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Hybrid Subgradient Algorithm for Equilibrium and Fixed Point Problems by Approximation of Nonexpansive Mapping

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Abstract. In this paper a new algorithm considered on a real Hilbert space for finding a common point in the solution set of a class of pseudomonotone equilibrium problem and the set of fixed points of nonexpansive mappings. We produce this algorithm by mappings T_k that are approximations of non-expansive mapping T. The strong convergence theorem of the proposed algorithms is investigated. Our results generalize some recent results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product h $\langle .,. \rangle$ and its reduced norm $\|.\|$. Let C be a nonempty closed convex subset of H. We recall that a mapping $T : C \to C$ is said to be a nonexpansive on C iff

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of all fixed points of T, i.e.

$$F(T) = \{x \in X : Tx = x\}.$$

It is well known that if $F(T) \neq \emptyset$, $F(T)$ is closed and convex. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ such that $f(x, x) = 0$ for all $x \in C$. An equilibrium problem $EP(f, C)$ in the sense of Blum and Oettli [3], is stated as follows:

$$\textit{Find } x^* \in C \ \textit{ such that } \ f(x^*, x) \geq 0 \quad \textit{for all } x \in C.$$

We denote the set of solutions $EP(f, C)$, by $Sol(C, f)$. This problem is also often called the Ky Fan inequality due to his contribution to this field. It is well known (see e.g. [3], [4], [5], [6]) that some important problems such as convex programs, variational inequalities, the Kakutani fixed point, minimax problems and Nash equilibrium models can be formulated as an equilibrium problem of the form $EP(f, C)$.

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Example 1.1. *Let $\varphi : C \to \mathbb{R}$ be given. The point x^* is a solution of optimization problem*

$$\text{Find } x^* \in C \text{ such that } \varphi(x^*) \leq \varphi(x) \quad \text{for all } x \in C.$$

if and only if it solves the equilibrium problem relative to $EP(f,C)$ where $f(x,y) = \varphi(y) - \varphi(x)$.

Example 1.2. *Let $T : C \to C$ be given. The point x^* is a fixed point of T if and only if it solves the equilibrium problem relative to $EP(f,C)$ where $f(x,y) = \langle x - Tx, y - x\rangle$.*

Many algorithms have been developed for solving problem $EP(f,C)$ combining diagonal subgradients with projections, see for instance ([1], [2], [7], [8], [10], [12]) and references therein. The problem $P(C,f,T)$ of finding a common point in the solution set of problem $EP(C,f)$ and the set $F(T)$ of a nonexpansive mapping T recently becomes an attractive subject, and various methods have been developed for solving this problem. Most of the existing algorithms for this problem are based on the proximal point method applying to equilibrium problem $EP(C,f)$ combining with a Mann's iteration to the problem of finding a fixed point of T. In 2006, Takahashi and Takahashi [11] proposed an iterative scheme under the name viscosity approximation methods for finding an element of the solutions set S of problem $P(C,f,T)$ of non-expansive mapping T in a real Hilbert space H. In 2014, Anh and Muu in [2] studied a new algorithm for solving $P(C,f,T)$ of nonexpansive mapping T in a real Hilbert space H which defined as follows:

$$\begin{cases} \text{Pick any } x_0 \in C \\ w_k \in \partial_{\varepsilon_k} f(x_k,.)(x_k), \\ \gamma_k = \max\{\lambda_k, \|w_k\|\} \text{ and } \alpha_k = \frac{\beta_k}{\gamma_k} \\ y_k = P_C(x_k - \alpha_k w_k) \text{ and let } x_{k+1} = \delta_k x_k + (1-\delta_k)T(y_k) \end{cases} \tag{1}$$

where the sequences $\{\lambda_k\}$, $\{\beta_k\}$, $\{\varepsilon_k\}$, $\{\delta_k\}$ of nonnegative numbers satisfy the following conditions

$$\begin{cases} 0 < \lambda_k < \overline{\lambda},\ 0 < a < \delta_k < b < 1,\ \delta_k \to \frac{1}{2}, \\ \beta_k > 0,\ \sum_{k=0}^{\infty} \beta_k = \infty,\ \sum_{k=0}^{\infty} \beta_k^2 < \infty, \\ \sum_{k=0}^{\infty} \beta_k \varepsilon_k < \infty. \end{cases}$$

The aim of this paper is to present a new algorithm for solving $P(C,f,T)$, by technical of Reich in [13], [14]. Anh and Muu in [2], produced an algorithm by non-expansive mapping T but we produces it by mappings T_k that are approximation of non-expansive mapping T. Our results complement some known recent results in the literature.

2. Preliminaries

Let C be a nonempty closed convex subset of a Hilbert space H. The bifunction $f : C \times C \to \mathbb{R}$ is pseudomonotone on a set $A \subseteq C$ with respect to x if and only if for every $y \in A$,

$$f(x,y) \geq 0 \text{ implies } f(y,x) \leq 0.$$

We say that f is pseudomonotone on A if it is pseudomonotone on A with respect to every $x \in A$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x as $n \to \infty$, and $x_n \to x$ means that x_n converges strongly to x. In a real Hilbert space H for every $x, y \in H$ and $\lambda \in \mathbb{R}$ we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Since C is closed and convex, for any $x \in H$ there exists a unique point in C, denoted by $P_C(x)$ satisfying

$$\|x - P_C(x)\| \le \|x - y\|, \forall y \in C.$$

P_C is called the metric projection of H to C. It is well known that P_C satisfies the following properties:

$$\langle x - y, P_C(x) - P_C(y)\rangle \ge \|P_C(x) - P_C(y)\|^2, \forall x, y \in H,$$
$$\langle x - P_C(x), P_C(x) - y\rangle \ge 0, \forall x \in H, y \in C,$$
$$\|x - y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \forall x \in H, y \in C.$$

Every Hilbert space satisfies the Opial condition, i.e., if the sequence $\{x_k\}$ in a Hilbert space H converges weakly to $x \in H$, Then

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\| \ \ \textit{for all } y \in H \textit{ that } y \neq x.$$

We denote by $\partial_\varepsilon f(x_0)$ the set of ε-subdifferential of the convex function $f : C \to \mathbb{R}$ at $x_0 \in C$, i.e.

$$\partial_\varepsilon f(x_0) = \{x \in C : f(y) - f(x_0) \ge \langle x, y - x_0\rangle - \varepsilon, \forall y \in C\}.$$

Also $\partial_\varepsilon f(x, .)(x)$ stands for ε-subdifferential of the convex function $f(x, .)$ at x. Let us assume that the bifunction $f : C \times C \to \mathbb{R}$ and the nonexpansive mapping $T : C \to C$ satisfy the following conditions:

A_1. For each x, $f(x, x) = 0$ and $f(x, .)$ is convex on C,

A_2. $\partial_\varepsilon f(x, .)(x)$ is nonempty for each $\varepsilon > 0$ and $x \in C$ and bounded on bounded subsets of C,

A_3. f is pseudomonotone on C,

A_4. f is paramonotonic i.e.

$$x \in Sol(C, f), y \in C, f(x, y) = f(y, x) = 0 \textit{ implies that } y \in Sol(C, f)$$

A_5. For each $x \in C$, $f(., x)$ is weakly upper semicontinuous on C,

A_6. The solution set S of Problem $P(C, f, T)$ is nonempty.

Suppose that the sequences $\{\lambda_k\}$, $\{\beta_k\}$, $\{\varepsilon_k\}$, $\{\delta_k\}$ and $\{\eta_k\}$ of nonnegative numbers satisfy the following conditions

$$\begin{cases} 0 < \underline{\lambda} < \lambda_k < \overline{\lambda}, \delta_k < b < 1, \delta_k \to \frac{1}{2}, \\ \beta > 0, \sum_{k=0}^{\infty} \beta_k = \infty, \sum_{k=0}^{\infty} \beta_k^2 < \infty, \\ \sum_{k=0}^{\infty} \beta_k \varepsilon_k < \infty, \\ \sum_{k=0}^{\infty} \eta_k < \infty. \end{cases} \tag{2}$$

and the mappings $T_k : C \to C$ such that for each integer $k \ge 0$, $\|T(x) - T_k(x)\| \le \eta_k$ for all $x \in C$. Now the iteration scheme for finding a common point in the set of solutions of Problem $P(C, f, T)$ can be written as follows:

$$\begin{cases} \textit{Pick } x_0 \in C; \\ w_k \in \partial_{\varepsilon_k} f(x_k, .)(x_k); \\ \gamma_k = \max\{\lambda_k, \|w_k\|\} \ \textit{and } \alpha_k = \frac{\beta_k}{\gamma_k}; \\ y_k = P_C(x_k - \alpha_k w_k) \ \textit{and let } x_{k+1} = \delta_k x_k + (1 - \delta_k) T_k(y_k). \end{cases} \tag{3}$$

To investigate the convergence of this scheme, we appeal the following results which will be used in the sequel.

Lemma 2.1. *Suppose that $\{\alpha_n\}$ be a sequence of nonnegative real number such that $\sum_0^\infty \alpha_n < \infty$. Then the sequence $\prod_{i=1}^n (1+\alpha_i)$ is convergent.*

Lemma 2.2. *Suppose that $\{s_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of nonnegative real numbers such that*

$$s_{n+1}^2 \leq s_n^2 + \alpha_n s_n + \beta_n,$$

where $\sum_0^\infty \alpha_n < \infty$ and $\sum_0^\infty \beta_n < \infty$. Then the sequence $\{s_n\}$ is convergent.

Proof. We claim that $\{s_n\}$ is bounded. Otherwise, there is a subsequence of $\{s_n\}$ (which we shall denote by $\{s_n\}$ again) such that for any $n \in \mathbb{N}$, $s_n > n^2$. Then we have

$$s_{n+1}^2 \leq s_n^2 + \alpha_n s_n + \beta_n \leq (1 + \frac{\alpha_n}{n^2}) s_n^2 + \beta_n.$$

Hence,

$$s_3^2 \leq (1 + \frac{\alpha_2}{4}) s_2^2 + \beta_2 \leq (1 + \frac{\alpha_2}{4})(1+\alpha_1) s_1^2 + (1 + \frac{\alpha_2}{4})\beta_1 + \beta_2$$

and

$$s_4^2 \leq (1 + \frac{\alpha_3}{9}) s_3^2 + \beta_3 \leq (1 + \frac{\alpha_3}{9})(1 + \frac{\alpha_2}{4})(1+\alpha_1) s_1^2 + (1 + \frac{\alpha_3}{9})(1 + \frac{\alpha_2}{4})\beta_1 + (1 + \frac{\alpha_3}{9})\beta_2 + \beta_3.$$

Now by induction we see that for every $n \geq 1$,

$$s_{n+1}^2 \leq s_1^2 \prod_{i=1}^{n} (1 + \frac{\alpha_i}{i^2}) + \beta_n + \sum_{k=1}^{n-1} (\beta_k \prod_{i=k+1}^{n} (1 + \frac{\alpha_i}{i^2})).$$

Then by lemma 2.1, we obtain

$$s_{n+1}^2 \leq s_1^2 \prod_{i=1}^{\infty} (1 + \frac{\alpha_i}{i^2}) + \sum_{k=1}^{\infty} (\beta_k \prod_{i=k+1}^{\infty} (1 + \frac{\alpha_i}{i^2})) < \infty,$$

which is a contradiction. Hence our claim is proved. Since $\{s_n\}$ is bounded, there is a number M such that for every $n \in \mathbb{N}$, $s_n < M$. So

$$s_{n+1}^2 \leq s_n^2 + \alpha_n s_n + \beta_n \leq s_n^2 + M\alpha_n + \beta_n.$$

Now for any $n, m \geq 1$, we have

$$s_{n+m+1}^2 \leq s_{n+m}^2 + M\alpha_{n+m} + \beta_{n+m} \leq \ldots \leq s_n^2 + \sum_{i=n}^{n+m} M\alpha_i + \beta_i$$

and then $\limsup_m s_m^2 \leq s_n^2 + \sum_{i=n}^\infty M\alpha_i + \beta_i$ which implies that

$$\limsup_m s_m^2 \leq \liminf_n s_n^2.$$

□

Lemma 2.3. *Suppose that f apply in A_3, A_4, A_5, $x^* \in Sol(C, f)$, $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ in H such that $x_{k_i} \rightharpoonup \overline{x}$ and*

$$\limsup_{k\to\infty} f(x_k, x^*) = \lim_{i\to\infty} f(x_{k_i}, x^*) = 0.$$

Then $\overline{x} \in Sol(C, f)$

Proof. Since $f(.,x^*)$ is weakly upper semicontinuous, we have

$$f(\overline{x},x^*) \geq \limsup_{i\to\infty} f(x_{k_i},x^*) = \lim_{i\to\infty} f(x_{k_i},x^*) = \limsup_{k\to\infty} f(x_k,x^*) = 0.$$

On the other hand, f is pseudomonotone. So $f(\overline{x},x^*) \leq 0$ and then $f(\overline{x},x^*) = 0$. Now we obtain $\overline{x}\in Sol(f,c)$. □

Lemma 2.4. *[9] Let H be a real Hilbert space, $\{\delta_k\}$ be a sequence of real numbers such that $0 < a \leq \delta_k \leq b < 1$ for all $k = 0, 1, ...,$ and let $\{v_k\}$, $\{w_k\}$ be sequences of H such that*

$$\limsup_{k\to\infty} \|v_k\| \leq c, \limsup_{k\to\infty} \|w_k\| \leq c,$$

and

$$\lim_{k\to\infty} \|\delta_k + (1-\delta_k)w_k\| = c. Then, \lim_{k\to\infty} \|v_k - w_k\| = 0$$

3. Main results

Now we are ready to prove the convergence of algorithm 3.

Theorem 3.1. *Suppose that Assumptions $A_1 - A_6$ are satisfied, the parameters δ, λ, and the sequences $\{\lambda_k\}$, $\{\beta_k\}$, $\{\varepsilon_k\}$, $\{\delta_k\}$ satisfy restrictions 2. Then the sequences $\{x_k\}$, $\{y_k\}$ and $\{P_C(x_k)\}$ generated by 3 strongly converge to the same point $\overline{x}$ and $\overline{x} = \lim_{k\to\infty} P_C(x_k)$.*

The theorem is proved through several claims.
Claim 1 For every $x^* \in C$ there is a real number c such that $\lim_{k\to\infty} \|x_k - x^*\| = c$.

Proof. It follows from $x_{k+1} = \delta_k x_k + (1-\delta_k)T_k(y_k)$ and $x^* \in F(T)$ that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 = \|\delta_k x_k + (1-\delta_k)T_k(y_k) - x^*\|^2 \\
&= \|\delta_k(x_k - x^*) + (1-\delta_k)(T_k(y_k) - T(x^*))\|^2 \\
&\leq \delta_k\|x_k - x^*\|^2 + (1-\delta_k)\|T_k(y_k) - T(x^*)\|^2 \\
&\leq \delta_k\|x_k - x^*\|^2 + (1-\delta_k)\|T_k(y_k) - T(y_k) + T(y_k) - T(x^*)\|^2 \\
&\leq \delta_k\|x_k - x^*\|^2 + (1-\delta_k)(\eta_k + \|y_k - x^*\|)^2 \\
&= \delta_k\|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + \|y_k - x^*\|^2 + 2\eta_k\|y_k - x^*\|) \\
&= \delta_k\|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + \|x_k - x^*\|^2 - \|y_k - x_k\|^2 + 2\langle x_k - y_k, x^* - y_k\rangle + 2\eta_k\|y_k - x^*\|) \\
&\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\langle x_k - y_k, x^* - y_k\rangle + 2\eta_k\|y_k - x^*\|).
\end{aligned} \tag{4}$$

Since $y_k = P_C(x_k - \alpha_k w_k)$ and $x^* \in C$,

$$\langle x_k - y_k, x^* - y_k\rangle \leq \alpha_k\langle w_k, x^* - y_k\rangle.$$

Also since $x_k \in C$, we have

$$\langle x_k - \alpha_k w_k - y_k, y_k - x_k\rangle \geq 0.$$

Hence

$$\begin{aligned}
&\|x_k - y_k\|^2 \leq \alpha_k\langle w_k, x_k - y_k\rangle \leq \alpha_k\|w_k\|\|x_k - y_k\| \\
&= \frac{\beta_k}{\gamma_k}\|w_k\|\|x_k - y_k\| \leq \frac{\beta_k}{\|w_k\|}\|w_k\|\|x_k - y_k\| = \beta_k\|x_k - y_k\|,
\end{aligned}$$

which implies that $\|x_k - y_k\| \leq \beta_k$. Combining this inequality with 4 yields

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\langle x_k - y_k, x^* - y_k\rangle + 2\eta_k\|y_k - x^*\|) \\ &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - y_k - x_k + x_k\rangle + 2\eta_k\|y_k - x^*\|) \\ &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - x_k\rangle + 2\alpha_k\|w_k\|\|x_k - y_k\| + 2\eta_k\|y_k - x^*\|) \\ &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - x_k\rangle + 2\beta_k^2 + 2\eta_k\|y_k - x^*\|) \\ &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - x_k\rangle + 2\beta_k^2 + 2\eta_k(\|y_k - x_k\| + \|x_k - x^*\|)) \\ &\leq \|x_k - x^*\|^2 + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - x_k\rangle + 2\beta_k^2 + 2\eta_k(\beta_k + \|x_k - x^*\|)) \\ &\leq \|x_k - x^*\|^2 + 2\eta_k(1-\delta_k)\|x_k - x^*\| + (1-\delta_k)(\eta_k^2 + 2\alpha_k\langle w_k, x^* - x_k\rangle + 2\beta_k^2 + 2\eta_k\beta_k). \end{aligned} \tag{5}$$

Since $w_k \in \partial_{\varepsilon_k} f(x_k, .)(x_k)$, $x^* \in C$ and $f(x, x) = 0$ for all $x \in C$, we have

$$\langle w_k, x^* - x_k\rangle \leq f(x_k, x^*) - f(x_k, x_k) + \varepsilon_k = f(x_k, x^*) + \varepsilon_k. \tag{6}$$

Combining 5 and 6, we obtain that

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + 2\eta_k(1-\delta_k)\|x_k - x^*\| + (1-\delta_k)(\eta_k^2 + 2\alpha_k(f(x_k, x^*) + \varepsilon_k) + 2\beta_k^2 + 2\eta_k\beta_k). \tag{7}$$

On the other hand, since $x^* \in S$, $f(x^*, x_k) \geq 0$ and then by pseudomonotonicity of f, we have $f(x_k, x^*) \leq 0$. Then

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + 2\eta_k(1-\delta_k)\|x_k - x^*\| + (1-\delta_k)(\eta_k^2 + 2\alpha_k\varepsilon_k + 2\beta_k^2 + 2\eta_k\beta_k). \tag{8}$$

Now applying Lemma 2.1 to 8, by Assumption 2, we obtain the existence of $\lim_{n\to\infty} \|x_{k+1} - x^*\|$. □

Claim 2 $\limsup_{k\to\infty} f(x_k, x^*) = 0$.

Proof. By 7, for every k, one has

$$\begin{aligned} &-2\alpha_k(1-\delta_k)f(x_k, x^*) \\ &\leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + 2\eta_k(1-\delta_k)\|x_k - x^*\| + (1-\delta_k)(\eta_k^2 + 2\alpha_k\varepsilon_k + 2\beta_k^2 + 2\eta_k\beta_k). \end{aligned}$$

Since $\|x_k - x^*\|$ is convergent, there is an M such that $\|x_k - x^*\| < M$. So

$$-2\alpha_k(1-\delta_k)f(x_k, x^*) \leq \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 + 2\eta_k M + (\eta_k^2 + 2\alpha_k\varepsilon_k + 2\beta_k^2 + 2\eta_k\beta_k).$$

Summing up the above inequalities for every k, we obtain that

$$\begin{aligned} 0 &\leq -2\sum_{k=0}^{\infty} \alpha_k(1-\delta_k)f(x_k, x^*) \\ &\leq \|x_0 - x^*\|^2 + \sum_{k=0}^{\infty} 2\eta_k(1-\delta_k)M + \sum_{k=0}^{\infty}(1-\delta_k)(\eta_k^2 + 2\alpha_k\varepsilon_k + 2\beta_k^2 + 2\eta_k\beta_k) < \infty. \end{aligned}$$

On the other hand, by A_2 we have that $\{\|w_k\|\}$ is bounded. In fact, by claim 1, we get that $\|x_k\|$ is bounded. Therefore, the assertion follows from A_2. In consequence, using 2 we conclude that there exists L such that $\alpha_k \geq \frac{\beta_k}{L}$. Therefore

$$0 \leq -\sum_{k=0}^{\infty} \frac{\beta_k}{L} f(x_k, x^*) \leq -2\sum_{k=0}^{\infty} \alpha_k(1-\delta_k)f(x_k, x^*) < \infty.$$

Now by $\sum_{k=0}^{\infty} \beta_k = \infty$ and $-f(x_k, x^*) \geq 0$ we have $\limsup_{k\to\infty} f(x_k, x^*) = 0$. □

Claim 3 $\lim_{k\to\infty}\|T(x_k)-x_k\|=0$.

Proof. By nonexpansiveness of T, we can write

$$\begin{aligned}&\|T_k(y_k)-x^*\|\leq\|T_k(y_k)-T(y_k)\|+\|T(y_k)-x^*\|\leq\eta_k+\|y_k-x^*\|\\&\leq\eta_k+\|x_k-x^*\|+\|y_k-x_k\|\leq\eta_k+\beta_k+\|x_k-x^*\|,\end{aligned}$$

which implies

$$\limsup_{k\to\infty}\|T_k(y_k)-x^*\|\leq\lim_{k\to\infty}\|x_k-x^*\|+\beta_k+\eta_k=c.$$

On the other hand

$$\lim_{k\to\infty}\|\delta_k(x_k-x^*)+(1-\delta_k)T_k(y_k)-x^*\|=\lim_{k\to\infty}\|x_{k+1}-x^*\|=c.$$

Then, applying Lemma 2.4 with $v_k:=x_k-x^*$ and $w_k:=T_k(y_k)-x^*$ it results

$$\lim_{k\to\infty}\|T_k(y_k)-x_k\|=0. \tag{9}$$

Since

$$\begin{aligned}&\|T(x_k)-x_k\|\leq\|T_k(x_k)-T(x_k)\|+\|T_k(x_k)-x_k\|\leq\eta_k+\|T_k(y_k)-T_k(x_k)\|+\|T_k(y_k)-x_k\|\\&\leq 3\eta_k+\|T(y_k)-T(x_k)\|+\|T_k(y_k)-x_k\|\\&\leq 3\eta_k+\|y_k-x_k\|+\|T_k(y_k)-x_k\|\\&\leq 3\eta_k+\beta_k+\|T_k(y_k)-x_k\|,\end{aligned}$$

we have

$$\lim_{k\to\infty}\|T(x_k)-x_k\|=0.$$

□

Claim 4 There is subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i}\to\overline{x}$ and $\overline{x}\in C$.

Proof. Since $\{x_k\}$ is bounded in Hilbert space H, there is a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $\overline{x}\in C$ such that $x_{k_i}\rightharpoonup\overline{x}$ and

$$\limsup_{k\to\infty}f(x_k,x^*)=\lim_{i\to\infty}f(x_{k_i},x^*).$$

So by lemma 2.3, $\overline{x}\in Sol(C,f)$. Now we show that $\overline{x}\in F(T)$. Suppose in contrary that $T(\overline{x})\neq\overline{x}$. Then by Opial condition we have

$$\begin{aligned}&\limsup_{i\to\infty}\|x_{k_i}-\overline{x}\|<\limsup_{i\to\infty}\|x_{k_i}-T(\overline{x})\|\\&\leq\limsup_{i\to\infty}\|x_{k_i}-T(x_{k_i})\|+\limsup_{i\to\infty}\|T(x_{k_i})-T(\overline{x})\|\\&\leq\limsup_{i\to\infty}\|x_{k_i}-\overline{x}\|,\end{aligned}$$

which is a contradiction. Hence $T(\overline{x})=\overline{x}$. □

Claim 5 $\lim_{k\to\infty}x_k=\overline{x}$.

Proof. By definition of x_{k+1}, we have

$$\begin{aligned}
&\|x_{k+1} - P_C(x_{k+1})\|^2 \leq \|x_{k+1} - P_C(x_k)\|^2 \\
&= \|\delta_k(x_k - P_C(x_k)) + (1-\delta_k)(T_k(y_k) - P_C(x_k))\|^2 \\
&\leq \delta_k\|x_k - P_C(x_k)\|^2 + (1-\delta_k)\|T_k(y_k) - P_C(x_k)\|^2 \\
&\leq \delta_k\|x_k - P_C(x_k)\|^2 + (1-\delta_k)(\|T_k(y_k) - x_k\|^2 - \|x_k - P_C(x_k)\|^2) \\
&\leq (2\delta_k - 1)\|x_k - P_C(x_k)\|^2 + (1-\delta_k)\|T_k(y_k) - x_k\|^2.
\end{aligned}$$

Now by $\|x_k - P_C(x_k)\| \leq \|x_k - x^*\|$ and (9), we obtain

$$\lim_{k\to\infty} \|x_{k+1} - P_C(x_{k+1})\| = 0. \tag{10}$$

Now we claim that $\{P_C(x_k)\}$ is Cauchy. For simplicity, let $P_k := P_C(x_k)$. By relation (8) with $x^* = P_k$ for $m > k$, we have

$$\begin{aligned}
&\|x_m - P_k\|^2 \leq \|x_{m-1} - P_k\|^2 + MA_{m-1} + B_{m-1} \\
&\leq \|x_{m-2} - P_k\|^2 + M(A_{m-1} + A_{m-2}) + B_{m-1} + B_{m-2} \\
&\leq \dots \\
&\leq \|x_k - P_k\|^2 + M\sum_{i=k}^{m-1} A_i + \sum_{i=k}^{m-1} B_i,
\end{aligned}$$

where $A_m := 2\eta_m(1-\delta_m)$, $B_m = (1-\delta_m)(\eta_m^2 + 2\alpha_m\varepsilon_m + 2\beta_m^2 + 2\eta_m\beta_m)$ and M is a bound of $\|x_k - x^*\|$. By convexity of C, we have $\frac{1}{2}(P_m + P_k) \in C$ and then

$$\begin{aligned}
&\|P_m - P_k\|^2 \leq 2\|x_m - P_m\|^2 + 2\|x_m - P_k\|^2 \\
&\leq 2\|x_m - P_m\|^2 + 2\|x_k - P_k\|^2 + M\sum_{i=k}^{m-1} A_i + \sum_{i=k}^{m-1} B_i,
\end{aligned}$$

which, together with $\sum_{i=k}^{m-1} A_i < \infty$, $\sum_{i=k}^{m-1} B_i < \infty$ and (10), implies that $\{P_k\}$ is a Cauchy sequence. Hence there is $P \in C$ such that $P_k \to P$ and then

$$P = \lim_{i\to\infty} P_{k_i} = \lim_{i\to\infty} P(x_{k_i}) = P(\lim_{i\to\infty} x_{k_i}) = P(\overline{x}) = \overline{x} \in C.$$

So $P_k \to \overline{x}$. Finally by

$$\|x_k - \overline{x}\| \leq \|x_k - P_k\| + \|P_k - \overline{x}\|,$$

we can conclude that $x_k \to \overline{x}$. □

Remark 3.2. *Let for map T in theorem 3.1, $T_i(x) = T(x)$. Then we obtain the iteration processes in [2].*

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