N and a non-trivial element $t \in G$ such that $N \cong PSL_2(q) \times Z(G)$, $t^2 \in N$ and $G = N.\langle t \rangle$.

2010 *Mathematics Subject Classification*. Primary 20E45; Secondary 05C25, 20D05
Keywords. Conjugacy classes, prime graph, non-solvable groups, Schur cover
Received: 20 November 2018; Accepted: 07 February 2020
Communicated by Dragan S. Djordjević
Thanks are due to Shahrekord University for financial support (Grant 97GRD1M1023). The author would like to thank Milad Ahanjideh for his constructive comments.
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In this paper, all groups are finite. For simplicity of notation, throughout this paper let $q > 4$ be a power of an odd prime p, $GF(q)$ be a field with q elements and G be a group with $cs^*(G) = cs^*(PGL_2(q))$. Throughout this paper, we use the following notation: For a natural number n, let $\pi(n)$ be the set of prime divisors of n, C_n denote a cyclic group of order n and for a group H, let $\pi(H) = \pi(|H|)$. Also, $H.G$ denotes an extension of H by G. For a prime r and natural numbers a and b, $|a|_r$ is the r-part of a, i.e., $|a|_r = r^t$ when $r^t \mid a$ and $r^{t+1} \nmid a$ and, $\gcd(a, b)$ and $\mathrm{lcm}(a, b)$ are the greatest common divisor of a and b and the lowest common multiple of a and b, respectively. For the set π of some primes, x is named a π-element (π'-element) of a group H if $\pi(o(x)) \subseteq \pi$ ($\pi(o(x)) \subseteq \pi(H) - \pi$).

2. Definitions and preliminary results

Lemma 2.1. [12, Proposition 4] *Let H be a group. If there exists $p \in \pi(H)$ such that p does not divide any conjugacy class sizes of H, then the p-Sylow subgroup of H is central in H.*

Definition 2.2. *For a group H, the prime graph $GK(H)$ of H is a simple graph whose vertices are the prime divisors of the order of H and two distinct prime numbers p and q are joined by an edge if G contains an element of order pq. Denote by $t(H)$ the number of connected components of the graph $GK(H)$ and denote by $\pi_i = \pi_i(H)$, $i = 1, ..., t(H)$, the i-th connected component of $GK(H)$. For a group H of an even order, let $2 \in \pi_1$. If $GK(H)$ is disconnected, then $|H|$ can be expressed as a product of co-prime positive integers $m_i(H)$, $i = 1, 2, \dots, t(H)$, where $\pi(m_i(H)) = \pi_i(H)$, and if there is no ambiguity write m_i for showing $m_i(H)$. These m_is are called the order components of H and the set of order components of H will be denoted by $OC(H)$. The list of all simple groups with disconnected prime graph and the sets of their order components have been obtained in* [15] *and* [18].

Lemma 2.3. [14] *If H is a group with $OC(H) = OC(PGL_2(q))$, then $H \cong PGL_2(q)$.*

Lemmas 2.4, 2.5 and 2.6 are easy to prove for a group H:

Lemma 2.4. *For $x \in H - Z(H)$, let $C/Z(H) = C_{H/Z(H)}(xZ(H))$. Then $C_H(x) \trianglelefteq C$.*

Lemma 2.5. *For every $x \in H$ and natural number n,*

(i) *$C_H(x) \leq C_H(x^n)$ and $|cl_H(x^n)| \mid |cl_H(x)|$;*

(ii) *if $|cl_H(x)|$ is maximal in $cs^*(H)$ by divisibility and $\pi = \pi(o(x))$, then for every π'-element $y \in C_H(x)$, $C_H(xy) = C_H(x)$. In particular, if $|cl_H(x)|$ is maximal and minimal in $cs^*(H)$ by divisibility and $\pi = \pi(o(x))$, then for every π'-element $y \in C_H(x) - Z(H)$, $C_H(y) = C_H(x)$.*

Lemma 2.6. *Let K be a normal subgroup of H and $\overline{H} = H/K$. Let $\overline{x}$ be the image of the element x of H in $\overline{H}$. Then,*

(i) *$|cl_K(x)|$ divides $|cl_H(x)|$;*

(ii) *$|cl_{\overline{H}}(\overline{x})|$ divides $|cl_H(x)|$;*

(iii) *for every abelian group A, $cs^*(H \times A) = cs^*(H)$.*

Lemma 2.7. *For a group H, $\mathrm{lcm}\{\alpha : \alpha \in cs^*(H)\} \mid [H : Z(H)]$.*

Proof. Since for every $x \in H$, $Z(H) \leq C_H(x)$, we get that $|cl_H(x)| \mid [H : Z(H)]$. Thus, $\mathrm{lcm}\{\alpha : \alpha \in cs^*(H)\} \mid [H : Z(H)]$, as wanted. □

Lemma 2.8. *Let π be a set of primes, x be a non-central π-element of the group H and $C/Z(H) = C_{H/Z(H)}(xZ(H))$. Then, for a π'-element $y \in H$, $y \in C$ if and only if $y \in C_H(x)$.*

Proof. Obviously, $C_H(x) \leq C$. Now let $y \in C$ be a π'-element. Then, $yZ(H) \in C/Z(H)$, so there exists $z \in Z(H)$ such that $y^{-1}xy = xz$. This shows that $o(x) = o(xz) = \mathrm{lcm}(o(x), o(z))$, hence $o(z) \mid o(x)$. On the other hand, $xyx^{-1} = yz$. Thus, $o(y) = o(yz) = \mathrm{lcm}(o(y), o(z))$, so $o(z) \mid o(y)$. This forces $o(z) \mid \gcd(o(x), o(y)) = 1$. Therefore, $z = 1$. Consequently, $y^{-1}xy = x$. This shows that $y \in C_H(x)$, as desired. □

Lemma 2.9. *For a group H, let $t, s \in \pi(H)$ and $S \in \mathrm{Syl}_s(H)$. If for every t-element $y \in H - Z(H)$, $|cl_H(y)|_s > 1$ and if x is a t-element of H such that $|cl_H(x)|$ is maximal and minimal in $cs^*(H)$ by divisibility, then either $|H/Z(H)|_s = |cl_H(x)|_s$ or $C_H(S) \leq Z(H)$.*

Proof. Let $C_H(S) \nleq Z(H)$. Thus, by assumption and Lemma 2.5(i), there exists a t'-element $z \in C_H(S) - Z(H)$. Now we claim that $|\frac{H}{Z(H)}|_s = |cl_H(x)|_s$. If not, then $C_H(x)$ contains a non-central s-element w. Hence, by Lemma 2.5(ii), $C_H(x) = C_H(w)$. Obviously, $z \in C_H(S) \leq C_H(w) = C_H(x)$. Consequently, Lemma 2.5(ii) forces $C_H(x) = C_H(z)$. Therefore, $|cl_H(x)|_s = |cl_H(z)|_s = 1$, which is a contradiction. So, $|H/Z(H)|_s = |cl_H(x)|_s$, as claimed. □

Lemma 2.10. *For a group H and $t \in \pi(H)$, let $\{|cl_H(x)| : x \in H - Z(H),\ o(x) \text{ is a power of } t\} = \{\alpha\}$ and $|cs^*(H)| > 1$. If α is maximal and minimal in $cs^*(H)$ by divisibility, then $|H/Z(H)|_t = \mathrm{Max}\{|\beta|_t : \beta \in cs^*(H)\}$.*

Proof. Working towards a contradiction, let $|H/Z(H)|_t \neq \mathrm{Max}\{|\beta|_t : \beta \in cs^*(H)\}$. Thus for every $\gamma \in cs^*(H) - \{\alpha\}$, $|\gamma|_t < |H/Z(H)|_t$. Let $\gamma = |cl_H(y)|$, for some $y \in H - Z(H)$. Then, by our assumption and Lemma 2.5(i), we can assume that y is a t'-element. Also, $|cl_H(y)|_t < |H/Z(H)|_t$. Hence, $C_H(y)$ contains a non-central t-element z. Since $|cl_H(z)| = \alpha$, Lemma 2.5(ii) shows that $|cl_H(y)| = |cl_H(z)| = \alpha$, which is a contradiction. This completes the proof. □

Lemma 2.11. *For a group H, $\pi(H/Z(H)) = \cup_{\alpha \in cs^*(H)} \pi(\alpha)$.*

Proof. By Lemma 2.7, $\cup_{\alpha \in cs^*(H)} \pi(\alpha) \subseteq \pi(H/Z(H))$. Now if there exists $t \in \pi(H/Z(H)) - \cup_{\alpha \in cs^*(H)} \pi(\alpha)$, then for every $\alpha \in cs^*(H)$, $t \nmid \alpha$. Therefore, Lemma 2.1 forces the t-Sylow subgroup T of H to be an abelian direct factor of H. Thus, $T \leq Z(H)$ and hence, $t \nmid |H/Z(H)|$, which is a contradiction. This shows that $\pi(H/Z(H)) = \cup_{\alpha \in cs^*(H)} \pi(\alpha)$. □

Lemma 2.12. *For a group H, if there exists $\alpha \in cs^*(H)$ and $p, q \in \pi(H/Z(H))$ ($p \neq q$) such that $|\alpha|_p < |H/Z(H)|_p$ and $|\alpha|_q < |H/Z(H)|_q$, then there exists a path between p and q in $GK(H/Z(H))$.*

Proof. Let $x \in H - Z(H)$ with $\alpha = |cl_H(x)|$. By Lemma 2.5(i), we can assume that x is of the prime power order. Since $|\alpha|_p < |H/Z(H)|_p$ and $|\alpha|_q < |H/Z(H)|_q$, we get that $p, q \mid |C_H(x)/Z(H)|$. Thus, $C_H(x)$ contains a non-central p-element x_1 and a non-central q-element x_2. If $p \mid o(x)$, then since $x_2 \in C_H(x)$, we get that $xx_2Z(H) \in H/Z(H)$ is of order pq, so the proof is complete. The same reasoning completes the proof when $q \mid o(x)$. Now let $o(x)$ be a power of a prime r, where $r \notin \{p, q\}$. The same reasoning as above shows that $H/Z(H)$ contains elements of order pr and rq, so $p - r - q$ is a path in $GK(H/Z(H))$, as wanted. □

3. Main results

Theorem 3.1. $OC(G/Z(G)) = OC(PGL_2(q))$.

Proof. We are going to prove this theorem in the following steps:
Step 1. $|PGL_2(q)| \mid [G : Z(G)]$.
Proof. From Lemma 2.7, $\mathrm{lcm}\{\alpha : \alpha \in cs^*(G)\} \mid [G : Z(G)]$. On the other hand,

$$cs^*(G) = cs^*(PGL_2(q)) = \{q^2 - 1, q(q \pm 1), q(q \pm 1)/2\}. \quad (1)$$

Therefore, $|PGL_2(q)| \mid [G : Z(G)]$.

Step 2. For every p-element $x \in G - Z(G)$, $|cl_G(x)| = q^2 - 1$ and $|cl_{\bar{G}}(\bar{x})| = q^2 - 1$, where $\bar{G} = G/Z(G)$ and $\bar{x}$ is the image of x in $\bar{G}$.
Proof. We first show that for every p-element $x \in G - Z(G)$, $|cl_G(x)| = q^2 - 1$. Working towards a contradiction, assume that G contains a non-central p-element x such that $|cl_G(x)| \neq q^2 - 1$. Thus, by (1)

$$|cl_G(x)|_p = |PGL_2(q)|_p. \quad (2)$$

Also, $q^2-1 \in cs^*(G)$, so there exists a non-central element $y \in G$ such that $|cl_G(y)| = q^2-1$. Hence, we can assume that there exists a p-Sylow subgroup P of G such that $x \in P$ and $P \leq C_G(y)$. Since q^2-1 is maximal in $cs^*(G)$ by divisibility, Lemma 2.5 leads us to assume that y is of the prime power order. If y is a p'-element, then since $x \in C_G(y)$, we get from maximality and minimality of q^2-1 in $cs^*(G)$, and Lemma 2.5(ii) that $|cl_G(x)| = q^2-1$, which is a contradiction. This forces y to be a p-element and for every p'-element $z \in G$, $|cl_G(z)| \neq q^2-1$. Thus,

$$y \in Z(P) - Z(G). \tag{3}$$

Also, $x \in C_G(x) - Z(G)$. Thus, $p \mid |C_G(x)/Z(G)|$ and hence, (2) forces $|G/Z(G)|_p > |PGL_2(q)|_p$. Now let z be a p'-element of $G-Z(G)$. Then, the above statements show that $p \mid |C_G(z)/Z(G)|$, so $C_G(z)$ contains a non-central p-element w. We can assume that $w \in P$ and $P \cap C_G(wz) \in \mathrm{Syl}_p(C_G(wz))$. Moreover, Lemma 2.5(ii) shows that $|cl_G(zw)|, |cl_G(w)| \neq q^2-1$, so (1) forces $|C_G(w)|_p = |C_G(wz)|_p = |C_G(z)|_p$. Since $C_G(wz) \leq C_G(w), C_G(z)$, we get from (3) that $y \in P \cap C_G(w) = P \cap C_G(wz) \leq C_G(z)$. Thus, Lemma 2.5(ii) shows that $|cl_G(z)| = |cl_G(y)| = q^2-1$, which is a contradiction. This shows that for every p-element $x \in G - Z(G)$, $|cl_G(x)| = q^2-1$.

Let $x \in G - Z(G)$ be a p-element and $C/Z(G) = C_{\bar{G}}(\bar{x})$. Thus, by the above statements, $|cl_G(x)| = q^2-1$ and hence if $y \in C - C_G(x)$, then Lemmas 2.4 and 2.8 show that $o(yC_G(x))$ is a power of p. So, Lemma 2.4 guarantees that $p \mid |C/C_G(x)|$. However, $C \leq G$ and hence, $|C/C_G(x)| \mid [G : C_G(x)] = |cl_G(x)|$. This forces $p \mid |cl_G(x)|$, which is a contradiction. Therefore, $C = C_G(x)$ and hence, $|cl_{\bar{G}}(\bar{x})| = |cl_G(x)| = q^2-1$, as desired.

Step 3. $|G/Z(G)| = |PGL_2(q)|$.

Proof. From Step 1, $|PGL_2(q)| \mid [G : Z(G)]$. Let $s \in \pi(G/Z(G))$. Since by Lemma 2.11, $\pi(G/Z(G)) = \pi(PGL_2(q))$, we have $s \in \pi(PGL_2(q))$. Let $S_1 \in \mathrm{Syl}_s(G)$ and $S \in \mathrm{Syl}_s(PGL_2(q))$. Since $Z(S) \neq 1$ and $Z(PGL_2(q)) = \{1\}$, we get that there exists $\alpha \in cs^*(PGL_2(q)) = cs^*(G)$ such that $|\alpha|_s = 1$. This forces $C_G(S_1) \nleq Z(G)$. Thus, if $s \neq p$, then Step 2 and Lemma 2.9 show that $|G/Z(G)|_s = |\beta|_s$, for some $\beta \in cs^*(G)$. So $|G/Z(G)|_s \leq |PGL_2(q)|_s$. Also, Lemma 2.10 guarantees that $|G/Z(G)|_p \leq |PGL_2(q)|_p$ and hence, $|G/Z(G)| \mid |PGL_2(q)|$. Therefore, $|G/Z(G)| = |PGL_2(q)|$.

Step 4. $OC(G/Z(G)) = OC(PGL_2(q))$.

Proof. If there exists $t \in \pi(G/Z(G)) - \{p\}$ such that t and p are adjacent in $GK(G/Z(G))$, then there exist a non-central p-element x and a non-cental t-element y such that $xy = yx$. So, $y \in C_G(x) - Z(G)$ and hence $t \mid |C_G(x)/Z(G)|$. On the other hand, Steps 2 and 3 show that $|cl_G(x)| = q^2-1$ and $|G/Z(G)| = |PGL_2(q)|$. Thus, $t \in \pi(q^2-1)$ and $|G/Z(G)|_t = |cl_G(x)|_t |C_G(x)/Z(G)|_t > |q^2-1|_t = |PGL_2(q)|_t$, which is a contradiction. This forces $\{p\}$ to be an odd connected component of $GK(G/Z(G))$. Also, for every $t, s \in \pi(PGL_2(q))$ which are adjacent in $GK(PGL_2(q))$, Step 3 and Lemma 2.12 show that there exists a path between t and s in $GK(G/Z(G))$. Now since $\pi_1(PGL_2(q)) = \pi(q^2-1)$ is a connected component in $GK(PGL_2(q))$, $|G/Z(G)| = |PGL_2(q)|$ and $\{p\}$ is an odd connected component of $GK(G/Z(G))$, we get that $\pi(q^2-1)$ is a component of $GK(G/Z(G))$. Hence, $OC(G/Z(G)) = OC(PGL_2(q))$. □

Corollary 3.2. $G/Z(G) \cong PGL_2(q)$.

Proof. Since by Theorem 3.1, $OC(G/Z(G)) = OC(PGL_2(q))$, Lemma 2.3 shows that $G/Z(G) \cong PGL_2(q)$. □

Lemma 3.3. *For every subgroup Z_1 of $Z(G)$, $cs^*(G/Z_1) = cs^*(PGL_2(q))$.*

Proof. Let Z_1 be a subgroup of $Z(G)$. Put $\tilde{G} = G/Z_1$ and $\hat{G} = (G/Z_1)/(Z(G)/Z_1)$. For every $x \in G$, let $\tilde{x}$ and $\hat{x}$ be the images of x in $\tilde{G}$ and $\hat{G}$, respectively. By Corollary 3.2, $\hat{G} \cong G/Z(G) \cong PGL_2(q)$. By (1), there exist $x_1, x_2, x_3 \in G$ such that $|cl_{\hat{G}}(\hat{x}_1)| = q^2-1$, $|cl_{\hat{G}}(\hat{x}_2)| = q(q-1)$ and $|cl_{\hat{G}}(\hat{x}_3)| = q(q+1)$. Also for every $1 \leq i \leq 3$, Lemma 2.6 implies that $|cl_{\hat{G}}(\hat{x}_i)| \mid |cl_{\tilde{G}}(\tilde{x}_i)|$ and $|cl_{\tilde{G}}(\tilde{x}_i)| \mid |cl_G(x_i)|$. However, q^2-1 and $q(q \pm 1)$ are maximal in $cs^*(\hat{G}) = cs^*(PGL_2(q)) = cs^*(G)$ by divisibility. Thus, for every $1 \leq i \leq 3$, $|cl_{\hat{G}}(\hat{x}_i)| = |cl_{\tilde{G}}(\tilde{x}_i)| = |cl_G(x_i)| \in \{q^2-1, q(q \pm 1)\}$. Therefore, $q^2-1, q(q \pm 1) \in cs^*(\hat{G})$.

On the other hand, for $\varepsilon \in \{\pm 1\}$, there exists $y_\varepsilon \in G$ such that $|cl_G(y_\varepsilon)| = q(q+\varepsilon 1)/2$. Since $|cl_{\hat{G}}(\hat{y}_\varepsilon)| \mid |cl_{\tilde{G}}(\tilde{y}_\varepsilon)|$, $|cl_{\tilde{G}}(\tilde{y}_\varepsilon)| \mid |cl_G(y_\varepsilon)|$ and $q(q+\varepsilon 1)/2$ is minimal in $cs^*(PGL_2(q)) = cs^*(G)$, we get that $|cl_{\hat{G}}(\hat{y}_\varepsilon)| = |cl_{\tilde{G}}(\tilde{y}_\varepsilon)| = |cl_G(y_\varepsilon)| = q(q+\varepsilon 1)/2$. Therefore, $q(q \pm 1)/2 \in cs^*(\hat{G})$ and hence, $cs^*(G) \subseteq cs^*(\tilde{G})$. Now if $y \in G$ such that $|cl_{\tilde{G}}(\tilde{y})| \in cs^*(\tilde{G}) - cs^*(G)$, then since $|cl_{\hat{G}}(\hat{y})| \mid |cl_{\tilde{G}}(\tilde{y})|$, $|cl_{\tilde{G}}(\tilde{y})| \mid |cl_G(y)|$ and $|cl_{\hat{G}}(\hat{y})|, |cl_G(y)| \in cs^*(\hat{G}) = cs^*(PGL_2(q)) = cs^*(G)$, we get, by considering the maximal elements of $cs^*(G)$, that $|cl_{\hat{G}}(\hat{y})| \in \{q(q \pm 1)/2\}$.

Therefore, $|cl_G(y)| \in \{q(q \pm 1), q(q \pm 1)/2\}$. Hence, $|cl_{\tilde{G}}(\tilde{y})| \in \{q(q \pm 1), q(q \pm 1)/2\} \subseteq cs^*(G)$, a contradiction. This implies that $cs^*(\tilde{G}) = cs^*(G)$. □

Lemma 3.4. *If M is a normal subgroup of G with $M/Z(M) \cong PGL_2(q)$, then $cs^*(M) = cs^*(PGL_2(q))$.*

Proof. Put $\bar{M} = M/Z(M)$ and for $x \in M$, let $\bar{x}$ be the image of x in $\bar{M}$. Then, since $|cl_{\bar{M}}(\bar{x})| \mid |cl_M(x)|$ and $|cl_M(x)| \mid |cl_G(x)|$, arguing by analogy as the proof of Lemma 3.3 completes the proof. □

Lemma 3.5. *For a group H, if $x \in H$ and $Z(H) \leq \langle x \rangle$, then $C_{\bar{H}}(\bar{x}) \leq N_H(\langle x \rangle)/Z(H)$, where $\bar{H} = H/Z(H)$ and $\bar{x}$ is the image of x in $\bar{H}$.*

Proof. Let $\bar{y} = yZ(H) \in C_{\bar{H}}(\bar{x})$. Then, there exists $z \in Z(H)$ such that $y^{-1}xy = xz \in \langle x \rangle$. Thus, $y \in N_H(\langle x \rangle)$. Therefore, $yZ(H) \in N_H(\langle x \rangle)/Z(H)$, as wanted. □

Lemma 3.6. *Let $Z = Z(GL_2(q))$ and let $\bar{x}$ be the image of $x \in GL_2(q)$ in $PGL_2(q)$. If $q \equiv \varepsilon \pmod 4$ and $|cl_{PGL_2(q)}(\bar{x})| \mid q(q+\varepsilon)$, then either $|cl_{PGL_2(q)}(\bar{x})| = q(q+\varepsilon)$ or $\bar{x} \in SL_2(q)Z/Z$ and $|cl_{PGL_2(q)}(\bar{x})| = q(q+\varepsilon)/2$.*

Proof. Let $|cl_{PGL_2(q)}(\bar{x})| \mid q(q+\varepsilon)$ and $|cl_{PGL_2(q)}(\bar{x})| \neq q(q+\varepsilon)$. Then, $|cl_{PGL_2(q)}(\bar{x})| = q(q+\varepsilon)/2$ and hence, $|C_{PGL_2(q)}(\bar{x})| = 2(q-\varepsilon)$. Thus, $\bar{x}$ is a semi-simple element in $PGL_2(q)$ and hence $o(\bar{x}) \mid (q-\varepsilon)$. So, one of the following cases holds:
I. $\varepsilon = +$. Then, we can assume that for some $\mu \in GF(q) - \{0\}$, $x = \mathrm{diag}(\mu, 1)$. Since $|C_{PGL_2(q)}(\bar{x})| = 2(q-\varepsilon)$, we can check at once that $wZ \in C_{PGL_2(q)}(\bar{x})$, where

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, there exists $1 \neq z \in Z$ such that

$$x^{-1}wx = wz \tag{4}$$

and hence $\mathrm{lcm}(o(z), o(w)) = o(wz) = o(w) = 2$. This forces $o(z) = 2$. Therefore, $z = \mathrm{diag}(-1,-1)$. So, (4) guarantees that $\mu = \mu^{-1} = -1$. On the other hand, for a generator d of $GF(q) - \{0\}$, $d^{(q-1)/2} = -1$. However, $(q-1)/2$ is even. Hence, there exists $d' \in GF(q) - \{0\}$ such that $d'^2 = -1$. Therefore, $x = \mathrm{diag}(d'^2, 1) = \mathrm{diag}(d', d'^{-1})\mathrm{diag}(d', d') \in SL_2(q)Z$. This shows that $\bar{x} \in SL_2(q)Z/Z$.
II. $\varepsilon = -$. Let $\alpha \in GF(q^2) - \{0\}$ such that $o(\alpha) = o(x)$. Let σ be a Frobenius automorphism of $GL_2(\overline{GF(q)})$ such that $(GL_2(\overline{GF(q)}))_\sigma = GL_2(q)$, where $\overline{GF(q)}$ is an algebraic closure of $GF(q)$. Then, there exists $g \in GL_2(\overline{GF(q)})$ such that $g^{-1}g^\sigma = w$, where

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Set $t = \mathrm{diag}(\alpha, \alpha^q)$. We can check at once that $w^g, t^g \in GL_2(q)$ and $N_{GL_2(q)}(\langle t^g \rangle) = C_{q^2-1}.\langle w^g \rangle$ such that $Z \leq C_{q^2-1}$ and $t^g \in C_{q^2-1}$. Without loss of generality, let $t^g = x$. Thus by Lemma 3.5, $w^gZ \in C_{PGL_2(q)}(\bar{x})$. However, $o(w^g) = 2$ and $[x, w^g] = z \in Z$. So, $o(zw^g) = o(w^g) = 2$ and hence $o(z) = 2$. Therefore, $z = \mathrm{diag}(-1,-1)$. Since $w^{-g}t^gw^g = t^gz$, $w^{-1}tw = tz$, consequently, $\alpha^q = -\alpha$. This forces $\alpha^{2(q-1)} = 1$. Thus, $o(\bar{x}) = o(\overline{t^g}) = 2$. Since $[PGL_2(q) : SL_2(q)Z/Z] = 2$, we get from $4 \mid q+1$ that $\bar{x} \in SL_2(q)Z/Z$, as wanted. □

Lemma 3.7. *If $G = (PSL_2(q) \times Z(G)).\langle t \rangle$, where $t \in G - (PSL_2(q) \times Z(G))$ and $t^2 \in PSL_2(q) \times Z(G)$, then $cs^*(G/Z(G)) = cs^*(G)$.*

Proof. Since $PSL_2(q) \trianglelefteq PSL_2(q) \times Z(G)$, for every $\sigma \in \mathrm{Aut}(PSL_2(q) \times Z(G))$, $\sigma(PSL_2(q)) \cap PSL_2(q) \trianglelefteq PSL_2(q)$. However, $PSL_2(q)$ is simple. Thus, $\sigma(PSL_2(q)) \cap PSL_2(q) = \{1\}$ or $PSL_2(q)$. In the first case, $PSL_2(q) \times \sigma(PSL_2(q)) \leq PSL_2(q) \times Z(G)$, which is impossible. Consequently, $\sigma(PSL_2(q)) = PSL_2(q)$. This shows that $PSL_2(q)$ is a characteristic subgroup of $PSL_2(q) \times Z(G)$. On the other hand, $[G : PSL_2(q) \times Z(G)] = 2$. Therefore, $PSL_2(q) \times Z(G) \trianglelefteq G$ and hence $PSL_2(q) \trianglelefteq G$. Thus, for every $x \in G$ and $y \in PSL_2(q)$, $x^{-1}yx \in PSL_2(q)$. This forces $C_{G/Z(G)}(yZ(G)) = C_G(y)/Z(G)$. Consequently, $|cl_{G/Z(G)}(yZ(G))| = |cl_G(y)|$.

Now let $y \in G - (PSL_2(q) \times Z(G))$. So, $y = gt$ for some $g \in PSL_2(q) \times Z(G)$. Without loss of generality, let $g \in PSL_2(q)$. Then, since $PSL_2(q) \trianglelefteq G$, we can see at once that there do not exist $g' \in PSL_2(q) \times Z(G)$ and $z' \in Z(G) - \{1\}$ such that $yg'y^{-1} = g'z'$. Also, if there exists $g' \in PSL_2(q)$ and $z', z'' \in Z(G)$ such that $(g'z't)^{-1}y(g'z't) = yz''$, then $t^{-1}g'^{-1}yg't = yz''$, so $t^{-1}g'^{-1}gtg' = gz''$. However, $g'^{-1}g \in PSL_2(q) \trianglelefteq G$. Therefore, $t^{-1}g'^{-1}gt = g'' \in PSL_2(q)$ and hence, $g''g' = gz''$. This forces $z'' \in Z(G) \cap PSL_2(q) = \{1\}$, so $z'' = 1$. This shows that $C_{G/Z(G)}(yZ(G)) = C_G(y)/Z(G)$ and consequently, $|cl_{G/Z(G)}(yZ(G))| = |cl_G(y)|$. This guarantees that $cs^*(G/Z(G)) = cs^*(G)$, as wanted. □

Proof of the main theorem. Let G be the smallest counterexample. Then, it is obvious that $Z(G) \neq 1$. We claim that $|Z(G)|$ is prime. If not, $Z(G)$ contains a non-trivial subgroup Z_1 of the prime order. Thus, by Lemma 3.3, $cs^*(G/Z_1) = cs^*(PGL_2(q))$. On the other hand, $(G/Z_1)/(Z(G)/Z_1) \cong G/Z(G) \cong PGL_2(q)$, by Corollary 3.2. Consequently, $Z(G/Z_1) = Z(G)/Z_1$. Also, $|G/Z_1| < |G|$. Hence, our assumption shows that one of the following cases occurs:
Case 1. $G/Z_1 \cong PGL_2(q) \times Z(G)/Z_1$. Then, G contains a non-trivial normal subgroup M with $M/Z_1 \cong PGL_2(q)$. Thus, $Z(M) = Z_1$ and Lemma 3.4 shows that $cs^*(M) = cs^*(PGL_2(q))$. Hence, our assumption shows that M is as follows:

(i) $M \cong PGL_2(q) \times Z_1$. Thus, M contains a normal subgroup N such that $N \cong PGL_2(q)$ and $M = N \times Z_1$. So, $G = MZ(G) = NZ(G)$. However, $N \cap Z(G) = N \cap (M \cap Z(G)) = N \cap Z_1 = \{1\}$. Therefore, $G = N \times Z(G) \cong PGL_2(q) \times Z(G)$, a contradiction.

(ii) $M \cong (PSL_2(q) \times Z_1).C_2$. Then, M contains a characteristic subgroup N such that $N \cong PSL_2(q)$ and $M = (N \times Z_1).C_2$. Since $N ch M \trianglelefteq G$, we have $N \trianglelefteq G$. Thus, $NZ(G) \trianglelefteq G$ and $N \cap Z(G) = N \cap (M \cap Z(G)) = N \cap Z_1 = \{1\}$. Consequently, $N \times Z(G) \trianglelefteq G$. Since $[G : N \times Z(G)] = 2$, we get that G contains a 2-element t such that $t^2 \in N \times Z(G)$ and $G = (N \times Z(G)).\langle t\rangle \cong (PSL_2(q) \times Z(G)).C_2$, a contradiction.

Case 2. $G/Z_1 \cong (PSL_2(q) \times (Z(G)/Z_1)).C_2$. Then, G contains a normal subgroup M and a subgroup N such that $Z_1 \leq N$, $N/Z_1 \cong PSL_2(q)$ and $M/Z_1 = N/Z_1 \times Z(G)/Z_1$. Since $N/Z_1 \cong PSL_2(q)$, we have $Z(N) = Z_1$. Also, $|Z_1|$ is prime. Thus, $N' \cap Z_1 = Z_1$ or $\{1\}$. If $N' \cap Z_1 = \{1\}$, then $N' \times Z_1 \trianglelefteq N$. However, $N' \cong N'Z_1/Z_1 \trianglelefteq N/Z_1 \cong PSL_2(q)$ and $PSL_2(q)$ is simple, so $N' \cong PSL_2(q)$. Hence, $N \cong PSL_2(q) \times Z_1$. Since $Z(PSL_2(q)) = \{1\}$, we have $M \cong PSL_2(q) \times Z(G)$. Also, $[G : M] = 2$. Therefore, G contains a 2-element t such that $t^2 \in M$ and $G = M.\langle t\rangle \cong (PSL_2(q) \times Z(G)).C_2$, a contradiction. This forces $N' \cap Z_1 = Z_1$. Thus, $Z_1 \leq N'$. If $|Z_1|$ is odd, then we have $N \cong PSL_2(q) \times Z_1$. Hence, the above argument leads us to get a contradiction. Now let $|Z_1| = 2$ and N be a Schur cover of $PSL_2(q)$. Therefore, $N \cong SL_2(q)$, $Z_1 = Z(N)$ and $M \cong SL_2(q)Z(G)$. On the other hand, $[G : M] = [G/Z_1 : M/Z_1] = 2$. This shows that G contains a 2-element t such that $t^2 \in M$ and $G \cong (SL_2(q)Z(G)).\langle t\rangle$. It is known that

$$cs^*(SL_2(q)) = \{q(q \pm 1), q^2 - 1\}. \tag{5}$$

Let $q \equiv \varepsilon \pmod 4$. Then, since $q(q+\varepsilon)/2 \in cs^*(G)$, we get that G contains an element x with $|cl_G(x)| = q(q+\varepsilon)/2$. Now we have two following possibilities:

- $x \in N$. Then, since $N \cong SL_2(q)$ and $|cl_N(x)| \mid |cl_G(x)|$, we get from (5) that $|cl_N(x)| = 1$, so $x \in Z(N) = Z_1 \leq Z(G)$, a contradiction.

- $x \in G - NZ(G)$. Then, $xZ(G) \in G/Z(G) \cong PGL_2(q)$. Thus, Lemma 3.6 shows that $|cl_{G/Z(G)}(xZ(G))| = q(q + \varepsilon 1)$. So, by Lemma 2.6, $q(q + \varepsilon 1) \mid |cl_G(x)|$, which is impossible.

The above contradictions show that $|Z(G)|$ is prime. Thus, we apply the same reasoning as one used in Case 2 as follows: Since $G/Z(G) \cong PGL_2(q)$ and $PGL_2(q)$ contains a normal subgroup of index 2 which is isomorphic to $PSL_2(q)$, we can assume that G contains a normal subgroup N containing $Z(G)$ such that $N/Z(G) \cong PSL_2(q)$. Since $|Z(G)|$ is prime, we have $N' \cap Z(G) = \{1\}$ or $N' \cap Z(G) = Z(G)$. If $N' \cap Z(G) = \{1\}$, then $N' \times Z(G) \trianglelefteq N$. However, $N' \cong N'Z(G)/Z(G) \trianglelefteq N/Z(G) \cong PSL_2(q)$ and $PSL_2(q)$ is simple, so $N' \cong PSL_2(q)$. Consequently, $N \cong PSL_2(q) \times Z(G)$. Moreover, $[G : N] = 2$ and hence, G contains a 2-element t such that $t^2 \in M$ and $G = N.\langle t\rangle \cong (PSL_2(q) \times Z(G)).C_2$, a contradiction. This forces $N' \cap Z(G) = Z(G)$. Thus, $Z(G) \leq N'$.

So, $|Z(G)| = 2$ and N is a Schur cover of $PSL_2(q)$. Therefore, $N \cong SL_2(q)$ and $Z(G) = Z(N)$. It follows that $[G : N] = [G/Z(G) : N/Z(G)] = 2$. This shows that G contains a 2-element $t \in G$ such that $t^2 \in N$ and $G = SL_2(q).\langle t\rangle$. It is known that

$$cs^*(SL_2(q)) = \{q(q \pm 1), q^2 - 1\}. \quad (6)$$

Let $q \equiv \varepsilon \pmod 4$. Then, since $q(q+\varepsilon)/2 \in cs^*(G)$, we get that G contains an element x with $|cl_G(x)| = q(q+\varepsilon)/2$. Now we have two following possibilities:

- $x \in N$. Then, since $N \cong SL_2(q)$ and $|cl_N(x)| \mid |cl_G(x)|$, we get from (6) that $|cl_N(x)| = 1$. So $x \in Z(N) = Z(G)$, a contradiction.
- $x \in G - NZ(G)$. Then, $xZ(G) \in G/Z(G) \cong PGL_2(q)$. Thus, Lemma 3.6 shows that $|cl_{G/Z(G)}(xZ(G))| = q(q + \varepsilon 1)$. So, by Lemma 2.6, $q(q + \varepsilon 1) \mid |cl_G(x)|$, which is impossible.

The above contradictions complete the proof as well.

Remark 3.8. *Let A be an abelian group containing a proper subgroup, say A', and $a \in A - A'$ such that $1 \neq a^2 \in A'$ and $A = A'.\langle a\rangle$. Also, let σ be a diagonal automorphism of $PSL_2(q)$. Set $t = (\sigma, a)$ and $H = (PSL_2(q) \times A').\langle t\rangle$. Then, since $1 \neq t^2 = (\sigma^2, a^2) \in PSL_2(q) \times A'$ and $A' = Z(H)$, Lemma 3.7 shows that $cs^*(H) = cs^*(H/Z(H)) = cs^*(PGL_2(q))$. Note that $H \ncong B \times PGL_2(q)$, for every abelian group B. Also, if $H \cong PGL_2(q) \times Z(H)$, then it is obvious that $cs^*(H) = cs^*(PGL_2(q))$. Thus, if $q > 5$ is odd, then $PGL_2(q)$ cannot be determined uniquely by its conjugacy class sizes under an abelian direct factor.*

Remark 3.9. *If $G \cong (PSL_2(q) \times Z(G)).C_2$, then we can check easily that $G \cong ((PSL_(q) \times Z(G)_2).C_2) \times Z(G)_{2'}$, where $Z(G)_2 \in \mathrm{Syl}_2(Z(G))$ and $Z(G)_{2'}$ is a $(\pi(Z(G)) - \{2\})$-Hall subgroup of $Z(G)$.*

References

[1] N. Ahanjideh, On the Thompson's conjecture for simple groups, Journal of Algebra 344(1) (2011) 205-228.
[2] N. Ahanjideh, On the Thompson's conjecture on conjugacy class sizes, International Journal of Algebra and Computation 23(1) (2013) 37-68.
[3] N. Ahanjideh, Thompson's conjecture for some simple groups with connected prime graph, Algebra and Logic 51(6) (2013) 451-478.
[4] N. Ahanjideh, M. Ahanjideh, On the validity of Thompson's conjecture for simple groups, Communications in Algebra 41(11) (2013) 4116-4145.
[5] N. Ahanjideh, Thompson's conjecture for simple groups of Lie type B_n and C_n, Journal of Group Theory 19(4) (2016) 713-733.
[6] N. Ahanjideh, Thompson's conjecture on conjugacy class sizes for the simple group $PSU_n(q)$, International Journal of Algebra and Computation 27(6) (2017) 769-792.
[7] N. Ahanjideh, Finite groups with the same conjugacy class sizes as a simple group, International Journal of Group Theory 8 (2019) 23-33.
[8] Z. Akhlaghi, M. Khatami, T. Le, J. Moori, H. P. Tong-Viet, A dual version of Huppert's conjecture on conjugacy class sizes, Journal of Group Theory 18 (2015) 115-131.
[9] A. Babai, A. Mahmoudifar, Thompson's conjecture for the alternating group of degree $2p$ and $2p+1$, Czechoslovak Mathematical Journal 67 (4) (2017) 1049-1058.
[10] A. R. Camina and R. D. Camina, The influence of conjugacy class sizes on the structure of groups: a survey, Asian-European Journal of Mathematics 4(4) (2011) 559-588.
[11] G. Chen, On Thompson's conjecture, Journal of Algebra 185 (1996) 184-193.
[12] D. Chillag, M. Herzog, On the length of the conjugacy classes of groups, Journal of Algebra 131 (1990) 110-125.
[13] I.B. Gorshkov, Towards Thompson's conjecture for alternating and symmetric groups, Journal of Group Theory 19 (2016) 331-336.
[14] B. Khosravi, M. Khatami, Z. Akhlaghi, Some new characterizations for $PGL(2, q)$, CUBO, A Mathematical Journal 13 (2) (2011) 151-161.
[15] A.S. Kondratiev, Prime graph components of simple groups, Mathematics of the USSR-Sbornik 67 (1) (1990) 235-247.
[16] A.V. Vasilev, On Thompson's conjecture, Siberian Electronic Mathematical Reports 6 (2009) 457-464.
[17] L. Wang, W. Shi, On Thompson's conjecture for almost sporadic simple groups, Journal of Algebra and Its Applications 13 (2014) DOI:10.1142/S0219498813500898.
[18] J.S. Williams, Prime graph components of groups, Journal of Algebra 69 (2) (1981) 487-513.