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Some Inclusion Relationships of Meromorphic Functions Associated to New Generalization of Mittag-Leffler Function

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Abstract. This article is devoted to introduce a new operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$ using the generalized Mittag-Leffler function. Then, we give meromorphic subclasses associated $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$. Finally, we calculated inclusion relations by using $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$ and integral operator F_μ.

1. Introduction

We need to define some essentially definitions. Firstly, $\sum$ is the class of functions of the form:

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

which is analytic in the punctured unit disk $\mathbb{U}^* = \mathbb{U}\backslash\{0\} = \mathbb{U}\backslash\{0\}$.
The Hadamard product (or convolution) of $f(z)$ given by (1) and g given by

$$g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n, \tag{2}$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{3}$$

A function $f(z) \in \Sigma$ is said meromorphically starlike function of order δ in $\mathbb{U}^*$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < -\delta, \quad 0 \le \delta < 1;\ z \in \mathbb{U}^*. \tag{4}$$

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We denote by $\Sigma S^{*}(\delta)$ the class of all meromorphically starlike functions of order δ. A function $f(z) \in \Sigma$ is said to be in the class $\Sigma C(\delta)$ of meromorphically convex function of order δ in $\mathbb{U}^{*}$, if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < -\delta, \quad (0 \leq \delta < 1;\ z \in \mathbb{U}^{*}). \tag{5}$$

It is easy to observe from (4) and (5) that

$$f(z) \in \Sigma C(\delta) \Longleftrightarrow -zf'(z) \in \Sigma S^{*}(\delta). \tag{6}$$

A function $f(z) \in \Sigma$ is said to be meromorphically close-to-convex function of order σ and type δ in $\mathbb{U}^{*}$, if there exists a function $g \in \Sigma S^{*}(\delta)$ such that

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} < -\sigma \quad (0 \leq \delta, \sigma < 1;\ z \in \mathbb{U}^{*}). \tag{7}$$

We denote by $\Sigma K(\sigma,\delta)$ the class of all meromorphically close-to-convex function of order σ and type δ. A function $f(z) \in \Sigma$ is said to be meromorphically quasi-convex functions of order σ and type δ in $\mathbb{U}^{*}$, if there exists a function $g \in \Sigma C(\delta)$ such that

$$\Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} < -\sigma \quad (0 \leq \delta, \sigma < 1; z \in \mathbb{U}^{*}). \tag{8}$$

$\Sigma K^{*}(\sigma,\delta)$ is the class of all meromorphically quasi-convex functions of order σ and type δ. It follows from (7) and (8) that

$$f(z) \in \Sigma K^{*}(\sigma,\delta) \Longleftrightarrow -zf'(z) \in \Sigma K(\sigma,\delta). \tag{9}$$

Let the Mittag-Leffler function $E_{\alpha}(z)$ (see [9]) defined as follows:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n+1)} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0). \tag{10}$$

A more general function $E_{\alpha}(z)$ is $E_{\alpha,\beta}(z)$ was introduced by Wiman (see [12, 13]) and given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n+\beta)} \quad (z \in \mathbb{C}). \tag{11}$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k)-1\}$ and $\Re(k) > 0$, Srivastava and Tomovski [15] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$ in the form

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}\, z^{n}}{\Gamma(\alpha n+\beta)n!} \quad (z \in \mathbb{C}). \tag{12}$$

Now, by using (12) we define the function $\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z)$ as follows:

$$\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z) = \Gamma(\beta) z^{-1} E_{\alpha,\beta}^{\gamma,k}(z).$$

It follows that, for $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k)-1\}$ and $\Re(k) > 0$ that

$$\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{(\gamma)_{(n+1)k}\ \Gamma(\beta) z^{n}}{\Gamma[\alpha(n+1)+\beta](n+1)!} (z \in \mathbb{C}). \tag{13}$$

We can define the function $\mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z)$ as a convolution product given by:

$$\begin{aligned} \mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z) &= \mathfrak{K}^{\gamma,k}_{\alpha,\beta}(z) * f(z), \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!} a_n z^n. \end{aligned} \tag{14}$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k) - 1\}, \Re(k) > 0, \eta \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, ...\}$, we define a new linear operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) : \sum \to \sum$ as follows:

$$\mathbb{I}^{\gamma,k,0}_{\alpha,\beta,\eta}(f)(z) = \mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z),$$

$$\begin{aligned} \mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z) &= (1-\eta)\mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z) + \eta z^{-1}[z^2 \mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z)]' \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\,[1 + \eta(n+1)] a_n z^n, \end{aligned}$$

$$\begin{aligned} \mathbb{I}^{\gamma,k,2}_{\alpha,\beta,\eta}(f)(z) &= \mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}[\mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z)] = (1-\eta)\mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z) + \eta z^{-1}[z^2 \mathbb{I}^{\gamma,k,2}_{\alpha,\beta,\eta}(f)(z)]' \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\,[1 + \eta(n+1)]^2 a_n z^n. \end{aligned}$$

By induction we prove that

$$\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{\gamma[\Gamma + k(n+1)]\,\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\,[1 + \eta(n+1)]^m a_n z^n. \tag{15}$$

Note that by taking $m = 0$ in (15), we obtain (14).

Remark 1.1. *We note that*

(i) $\mathbb{I}^{1,1,0}_{0,\beta,\eta,}(f)(z) = f(z)$;

(ii) $\mathbb{I}^{2,1,0}_{0,\beta,\eta}(f)(z) = 2f(z) + zf'(z)$;

(iii) $\mathbb{I}^{1,1,0}_{1,1,\eta}(\frac{1}{z(1-z)}) = z^{-1}e^z$;

(iv) $\mathbb{I}^{1,1,0}_{2,1,\eta}(\frac{1}{z(1-z)}) = z^{-1}\cosh\left(\sqrt{z}\right)$;

(v) $\mathbb{I}^{1,1,0}_{2,2,\eta}(\frac{1}{z(1-z)}) = \frac{\sinh(\sqrt{z})}{\sqrt{z^3}}$.

Observe that:

(a) $\mathbb{I}^{1,1,m}_{0,\beta,\eta}(f)(z) = D^m_\eta f(z)$ (see [1, 5], with $l = p = 1$ and [4] with $p = 1$);

(b) $\mathbb{I}^{1,1,m}_{0,\beta,1}(f)(z) = D^m_{1,1} f(z)$ (see [2]);

(c) $\mathbb{II}^{\gamma,k,0}_{\alpha,\beta,\eta}(f)(z) = M^{\gamma,k}_{1,\beta,\eta} f(z)$ (see [10, 11] with p=1);

(d) $\mathbb{I}^{c,1,0}_{1,k+1,\eta}(f)(z) = I_{k,c} f(z)$ $(c > 1)$ (see Yuan et al. [19]).

2. Preliminaries

The following Lemmas will be required in our investigation.

Lemma 2.1. *Let $f \in \sum$ then the operator $I^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)$ achieve the following relations*

$$(i) \quad z[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)]' = \frac{\gamma}{k}\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}(f)(z) - \left(\frac{\gamma+k}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z), \tag{16}$$

$$(ii) \quad z\alpha[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta+1,\eta}(f)(z)]' = \beta\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) - (\alpha+\beta)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta+1,\eta}(f)(z), \tag{17}$$

$$(iii) \quad z[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)]' = \frac{1}{\eta}\mathbb{I}^{\gamma,k,m+1}_{\alpha,\beta,\eta}(f)(z) - \left(1+\frac{1}{\eta}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z), \tag{18}$$

for all $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k) - 1\}$, $\Re(k) > 0$, $\eta \geq 0$ and $m \in \mathbb{N}_0$.

Next, by using the operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$, the classes $\Sigma S^*(\delta)$, $\Sigma C(\delta)$, $\Sigma K(\delta,\sigma)$ and $\Sigma K^*(\delta,\sigma)$ which defined, respectively, by relations (4), (5), (7) and (8), we introduce the following new classes of meromorphic functions for $0 \leq \delta, \sigma < 1$:

$$\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma S^*(\delta)\right\},$$
$$\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma C(\delta)\right\},$$
$$\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma K(\delta,\sigma)\right\}$$

and

$$\Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma K^*(\delta,\sigma)\right\}.$$

We can see that:

$$f(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \tag{19}$$

and

$$f(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}. \tag{20}$$

We note that

i) $\Sigma S^{*,1,1,0}_{0,\beta,\eta} = \Sigma S^*(\delta)$ $(0 \leq \delta < 1)$ (see Juneja and Reddy [7]);

ii) $\Sigma C^{1,1,0}_{0,\beta,\eta} = \Sigma C(\delta)$ $(0 \leq \delta < 1)$ (see Srivastava et al. [14]);

iii) $\Sigma S^{*,\gamma,k,0}_{\alpha,\beta,\eta} = \Sigma S^{\gamma,k}_{\alpha,\beta,\eta}$ (see Aouf and Seoudy [3], with $p = B = 1$ and $A = 2\delta - 1$).

Lemma 2.2. *[8] Let φ be complex-valued function such that,*

$$\varphi : D \longrightarrow \mathbb{C}, \ (D \subset \mathbb{C} \times \mathbb{C}).$$

For $u = u_1 + iu_2$ and $v = v_1 + iv_2$ in $\mathbb{C}$, we suppose that φ satisfies the following conditions:

i) *$\varphi(u, v)$ is continuous in D;*

ii) $(1,0) \in D$ *and* $\Re\{\varphi(1,0)\} > 0$;

iii) $\Re\{\varphi(iu_2, v_1)\} \le 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \le -\frac{(1+u_2^2)}{2}$.

Let

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots, \tag{21}$$

be regular in $\mathbb{U}$ *such that* $(h(z),\ zh'(z)) \in D$ *for all* $z \in \mathbb{U}$.
If

$$\Re\{\varphi(h(z),\ zh'(z))\} > 0\ (z \in \mathbb{U}),$$

then

$$\Re\{h(z)\} > 0\ (z \in \mathbb{U}).$$

Lemma 2.3. *[6] Let the (nonconstant) function* $w(z)$ *be analytic in* $\mathbb{U}$, *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \mathbb{U}$, *then*

$$z_0 w'(z_0) = \xi w(z_0)\,,$$

where ξ *is a real number and* $\xi \ge 1$.

In the following section, we will get inclusion properties which associate the operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$ with the classes $\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$ and $\Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.

3. Main results

Unless otherwise mentioned, we shall assume in this paper that $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max\{0, \Re(k) - 1\}$, $\Re(k) > 0$, $\eta > 0$ and $m \in \mathbb{N}_0$.

Theorem 3.1. *If* $f(z) \in \Sigma$, $\Re(\beta) > 1$, $\Re(\frac{\gamma}{k}) > 0$, *then*

$$\Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \subset \Sigma S^{*,\gamma,k,m}_{\alpha,\beta+1,\eta} \tag{22}$$

and

$$\Sigma S^{*,\gamma,k,m+1}_{\alpha,\beta,\eta} \subset \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}. \tag{23}$$

Proof. To prove the first part of (22), let $f \in \Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}$ and

$$\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} = -\delta - (1-\delta)h(z)\,, \tag{24}$$

where h is given by (21). Applying (16) in (24), we obtain

$$\frac{\gamma}{k}\frac{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z)}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z)} = -\delta - (1-\delta)h(z) + \frac{k+\gamma}{k}. \tag{25}$$

Differentiating (25) logarithmically with respect to z, we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z)} = \frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} + \frac{(1-\delta)zh'(z)}{(1-\delta)h(z) + \delta - \left(\frac{k+\gamma}{k}\right)}\quad (z \in \mathbb{U}),$$

which, by (24), we get

$$\frac{z(\mathbb{I}_{\alpha,\beta,\eta}^{\gamma+1,k,m}f(z))'}{\mathbb{I}_{\alpha,\beta,\eta}^{\gamma+1,k,m}f(z)} = -\delta-(1-\delta)h(z)+\frac{(1-\delta)zh'(z)}{(1-\delta)h(z)+\delta-\left(\frac{k+\gamma}{k}\right)}. \tag{26}$$

Let

$$\varphi(u,v)=(1-\delta)u-\frac{(1-\delta)v}{(1-\delta)u+\delta-\left(\frac{k+\gamma}{k}\right)}, \tag{27}$$

with $h(z)=u=u_1+iu_2,\ zh'(z)=v=v_1+iv_2$. Then

i) $\varphi(u,v)$ is continuous in $D=\mathbb{C}\backslash\left\{1+\frac{(\gamma/k)}{1-\delta}\right\}\times\mathbb{C}$,

ii) $(1,0)\in D$ and $\Re\{\varphi(1,0)\}=1-\delta$,

iii) $\Re\{\varphi(iu_2,v_1)\}\leq 0$ for all $(iu_2,v_1)\in D$ and such that $v_1\leq-\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}\Re\{\varphi(iu_2,v_1)\} &= \Re\left\{\frac{-(1-\delta)v_1}{(1-\delta)iu_2+\delta-\left(\frac{k+\gamma}{k}\right)}\right\}\\ &= \frac{(1-\delta)\left[\left(\frac{k+\gamma}{k}\right)-\delta\right]v_1}{\left[\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+(1-\delta)^2u_2^2}\\ &\leq -\frac{(1-\delta)(1+u_2^2)\left[\left(\frac{k+\gamma}{k}\right)-\delta\right]}{2\left[\left[\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+(1-\delta)^2u_2^2\right]}\\ &< 0.\end{aligned}$$

Therefore, the function $\varphi(u,v)$ satisfies the conditions in Lemma 2.2. Thus, we have $\Re\{h(z)\}>0$, that is, $f\in\Sigma S_{\alpha,\beta,\eta}^{*,\gamma,k,m}$. By using the similar arguments to those details above with (17) instead of (16), we can see that the conditions of Lemma 2.2 are satisfied for the second part in (22) with $\mathcal{D}=\mathbb{C}\backslash\left\{1+\frac{\alpha+\beta-1}{1-\delta}\right\}\times\mathbb{C}$.
We can prove (23) by using the similar arguments to those detailed above with (18) instead of (16) with $\mathbf{D}=\mathbb{C}\backslash\left\{1+\frac{1/\eta}{1-\delta}\right\}\times\mathbb{C}$, so we omitted the proof of (23). Therefore, the proof of Theorem 3.1 is completed. □

Theorem 3.2. *If $f(z)\in\Sigma$, then*

$$\Sigma C_{\alpha,\beta,\eta}^{\gamma+1,k,m}\subset\Sigma C_{\alpha,\beta,\eta}^{\gamma,k,m}\subset\Sigma C_{\alpha,\beta+1,\eta}^{\gamma,k,m}, \tag{28}$$

and

$$\Sigma C_{\alpha,\beta,\eta}^{\gamma,k,m+1}\subset\Sigma C_{\alpha,\beta,\eta}^{\gamma,k,m}. \tag{29}$$

Proof. To prove (28) applying (19) and using Theorem 3.1, we observe that

$$f(z)\in\Sigma C_{\alpha,\beta,\eta}^{\gamma+1,k,m} \iff -zf'(z)\in\Sigma S_{\alpha,\beta,\eta}^{*,\gamma+1,k,m}$$

$$\Longrightarrow -zf'(z)\in\Sigma S_{\alpha,\beta,\eta}^{*,\gamma,k,m} \iff f(z)\in\Sigma C_{\alpha,\beta,\eta}^{\gamma,k,m}.$$

Also

$$f(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$$

$$\Longrightarrow -zf'(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta+1,\eta} \iff f(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta+1,\eta}.$$

By the same manner we can prove (29) which evidently completes Theorem 3.2. □

Theorem 3.3. *If $f(z) \in \Sigma$, then*

$$\Sigma K^{\gamma+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta} \subset \Sigma K^{\gamma,k,m}_{\alpha,\beta+1,\eta}, \tag{30}$$

and

$$\Sigma K^{\gamma,k,m+1}_{\alpha,\beta,\eta} \subset \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}. \tag{31}$$

Proof. To prove the first inclusion, let $f(z) \in \Sigma K^{\gamma+1,k,m}_{\alpha,\beta,\eta}$. Then, there exists a function $g(z) \in \Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}$ such that

$$\Re\left(\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)}\right) < -\sigma.$$

Let

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = -\sigma - (1-\sigma)h(z), \tag{32}$$

where $h(z)$ is given by (21). Using (16), we have

$$\begin{aligned}\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} &= \frac{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}(zf'(z))}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} \\ &= \frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)' + \left(\frac{k+\gamma}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))}{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)\right)' + \left(\frac{k+\gamma}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} \\ &= \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \left(\frac{k+\gamma}{k}\right)\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)}}{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \left(\frac{k+\gamma}{k}\right)}.\end{aligned}$$

Since $g(z) \in \Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, from Theorem 3.1, we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = -\delta - (1-\delta)\,\chi(z), \tag{33}$$

where $\chi(z) = g_1(x,y) + ig_2(x,y)$ and $\Re\{\chi(z)\} = g_1(x,y) > 0$ in $\mathbb{U}$. Then, by using (32), we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} = \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} - \left(\frac{k+\gamma}{k}\right)[\sigma + (1-\sigma)h(z)]}{-\delta - (1-\delta)\chi(z) + \left(\frac{k+\gamma}{k}\right)}. \tag{34}$$

Differentiating (32) with respect to z, we have

$$\frac{z\left(z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z))'\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}g(z)} = -(1-\sigma)zh'(z) + [\delta + (1-\delta)\,\chi(z)]\,[\sigma + (1-\sigma)h(z)]\,. \tag{35}$$

By substituting (35) into (34), we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}g(z)} + \sigma = -\left\{(1-\sigma)h(z) - \frac{(1-\sigma)zh'(z)}{(1-\delta)\,\chi(z) + \delta - \left(\frac{k+\gamma}{k}\right)}\right\}.$$

Let

$$\varphi\,(u,v) = (1-\sigma)u - \frac{(1-\sigma)v}{(1-\delta)\chi(z) + \delta - \left(\frac{k+\gamma}{k}\right)},$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = v = v_1 + iv_2$. Then

i) $\varphi(u,v)$ is continuous in $\mathring{D} = \mathbb{C}\backslash D^* \times \mathbb{C}$, where

$$D^* = \{z : z \in \mathbb{C} \text{ and } \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\gamma}{k(1-\delta)}\},$$

ii) $(1,0) \in D$ and $\Re\{\varphi(1,0)\} = (1-\sigma)$,

iii) $\Re\{\varphi(iu_2,v_1)\} \leq 0$ for all $(iu_2,v_1) \in D$ and such that $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}
\Re\{\varphi(iu_2,v_1)\} &= \Re\left\{\frac{-(1-\sigma)v_1}{(1-\delta)\chi(z) + \delta - \left(\frac{k+\gamma}{k}\right)}\right\} \\
&= \frac{(1-\sigma)\left[\left(\frac{k+\gamma}{k}\right) - \delta - (1-\delta)\,g_1(x,y)\right]v_1}{\left[(1-\delta)\,g_1(x,y) + \delta - \left(\frac{k+\gamma}{k}\right)\right]^2 + [(1-\delta)g_2(x,y)]^2} \\
&\leq -\frac{(1-\sigma)(1+u_2^2)\left[\left(\frac{k+\gamma}{k}\right) - \delta - (1-\delta)\,g_1(x,y)\right]}{2\left\{\left[(1-\delta)\,g_1(x,y) + \delta - \left(\frac{k+\gamma}{k}\right)\right]^2 + [(1-\delta)g_2(x,y)]^2\right\}} \\
&< 0.
\end{aligned}$$

Therefore, the function $\varphi\,(u,v)$ satisfies the conditions of Lemma 2.2. Thus we have $\Re\{h(z)\} > 0$, that is, $f \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$. By using the similar arguments to those details above with (17) instead of (16), we can see that the conditions of Lemma 2.2 are satisfied for the second part of (30) with $Q = \mathbb{C}\backslash Q^* \times \mathbb{C}$, where $Q^* = \{z : z \in \mathbb{C}$ *and* $\Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\alpha+\beta-1}{1-\delta}\}$.

We can prove (31) by using the similar arguments to these precedent details with (18) instead of (16) with $B = \mathbb{C}\backslash B^* \times \mathbb{C}$, where $B^* = \{z : z \in \mathbb{C}$ *and* $\Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{1/\eta}{1-\delta}\}$, so we omitted the proof of (31). Therefore, the proof of Theorem 3.3 is completed. □

Theorem 3.4. *If* $f(z) \in \Sigma$, *then*

$$\Sigma K^{*,\gamma+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta} \subset \Sigma K^{*,\gamma,k,m}_{\alpha,\beta+1,\eta},$$

and

$$\Sigma K^{*,m+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$$

Proof. Just as we derived Theorem 3.2 as a consequence of Theorem 3.1 by using the equivalence (19), we can also prove Theorem 3.4 by using Theorem 3.3 in conjunction with the equivalence (20).

Let F_μ be the integral operator

$$F_\mu(f)(z) = \frac{\mu}{z^\mu}\int_0^z t^\mu f(t)dt = (z^{-1} + \sum_{k=0}^{\infty} \frac{\mu}{\mu+k+1} z^k) * f(z)$$

$(f \in \Sigma; \mu > 0; z \in \mathbb{U}^*)$.

$$z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z))' = \mu \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) - (\mu+1)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z) \ (\mu > 0)\,. \tag{36}$$

In the following theorems we will get inclusion properties which associate the operator F_μ with the classes $\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma C^{gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$ and $\Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$. □

Theorem 3.5. *If $f(z) \in \Sigma$, $\mu > 0$ and $f(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, then $F_\mu(f)(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.*

Proof. Let $f \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$ and set

$$\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)} = -\frac{1+(1-2\delta)w(z)}{1-w(z)}\,, \tag{37}$$

where $w(0) = 0$. Using (36) in (37), we obtain

$$\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)} = \frac{\mu - (\mu+2-2\delta)w(z)}{\mu\,[1-w(z)]}. \tag{38}$$

Differentiating (38) logarithmically with respect to z, we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} = -\frac{1+(1-2\delta)w(z)}{1-w(z)} + \frac{zw'(z)}{1-w(z)} - \frac{(\mu+2-2\delta)zw'(z)}{\mu-(\mu+2-2\delta)w(z)}\,, \tag{39}$$

so that

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} + \delta = \frac{(1-\delta)(1+w(z))}{1-w(z)} + \frac{zw'(z)}{1-w(z)} - \frac{(\mu+2-2\delta)zw'(z)}{\mu-(\mu+2-2\delta)w(z)}. \tag{40}$$

Let $max_{|z|\le|z_0|}|w(z)| = |w(z_0))| = 1,\ z_0 \in \mathbb{U}$ and applying Lemma 2.3, we have

$$z_0 w'(z_0) = \zeta w(z_0)\ \zeta \ge 1.$$

If we set $w(z_0) = e^{i\theta},\ \theta \in \mathbb{R}$ in (40) and observe that

$$\Re\left\{\frac{(1-\delta)(1+w(z_0))}{1-w(z_0)}\right\} = 0,$$

then, we have

$$\begin{aligned}
\Re\left\{\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z_0))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z_0)}+\delta\right\} &= \Re\left\{\frac{z_0w'(z_0)}{1-w(z_0)}-\frac{(\mu+2-2\delta)z_0w'(z_0)}{\mu-(\mu+2-2\delta)w(z_0)}\right\}\\
&= \Re\left\{-\frac{2(1-\delta)\zeta e^{i\theta}}{(1-e^{i\theta})\left(\mu-(\mu+2-2\delta)e^{i\theta}\right)}\right\}\\
&= \frac{2\zeta(1-\delta)(\mu+1-\delta)}{\mu^2-2\mu(\mu+2-2\delta)\cos\theta+(\mu+2-2\delta)^2}\\
&\geq 0,
\end{aligned}$$

which obviously contradicts the hypothesis $f(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$. Consequently, we can deduce that $|w(z)|<1$ for any $z\in\mathbb{U}$, which, in view of (37), proves the integral-preserving property asserted by Theorem 3.5. □

Theorem 3.6. *If $f(z)\in\Sigma,\ \mu>0$ and $f(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$, then $F_\mu(f)(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$.*

Proof. Applying (19) and using Theorem 3.5, we observe that

$$\begin{aligned}
f(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} &\iff -zf'(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \implies F_\mu\left(-zf'(z)\right)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}\\
\iff\quad -z\left(F_\mu f(z)\right)' &\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \implies F_\mu(f)(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}
\end{aligned}$$

which evidently proves Theorem 3.6.

□

Theorem 3.7. *If $f(z)\in\Sigma,\ \mu>0$ and $f(z)\in\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$, then $F_\mu(f)(z)\in\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$.*

Proof. Let $f(z)\in\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$. Then, there exists a function $g(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$ such that

$$\Re\left(\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}g(z)}\right)<-\sigma.$$

Let

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(f)(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(g)(z)}=-\sigma-(1-\sigma)h(z), \tag{41}$$

where $h(z)$ is given by (21). Using (36), we have

$$\begin{aligned}
\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}g(z)} &= -\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}\left(-zf'(z)\right)}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}g(z)}\\
&= -\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(-zf'(z)\right)\right)'+(\mu+1)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(-zf'(z)\right)}{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(g\right)(z)\right)'+(\mu+1)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(g\right)(z)}\\
&= -\frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(-zf'(z)\right)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(g)(z)}+(\mu+1)\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu\left(-zf'(z)\right)}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(g)(z)}}{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(g)(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}F_\mu(g)(z)}+\mu+1}.
\end{aligned}$$

Since $g(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, then from Theorem 3.5, we have $F_\mu(f)(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, we set

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z)} = -\delta - (1-\delta)\,\chi(z), \tag{42}$$

where $\chi(z) = g_1(x,y) + ig_2(x,y)$. Then

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu\left(-zf'(z)\right)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} + (\mu+1)[\sigma+(1-\sigma)h(z)]}{\delta+(1-\delta)\chi(z)-\mu-1}. \tag{43}$$

Differentiating (41) with respect to z, we have

$$\frac{z\left(z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu f(z))'\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z)} = -(1-\sigma)zh'(z) + [\delta + (1-\delta)\,\chi(z)]\,[\sigma + (1-\sigma)h(z)]\,. \tag{44}$$

By substituting (44) into (43), we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \sigma = -\left\{(1-\sigma)h(z) - \frac{(1-\sigma)zh'(z)}{(1-\delta)\,\chi(z) + \delta - \mu - 1}\right\}.$$

Let

$$\varphi(u,v) = (1-\sigma)u - \frac{(1-\sigma)v}{(1-\delta)\chi(z) + \delta - \mu - 1},$$

with $h(z) = u = u_1 + iu_2$, $zh'(z) = v = v_1 + iv_2$. We can see that the conditions of Lemma 2.2 are satisfied with $Q = \mathbb{C}\backslash Q^* \times \mathbb{C}$, where $Q^* = \{z : z \in \mathbb{C} \text{ and } \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\mu}{1-\delta}\}$. The remainder of our proof of Theorem 3.7 is similar to that of Theorem 3.3, so we choose to omit the analogous details involved. This completes the proof of Theorem 3.7.

□

Theorem 3.8. *If $f(z) \in \Sigma$, $\mu > 0$ and $f(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, then $F_\mu(f)(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.*

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