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Quasi-Uniform and Uniform Convergence of Riemann and Riemann-Type Integrable Functions With Values in a Banach Space

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Abstract. In this article, we study quasi-uniform and uniform convergence of nets and sequences of different types of functions defined on a topological space, in particular, on a closed bounded interval of $\mathbb{R}$, with values in a metric space and in some cases in a Banach space. We show that boundedness and continuity are inherited to the quasi-uniform limit, and integrability is inherited to the uniform limit of a net of functions. Given a sequence of functions, we construct functions with values in a sequence space and consequently we infer some important properties of such functions. Finally, we study convergence of partially equi-regulated* nets of functions which is shown to be a generalized notion of exhaustiveness.

1. Introduction

The object of the present paper is to study quasi-uniform and uniform convergence of nets and sequences of different types of functions defined on a topological space or on a closed bounded interval of $\mathbb{R}$ with values in a metric space or in a Banach space. It is known that pointwise convergence is too weak to inherit many properties, such as, boundedness, continuity, differentiability, integrability etc. to the limit function. This lacuna of pointwise convergence is met either by imposing some restrictions on the net or sequence, or by strengthening the nature of convergence. Quasi-uniform convergence is stronger than pointwise convergence while uniform convergence is stronger than quasi-uniform convergence.

We begin with the study of quasi-uniform convergence of a net of functions and show that boundedness and continuity are inherited to the limit function under such convergence. The result is seen to hold for Darboux integrability under an additional condition which is automatically satisfied in case of a sequence of Darboux integrable functions.

Next we consider uniform convergence of nets and sequences of Riemann and Riemann-type integrable functions. It is shown that uniform limit of a net of Riemann integrable functions is Riemann integrable. Similar results are shown to be true for Darboux, Riemann-Dunford and Riemann-Pettis integrable functions. We also show that uniformly convergent sequence of Riemann (resp. Darboux) integrable functions is equi-Riemann (resp. equi-Darboux) integrable.

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Then with the help of suitable sequences of functions, we construct functions with values in l_∞, c, l_1, and study how integrability and some related properties of the derived function are connected with those of the functions constituting the initial sequence.

We end the article with the study of convergence of nets and sequences in localized senses. We introduce the notion of partially equi-regulated* net of functions and investigate when such a net is uniformly convergent about a point. We show that partially equi-regulated*ness is a generalized notion of exhaustiveness.

2. Preliminaries

Throughout the paper, S stands for a topological space, (Y, d) for a metric space and X for a real Banach space with dual X^* (any other normed linear space or Banach space appeared in this article will also be assumed to be a real normed linear space or a real Banach space). The closed unit ball of X and X^* will be denoted by B_X and B_{X^*} respectively.

Throughout $[a, b]$ stands for a closed bounded interval of $\mathbb{R}$, Σ_L for the σ-algebra of the Lebesgue measurable subsets of $[a, b]$ and λ for the Lebesgue measure on Σ_L so that $([a, b], \Sigma_L, \lambda)$ becomes a complete finite measure space. For details on Bochner and Pettis integrable functions on a finite measure space with values in a Banach space, we refer to [3] and [14].

A partition of the interval $[a, b]$ is a finite set of points $\{t_i : 0 \le i \le n\}$ in $[a, b]$ that satisfy $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. The points $\{t_i : 0 \le i \le n\}$ are called the points of the partition and the intervals $\{[t_{i-1}, t_i] : 1 \le i \le n\}$ are called the intervals of the partition. A tagged partition of $[a, b]$ is a partition $\{t_i : 0 \le i \le n\}$ of $[a, b]$ together with a set of points $\{s_i : 1 \le i \le n\}$ that satisfy $s_i \in [t_{i-1}, t_i]$ for each i and it is denoted as $\{(s_i, [t_{i-1}, t_i]) : 1 \le i \le n\}$. The points $\{s_i : 1 \le i \le n\}$ are called the tags of the partition. The norm of a partition $\mathcal{P} = \{t_i : 0 \le i \le n\}$, denoted as $|\mathcal{P}|$, is defined as $|\mathcal{P}| = \max\{t_i - t_{i-1} : 1 \le i \le n\}$. For any $\delta > 0$, we say that a partition $\mathcal{P}$ of $[a, b]$ is δ-fine if $|\mathcal{P}| < \delta$. Finally, a partition $\mathcal{P}_1$ of $[a, b]$ is said to refine a partition $\mathcal{P}_2$ of $[a, b]$ if every point of $\mathcal{P}_2$ is a point of $\mathcal{P}_1$.

Let $f \in X^{[a,b]}$ and let $E \subset [a, b]$. Then the oscillation of f on E is defined as $\omega(f, E) = \sup\{\|f(u) - f(v)\| : u, v \in E\}$. For any tagged partition $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \le i \le n\}$ of $[a, b]$, $f(\mathcal{P})$ will denote the Riemann sum $\sum_{i=1}^{n} f(s_i)(t_i - t_{i-1})$, and for any partition $\mathcal{P} = \{t_i : 0 \le i \le n\}$ of $[a, b]$, $\omega(f, \mathcal{P})$ will denote the oscillatory sum $\sum_{i=1}^{n} \omega(f, [t_{i-1}, t_i])(t_i - t_{i-1})$.

For definitions and some standard results on Riemann, Darboux, scalarly Riemann or Riemann-Dunford and Riemann-Pettis integrable functions defined on a closed bounded interval of $\mathbb{R}$ with values in a Banach space, we refer to [6] and [16].

The collections of all Riemann, Darboux, Riemann-Dunford and Riemann-Pettis integrable functions defined on $[a, b]$ with values in X will be denoted by $R([a, b], X)$, $D([a, b], X)$, $RD([a, b], X)$ and $RP([a, b], X)$ respectively. If, in particular, $X = \mathbb{R}$, we write $R([a, b])$ for $R([a, b], X)$ and similarly for other such collections. It is well known that $D([a, b], X) \subset R([a, b], X) \subset RP([a, b], X) \subset RD([a, b], X) \subset l^\infty([a, b], X)$, $l^\infty([a, b], X)$ being the set of all bounded functions in $X^{[a,b]}$. Further, for a finite-dimensional space X, we have $D([a, b], X) = R([a, b], X) = RP([a, b], X) = RD([a, b], X)$.

If $f \in X^{[a,b]}$ is integrable on $[a, b]$ in any of the above senses, then it is so on every closed subinterval of $[a, b]$. If $f \in R([a, b], X)$, then the function $F \in X^{[a,b]}$, defined by $F(t) = R\text{-}\int_a^t f(t)dt$, $t \in [a, b]$ is called the indefinite Riemann integral of f. Indefinite integrals of other types of integrable functions in $X^{[a,b]}$ mentioned above are defined similarly.

Let $f \in R([a, b], X)$. Then the Alexiewicz norm of f, denoted as $\|f\|_A$, is defined as $\|f\|_A = \sup\left\{\left\|R\text{-}\int_a^t f(t)dt\right\| : a \le t \le b\right\}$, that is, $\|f\|_A = \|F\|_\infty$, F being the indefinite integral of f. Hence it follows that convergence of a net in $R([a, b], X)$ in Alexiewicz norm is equivalent to the uniform convergence of the net of indefinite integrals of its members. We define

$$\|f\|' = \sup\left\{\left\|R\text{-}\int_c^d f(t)dt\right\| : [c,d] \text{ is a closed subinterval of } [a,b]\right\}.$$

It can be shown that $\|\cdot\|'$ is a norm on $R([a,b],X)$. Also it can be verified that $\|f\|_A \leq \|f\|' \leq 2\|f\|_A$ which implies that the norms $\|\cdot\|_A$ and $\|\cdot\|'$ are equivalent in $R([a,b],X)$.

Also we recall the following definitions from [12]:

Let $\mathcal{P}$ be any partition of $[a,b]$. Then for any $f \in X^{[a,b]}$,

$$\theta_{\mathcal{P}}(f) = \sup\{\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| : \mathcal{P}_1, \mathcal{P}_2 \text{ are tagged partitions of } [a,b] \text{ that refine } \mathcal{P}\},$$

$$\omega_{\mathcal{P}}(f) = \sup\{\omega(f,\mathcal{P}') : \mathcal{P}' \text{ is a partition of } [a,b] \text{ that refines } \mathcal{P}\},$$

and for any $\mathcal{F} \subset X^{[a,b]}$, $\theta_{\mathcal{P}}(\mathcal{F}) = \sup_{f\in\mathcal{F}} \theta_{\mathcal{P}}(f)$ and $\omega_{\mathcal{P}}(\mathcal{F}) = \sup_{f\in\mathcal{F}} \omega_{\mathcal{P}}(f)$.

It follows from [6, p. 925, Theorem 5 ((1) $\Longleftrightarrow$ (3))] (resp. [6, p. 933, Definition 17 (b)]) that a function $f \in X^{[a,b]}$ is Riemann integrable (resp. Darboux integrable) on $[a,b]$ if and only if for each $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\theta_{\mathcal{P}}(f) < \varepsilon$ (resp. $\omega_{\mathcal{P}}(f) < \varepsilon$).

For definition of equi-Riemann integrable collection of functions, we refer to [12]. We recall that a collection of functions $\mathcal{F}$, in $X^{[a,b]}$, is equi-Riemann integrable on $[a,b]$ if and only if for each $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\theta_{\mathcal{P}}(\mathcal{F}) < \varepsilon$ [12, p. 310, Theorem 3.10 ((a) $\Longleftrightarrow$ (b))].

Also we recall the following definitions from [12]:

A collection of functions, $\mathcal{F}$, in $X^{[a,b]}$, is said to be

(a) equi-Darboux integrable on $[a,b]$ if for each $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\omega_{\mathcal{P}}(\mathcal{F}) < \varepsilon$ [12, p. 312, Definition 3.12],

(b) equi-Riemann-Dunford integrable on $[a,b]$ if for each $x^* \in X^*$, $\{x^*f : f \in \mathcal{F}\}$ is equi-Riemann integrable thereon [12, p. 316, Definition 3.26],

(c) equi-Riemann-Pettis integrable on $[a,b]$ if it is equi-Riemann-Dunford integrable thereon and each $f \in \mathcal{F}$ is Pettis integrable on $([a,b], \Sigma_L, \lambda)$ [12, p. 316, Definition 3.26].

We would like to mention here that every equi-Riemann integrable collection in $X^{[a,b]}$ is contained in $R([a,b],X)$ and similarly for other equi-integrable collections of functions.

For some important properties of different types of equi-integrable collections of functions as mentioned above, we refer to [12].

3. Main Results

Let us recall the following definition of quasi-uniformly convergent nets of functions from [5, p. 221]:

Definition 3.1. *A net $\{f_\iota\}_{\iota\in I}$ in Y^S is said to converge quasi-uniformly to $f \in Y^S$ at a point $s \in S$ if for each $\varepsilon > 0$, there exists an $\iota_0 \in I$ such that for each $\iota \in I$ with $\iota_0 \leq \iota$, there exists a neighbourhood U of s with the property*

$$d(f_\iota(t), f(t)) < \varepsilon$$

for all $t \in U$.

As usual, a net in Y^S is said to converge quasi-uniformly to $f \in Y^S$ on S if it converges quasi-uniformly to f at each point of S.

Clearly a uniformly convergent net of functions is quasi-uniformly convergent and a quasi-uniformly convergent net of functions is pointwise convergent to the same limit.

It is well known that the pointwise limit of a sequence of continuous functions is not necessarily continuous. On the other hand, the uniform limit of a net of continuous functions is continuous. The converse of this result is not necessarily true. The following lemma shows that the result as well as its converse are true for quasi-uniform convergence of a net of functions defined on a topological space with values in a metric space. The result can be found in [15, p. 32, Theorem 1] and [1, p. 74, Theorem 4.1] in a more generalized form. However for the sake of completeness we present a proof here.

Lemma 3.2. *Let* $\{f_\iota\}_{\iota\in I}$ *be a net in* Y^S *and let* f_ι *be continuous at* $s \in S$ *for each* $\iota \in I$. *Then* $\{f_\iota\}_{\iota\in I}$ *converges quasi-uniformly to* $f \in Y^S$ *at* s *if and only if* $\{f_\iota(s)\}_{\iota\in I}$ *converges to* $f(s)$ *and* f *is continuous at* s.

Proof. Let $\{f_\iota\}_{\iota\in I}$ converge quasi-uniformly to f at s. That $\{f_\iota(s)\}_{\iota\in I}$ converges to $f(s)$ follows from the definition and the fact that s belongs to each neighbourhood of itself.

Let $\varepsilon > 0$. Then there exist an $\iota_0 \in I$ and a neighbourhood U_0 of s with the property that

$$d(f_{\iota_0}(t), f(t)) < \frac{\varepsilon}{3}$$

for all $t \in U_0$.

In particular,

$$d(f_{\iota_0}(s), f(s)) < \frac{\varepsilon}{3}.$$

Since f_{ι_0} is continuous at s, there exists a neighbourhood U_1 of s such that

$$d(f_{\iota_0}(t), f_{\iota_0}(s)) < \frac{\varepsilon}{3}$$

for all $t \in U_1$.

Let $U = U_0 \cap U_1$. Then U is a neighbourhood of s. Now for all $t \in U$, we have

$$\begin{aligned} d(f(t), f(s)) &\leq d(f(t), f_{\iota_0}(t)) + d(f_{\iota_0}(t), f_{\iota_0}(s)) + d(f_{\iota_0}(s), f(s)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

which implies that f is continuous at s.

Conversely, let $\{f_\iota(s)\}_{\iota\in I}$ converge to $f(s)$ and let f be continuous at s. Let $\varepsilon > 0$. Then there exists an $\iota_0 \in I$ such that

$$d(f_\iota(s), f(s)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_0 \leq \iota$.

Let $\iota \in I$ be such that $\iota_0 \leq \iota$. Since both f and f_ι are continuous at s, there exists a neighbourhood V of s such that

$$d(f(t), f(s)) < \frac{\varepsilon}{3}$$

and

$$d(f_\iota(t), f_\iota(s)) < \frac{\varepsilon}{3}$$

for all $t \in V$.

Hence for all $t \in V$, we have

$$\begin{aligned} d(f_\iota(t), f(t)) &\leq d(f_\iota(t), f_\iota(s)) + d(f_\iota(s), f(s)) + d(f(s), f(t)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

which implies that $\{f_\iota\}_{\iota\in I}$ converges quasi-uniformly to f at s. □

Lemma 3.2 yields that for a net or a sequence of functions, continuity is not preserved by any convergence weaker than quasi-uniform convergence.

For $f \in Y^S$, C_f (resp. D_f) denote, as usual, the set of points of continuity (resp. discontinuity) of f in S. The following result follows trivially from Lemma 3.2:

Corollary 3.3. *Let $\{f_\iota\}_{\iota\in I}$ be a net in Y^S and let $f \in Y^S$. Let $\{f_\iota\}_{\iota\in I}$ converge quasi-uniformly to f on S. Then $\bigcap_{\iota\in I} C_{f_\iota} \subseteq C_f$ and hence $D_f \subseteq \bigcup_{\iota\in I} D_{f_\iota}$.*

A net $\{f_\iota\}_{\iota\in I}$ in X^S is said to converge weakly quasi-uniformly to $f \in X^S$ at $s \in S$ if for each $x^* \in X^*$, $\{x^* f_\iota\}_{\iota\in I}$ converges quasi-uniformly to $x^* f$ at s.

Corollary 3.4. *Let $\{f_\iota\}_{\iota\in I}$ be a net in X^S and let f_ι be weakly continuous at $s \in S$ for each $\iota \in I$. Then $\{f_\iota\}_{\iota\in I}$ converges weakly quasi-uniformly to f at s if and only if $\{f_\iota(s)\}_{\iota\in I}$ converges weakly to $f(s)$ and f is weakly continuous at s.*

Proof. Follows by an application of Lemma 3.2 to the net $\{x^* f_\iota\}_{\iota\in I}$ for each $x^* \in X^*$. □

A function $f \in Y^S$ is said to be locally bounded at a point in S if it is bounded in some neighbourhood of that point; if f is locally bounded at every point in S, then it is called locally bounded on S [11, p. 251, Definition 1].

It is evident that if a function f is bounded on S, then it is locally bounded on S. But the converse is not true, even if S is locally compact [11, p. 255].

However we have the following result which is an easy generalization of [11, p. 255, Theorem 5]:

Lemma 3.5. *Let $f \in Y^S$ be locally bounded on S. If S is compact, then f is bounded on S.*

Theorem 3.6. *Let $\{f_\iota\}_{\iota\in I}$ be a net in Y^S such that each f_ι is locally bounded at $s \in S$. If $\{f_\iota\}_{\iota\in I}$ converges quasi-uniformly to $f \in Y^S$ at s, then f is locally bounded at s.*

Proof. Let $\varepsilon = 1$. Since $\{f_\iota\}_{\iota\in I}$ converges quasi-uniformly to f at s, there exist an $\iota_s \in I$ and a neighbourhood U_1 of s such that

$$d(f_{\iota_s}(t), f(t)) < 1$$

for all $t \in U_1$.

Now f_{ι_s} is locally bounded at s. So there exist an $M_{\iota_s} > 0$ and a neighbourhood U_2 of s such that

$$d(f_{\iota_s}(t_1), f_{\iota_s}(t_2)) \leq M_{\iota_s}$$

for all $t_1, t_2 \in U_2$.

Let $U = U_1 \cap U_2$. Then U is a neighbourhood of s, and for any $t_1, t_2 \in U$, we have

$$\begin{aligned} & d(f(t_1), f(t_2)) \\ & \leq d(f(t_1), f_{\iota_s}(t_1)) + d(f_{\iota_s}(t_1), f_{\iota_s}(t_2)) + d(f_{\iota_s}(t_2), f(t_2)) \\ & < 1 + M_{\iota_s} + 1 = 2 + M_{\iota_s} \end{aligned}$$

which shows that f is locally bounded at s. □

Theorem 3.7. *Let $\{f_\iota\}_{\iota\in I}$ be a net of bounded functions in $X^{[a,b]}$ which converges quasi-uniformly to $f \in X^{[a,b]}$ on $[a, b]$ and let $\lambda\left(\bigcup_{\iota\in I} D_{f_\iota}\right) = 0$. Then $f \in D([a, b], X)$.*

Proof. Since $[a, b]$ is compact, it follows from Theorem 3.6 and Lemma 3.5 that f is bounded on $[a, b]$. Hence the result follows from Corollary 3.3 and [6, p. 933, Theorem 18]. □

For a sequence of functions, we have the following result which obviously follows from [6, p. 933, Theorem 18] and Theorem 3.7:

Corollary 3.8. *Let $\{f_n\}$ be a sequence in $D([a,b],X)$ and let $f \in X^{[a,b]}$. If $\{f_n\}$ converges quasi-uniformly to f on $[a,b]$, then $f \in D([a,b],X)$.*

Corollary 3.9. *Let $\{f_n\}$ be a sequence in $RD([a,b],X)$ which converges weakly quasi-uniformly to $f \in X^{[a,b]}$ on $[a,b]$. Then $f \in RD([a,b],X)$.*

If, moreover, each $f_n \in RP([a,b],X)$ and $\{x^* f_n : x^* \in B_{X^*}, n \in \mathbb{N}\}$ is uniformly integrable, then $f \in RP([a,b],X)$ and $\{RP\text{-}\int_c^d f_n dt\}$ converges weakly to $RP\text{-}\int_c^d f dt$ uniformly with respect to closed subintervals $[c,d]$ of $[a,b]$.

Proof. First part follows by an application of Corollary 3.8 to the sequence $\{x^* f_n\}$ for each $x^* \in X^*$.

For the second part, it follows from hypothesis that $\{x^* f_n\}$ converges to $x^* f$ pointwise on $[a,b]$ for each $x^* \in X^*$. Hence it follows from [14, p. 550, Theorem 5.2] that f is Pettis integrable and hence $f \in RP([a,b],X)$.

Last part follows by an application of Vitali's convergence theorem for Lebesgue integral to $\{x^* f_n\}$ for each $x^* \in X^*$. □

Let us consider the following example due to Doboš and Šalát [4, p. 222]:

Example 3.10. *For each n, let*

$$f_n(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{n+1} \\ \{2n(n+1)\}^2 (t - \frac{1}{n+1}) & \text{for } \frac{1}{n+1} < t < \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1}) \\ 2n(n+1) & \text{for } t = \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1}) \\ \{2n(n+1)\}^2 (\frac{1}{n} - t) & \text{for } \frac{1}{2}(\frac{1}{n} + \frac{1}{n+1}) < t < \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le t \le 1. \end{cases}$$

Also let

$$f(t) = 0, \ for \ \ 0 \le t \le 1.$$

Then it can be verified that each f_n and f are continuous and hence Darboux integrable on $[0,1]$ and the sequence $\{f_n\}$ converges pointwise to f on $[0,1]$. This implies by Lemma 3.2 that $\{f_n\}$ converges quasi-uniformly to the function f on $[0,1]$. But $\int_0^1 f_n(t)dt$ does not converge to $\int_0^1 f(t)dt$.

From this example, we observe in view of [13, p. 223–224, Corollary 3.36 (b)] and Vitali's Convergence Theorem for Lebesgue integrable functions that a quasi-uniformly convergent sequence of real-valued Riemann integrable functions defined on a closed interval need neither be equi-Riemann integrable nor be uniformly integrable.

It should be noted that pointwise limit of an equi-Riemann integrable sequence of functions is Riemann integrable [13, p. 223–224, Corollary 3.36 (b)]. The above example shows that for the pointwise limit of a sequence of Riemann integrable functions to be Riemann integrable, it is not necessary for the sequence to be equi-Riemann integrable.

The following result follows in a straightforward way:

Lemma 3.11. *Let $\{f_\iota\}_{\iota \in I}$ be a net in $X^{[a,b]}$ which converges uniformly to $f \in X^{[a,b]}$ on $[a,b]$. Then*

(a) $\{f_\iota(\mathcal{P})\}_{\iota \in I}$ converges to $f(\mathcal{P})$ uniformly with respect to tagged partitions $\mathcal{P}$ of $[a,b]$,

(b) $\{\omega_{\mathcal{P}}(f_\iota)\}_{\iota \in I}$ converges to $\omega_{\mathcal{P}}(f)$ uniformly with respect to partitions $\mathcal{P}$ of $[a,b]$.

A net $\{f_\iota\}_{\iota \in I}$ in $X^{[a,b]}$ is said to be partially equi-Riemann (resp. partially equi-Darboux) integrable on $[a,b]$ if for any $\varepsilon > 0$, there exist an $\iota_0 \in I$ and a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that $\theta_{\mathcal{P}_\varepsilon}(f_\iota) < \varepsilon$ (resp. $\omega_{\mathcal{P}_\varepsilon}(f_\iota) < \varepsilon$) for all $\iota \in I$ with $\iota_0 \le \iota$.

As $\theta_{\mathcal{P}}(\mathcal{F}) \le 2\omega_{\mathcal{P}}(\mathcal{F})$ for any $\mathcal{F} \subset X^{[a,b]}$ and for all partitions $\mathcal{P}$ of $[a,b]$ [12, p. 305, Theorem 3.3 (*a*)], it follows that a partially equi-Darboux integrable net of functions is partially equi-Riemann integrable.

We recall the following definitions from [13, p. 209, Definition 3.6]:

A net $\{f_\iota\}_{\iota\in I}$ in $X^{[a,b]}$ is said to be Riemann δ-Cauchy (resp. Riemann Δ-Cauchy) on $[a,b]$ if for each $\varepsilon > 0$, there exist a $\delta > 0$ (resp. a partition $\mathcal{P}_\varepsilon$ of $[a,b]$) and an $\iota_0 \in I$ such that

$$\|f_\iota(\mathcal{P}) - f_\kappa(\mathcal{P})\| < \varepsilon$$

for all $\iota, \kappa \in I$ with $\iota_0 \leq \iota, \kappa$ and for all δ-fine tagged partitions $\mathcal{P}$ of $[a,b]$ (resp. for all tagged partitions $\mathcal{P}$ of $[a,b]$ that refine $\mathcal{P}_\varepsilon$).

Theorem 3.12. *Let $\{f_\iota\}_{\iota\in I}$ be a net in $X^{[a,b]}$ which converges uniformly to $f \in X^{[a,b]}$ on $[a,b]$ and let $[c,d]$ be an arbitrary closed subinterval of $[a,b]$. Then the following statements hold good:*

(a) $\{f_\iota\}_{\iota\in I}$ is Riemann δ-Cauchy on $[c,d]$.

(b) If $\{f_\iota\}_{\iota\in I} \subset R([a,b],X)$ (resp. $D([a,b],X)$), then $f \in R([c,d],X)$ (resp. $D([c,d],X)$), $\{f_\iota\}_{\iota\in I}$ is partially equi-Riemann (resp. partially equi-Darboux) integrable on $[c,d]$, and $\{f_\iota\}_{\iota\in I}$ converges to f in Alexiewicz norm, and hence $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[c,d]$, F_ι, F being the indefinite integrals of f_ι, f respectively.

Proof. (*a*) Follows from the fact that $\{f_\iota(\mathcal{P})\}_{\iota\in I}$ is uniformly convergent by part (*a*) of Lemma 3.11 and hence uniformly Cauchy with respect to tagged partitions $\mathcal{P}$ of $[c,d]$.

(*b*) From part (*a*), it follows that $\{f_\iota\}_{\iota\in I}$ is Riemann δ-Cauchy and hence Riemann Δ-Cauchy on $[c,d]$ [13, p. 216, Lemma 3.17 ((a) $\Longrightarrow$ (b))].

For the Riemann part, it follows from [13, p. 214, Theorem 3.14] that $\{f_\iota\}_{\iota\in I}$ is partially equi-Riemann integrable on $[c,d]$, and hence it follows from [13, p. 217, Theorem 3.20 ((b) $\Longrightarrow$ (d))] that $f \in R([c,d],X)$.

For the Darboux part, let $\varepsilon > 0$. Then by part (*b*) of Lemma 3.11, there exists an $\iota_0 \in I$ such that

$$|\omega_{\mathcal{P}}(f_\iota) - \omega_{\mathcal{P}}(f)| < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_0 \leq \iota$ and for all partitions $\mathcal{P}$ of $[c,d]$.

Since $f_{\iota_0} \in D([a,b],X)$, we have $f_{\iota_0} \in D([c,d],X)$, and so there exists a partition $\mathcal{P}$ of $[c,d]$ such that

$$\omega_{\mathcal{P}}(f_{\iota_0}) < \frac{\varepsilon}{3}.$$

Now we have

$$\omega_{\mathcal{P}}(f) \leq |\omega_{\mathcal{P}}(f_{\iota_0}) - \omega_{\mathcal{P}}(f)| + \omega_{\mathcal{P}}(f_{\iota_0}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$$

which implies that $f \in D([c,d],X)$.

Again for all $\iota \in I$ with $\iota_0 \leq \iota$, we have

$$\omega_{\mathcal{P}}(f_\iota) \leq |\omega_{\mathcal{P}}(f_\iota) - \omega_{\mathcal{P}}(f)| + \omega_{\mathcal{P}}(f) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

which implies that $\{f_\iota\}_{\iota\in I}$ is partially equi-Darboux integrable on $[c,d]$.

That $\{f_\iota\}_{\iota\in I}$ converges to f in Alexiewicz norm follows from [13, p. 217, Theorem 3.20 ((b) $\Longrightarrow$ (d))] in both the cases as $D([a,b],X) \subset R([a,b],X)$. □

Corollary 3.13. *Let $\{f_\iota\}_{\iota\in I}$ be a net in $RD([a,b],X)$ which converges weakly uniformly to $f \in X^{[a,b]}$ on $[a,b]$. Then $f \in RD([a,b],X)$.*

Proof. Follows by an application of part (*b*) of Theorem 3.12 to $\{x^* f_\iota\}_{\iota\in I}$ for each $x^* \in X^*$. □

Corollary 3.14. *If $\{f_\iota\}_{\iota\in I}$ is a net in $RP([a,b],X)$ which converges uniformly to $f \in X^{[a,b]}$ on $[a,b]$, then $f \in RP([a,b],X)$.*

Proof. It follows from Corollary 3.13 that $f \in RD([a,b],X)$. Also each f_ι is Pettis integrable. It is easy to verify that for any closed subinterval $[c,d]$ of $[a,b]$, $\left\{RP\text{-}\int_c^d f_\iota dt\right\}_{\iota\in I}$ is Cauchy in X and hence convergent, and that $\lim_\iota RP\text{-}\int_c^d f_\iota dt = RD\text{-}\int_c^d f dt$ which implies that $RD\text{-}\int_c^d f dt \in X$. Hence $f \in RP([a,b],X)$ [6, p. 943–944, Definition 28]. □

We observed in Example 3.10 that a quasi-uniformly convergent sequence of Riemann integrable functions is not necessarily equi-Riemann integrable. In contrast with this, we have the following result:

Corollary 3.15. *A uniformly convergent sequence in* $R([a,b],X)$ (*resp.* $D([a,b],X)$) *is equi-Riemann integrable* (*resp. equi-Darboux integrable*).

Proof. It is easy to check that the notions of equi-Riemann (resp. equi-Darboux) integrability and partial equi-Riemann (resp. partial equi-Darboux) integrability coincide for a sequence in $R([a,b],X)$ (resp. $D([a,b],X)$). Hence the results follow from part (b) of Theorem 3.12. □

From Corollary 3.15, it follows that a uniformly convergent sequence $\{f_n\}$ in $R([a,b],X)$ (resp. $D([a,b],X)$) is pointwise convergent as well as equi-Riemann (resp. equi-Darboux) integrable. Lee presented an example to show that the converse of this result is not true [10, p. 18, Example 5].

Corollary 3.16. *Let* $\{f_n\}$ *be a sequence in* $RP([a,b],X)$ *which converges weakly uniformly to* $f \in X^{[a,b]}$ *on* $[a,b]$. *Then* $f \in RP([a,b],X)$ *and* $\{RP\text{-}\int_c^d f_n dt\}$ *converges weakly to* $RP\text{-}\int_c^d f dt$ *uniformly with respect to closed subintervals* $[c,d]$ *of* $[a,b]$.

Proof. It follows by an application of Corollary 3.15 to $\{x^* f_n\}$ that $\{x^* f_n\}$ is equi-Riemann integrable for each $x^* \in X^*$ which implies that $\{f_n\}$ is equi-Riemann-Dunford integrable and hence equi-Riemann-Pettis integrable. So the result follows from [13, p. 221, Corollary 3.30]. □

Gordon extended the uniform convergence results to sequences of real-valued functions that do not converge uniformly but for which the convergence is closed to being uniform [7, p. 143, Theorem 2]. In the next theorem we generalize this result for nets of functions with values in a Banach space:

Theorem 3.17. *Let* $\{f_\iota\}_{\iota\in I}$ *be a net in* $X^{[a,b]}$ *that converges pointwise to* $f \in X^{[a,b]}$ *on* $[a,b]$ *and uniformly bounded in some neighbourhood of* a *as well as in some neighbourhood of* b. *Let* $\{f_\iota\}_{\iota\in I}$ *converge uniformly to* f *on every closed subinterval of* (a,b). *If each* f_ι *is Darboux integrable on every closed subinterval of* (a,b), *then* $f_\iota, f \in D([a,b],X)$ *for each* $\iota \in I$ *and* $\{D\text{-}\int_c^d f_\iota dt\}_{\iota\in I}$ *converges to* $D\text{-}\int_c^d f dt$ *uniformly with respect to closed subintervals* $[c,d]$ *of* $[a,b]$.

Proof. From hypothesis, it follows that f is bounded in some neighbourhood of a as well as in some neighbourhood of b. Hence there exist a $\delta > 0$ and an $M > 0$ such that

$$\|f_\iota(t)\| \leq M \quad \text{and} \quad \|f(t)\| \leq M$$

for all $t \in [a, a+\delta) \cup (b-\delta, b]$ and for all $\iota \in I$.

Also by part (b) of Theorem 3.12, f is Darboux integrable on every closed subinterval of (a,b). Hence an application of [12, p. 318, Corollary 3.33 (a)] to a single function yields that $f_\iota, f \in D([a,b],X)$ for each $\iota \in I$.

Let F_ι, F be the indefinite integrals of f_ι, f respectively.

Let $\varepsilon > 0$. Let us choose c and d such that $a < c < d < b$ and $c - a = b - d < \min\left\{\delta, \frac{\varepsilon}{4M}\right\}$. Then for any $s \in [a,c]$ and for all $\iota \in I$, we have

$$\begin{aligned}\|F_\iota(s) - F(s)\| &\leq \int_a^s \|f_\iota(t) - f(t)\| dt \\ &\leq 2M(s-a) < \frac{2M\varepsilon}{4M} < \varepsilon\end{aligned}$$

which shows that $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[a,c]$.

Since $\{f_\iota\}_{\iota\in I}$ converges uniformly to f on $[c,d]$, it follows from part (b) of Theorem 3.12 that $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[c,d]$. Hence $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[a,d]$. So there exists an $\iota_0\in I$ such that

$$\|F_\iota(d)-F(d)\|<\frac{\varepsilon}{2}$$

for all $\iota\in I$ with $\iota_0\leq\iota$.

Now for any $s\in[d,b]$ and for all $\iota\in I$ with $\iota_0\leq\iota$, we have

$$\begin{aligned}\|F_\iota(s)-F(s)\| &= \left\|\left\{F_\iota(d)+D\text{-}\int_d^s f_\iota(t)dt\right\}-\left\{F(d)+D\text{-}\int_d^s f(t)dt\right\}\right\|\\ &\leq \|F_\iota(d)-F(d)\|+\int_d^s\|f_\iota(t)-f(t)\|dt\\ &< \frac{\varepsilon}{2}+2M(s-d)<\frac{\varepsilon}{2}+\frac{2M\varepsilon}{4M}=\varepsilon\end{aligned}$$

which implies that $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[d,b]$. Consequently $\{F_\iota\}_{\iota\in I}$ converges uniformly to F on $[a,b]$. Therefore $\{f_\iota\}_{\iota\in I}$ converges to f in the Alexiewicz norm and hence in the norm $\|\cdot\|'$ and the result follows. □

The following result on series of Riemann and Darboux integrable functions follows from part (b) of Theorem 3.12:

Lemma 3.18. *If $\sum f_n$ is a series of functions in $R([a,b],X)$ (resp. $D([a,b],X)$) and converges uniformly to f on $[a,b]$, then $f\in R([a,b],X)$ (resp. $D([a,b],X)$).*

Lemma 3.19. *Let $\{f_n\}$ be a sequence of uniformly bounded functions in $R([a,b],X)$ (resp. $D([a,b],X)$). Let $\sum\lambda_n$ be an absolutely convergent series of real numbers. Then the series $\sum\lambda_n f_n$ is absolutely uniformly convergent on $[a,b]$ and the uniform limit is contained in $R([a,b],X)$ (resp. $D([a,b],X)$).*

Proof. That the series $\sum\lambda_n f_n$ is absolutely uniformly convergent on $[a,b]$ follows very easily and the next part follows from Lemma 3.18. □

Let $l_\infty(X)$, the vector space of all bounded X-valued sequences, be equipped with the usual sup norm under which it is a Banach space.

Theorem 3.20. *Let Z be a closed subspace of $l_\infty(X)$ and let $\{f_n\}$ be a sequence in $X^{[a,b]}$ such that $(f_1(t),f_2(t),\cdots)\in Z$ for each $t\in[a,b]$. Let*

$$f(t)=(f_1(t),f_2(t),\cdots)$$

for each $t\in[a,b]$. Then $f\in Z^{[a,b]}$ and the following results hold good:

(a) f is bounded on $[a,b]$ if and only if $\{f_n\}$ is uniformly bounded on $[a,b]$.

(b) $f\in R([a,b],Z)$ if and only if $\{f_n\}$ is equi-Riemann integrable on $[a,b]$.

(c) If $f\in D([a,b],Z)$, then $\{f_n\}$ is equi-Darboux integrable on $[a,b]$.

(d) f is continuous at a point in $[a,b]$ if and only if $\{f_n\}$ is equicontinuous thereat.

(e) $f\in D([a,b],Z)$ if and only if $\{f_n\}$ is uniformly bounded and equicontinuous almost everywhere on $[a,b]$.

Proof. (a) Follows from the equality

$$\|f(t)\|_Z = \sup_n \|f_n(t)\|_X$$

for each $t \in [a, b]$.

(b) It is easy to verify that for any two tagged partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a, b]$

$$\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\|_Z = \sup_n \|f_n(\mathcal{P}_1) - f_n(\mathcal{P}_2)\|_X.$$

Hence the result follows from [6, p. 925, Theorem 5 ((1) $\iff$ (3))] and [12, p. 310, Theorem 3.10 ((a) $\iff$ (b))].

(c) It is easy to verify that

$$\omega_{\mathcal{P}}(f_n) \le \omega_{\mathcal{P}}(f)$$

for all n and for all partitions $\mathcal{P}$ of $[a, b]$. Hence the result follows.

(d) Follows from the equality

$$\|f(s) - f(t)\|_Z = \sup_n \|f_n(s) - f_n(t)\|_X$$

for all $s, t \in [a, b]$.

(e) Follows from [6, p. 933, Theorem 18], parts (a) and (d). □

We recall that a function $f \in (X^*)^{[a,b]}$ is said to be Riemann-Gelfand integrable on $[a, b]$, if $xf \in R([a, b])$ for each $x \in X$. The collection of all Riemann-Gelfand integrable functions in $(X^*)^{[a,b]}$ is denoted by $RG([a, b], X^*)$ [16, p. 432].

Theorem 3.21. *Let $\{\phi_n\}$ be a pointwise bounded sequence in $\mathbb{R}^{[a,b]}$ so that $(\phi_1(t), \phi_2(t), \cdots) \in l_\infty$ for each $t \in [a, b]$. Let*

$$f(t) = (\phi_1(t), \phi_2(t), \cdots)$$

for each $t \in [a, b]$. Then $f \in (l_\infty)^{[a,b]}$ and the following statements hold good:

(a) $f \in RG([a, b], l_\infty)$ if and only if $\{\phi_n\}$ is uniformly bounded on $[a, b]$ and $\{\phi_n\} \subset R([a, b])$.

(b) $f \in R([a, b], l_\infty)$ if and only if $\{\phi_n\}$ is equi-Riemann integrable on $[a, b]$ if and only if $\{\phi_n\}$ is equi-Darboux integrable on $[a, b]$.

Proof. (a) Let $f \in RG([a, b], l_\infty)$. Then f is bounded and hence $\{\phi_n\}$ is uniformly bounded by part (a) of Theorem 3.20 by taking $X = \mathbb{R}$ and $Z = l_\infty$. Next let $e_n, n = 1, 2, \cdots$, be the standard unit vectors of l_1, the pre-dual of l_∞. Then clearly $\phi_n(\cdot) = f(\cdot)(e_n) \in R([a, b])$ for $n = 1, 2, \cdots$ and the direct part follows.

Conversely, let $\{\phi_n\}$ be uniformly bounded and let $\{\phi_n\} \subset R([a, b])$. Now for each $\gamma = (\gamma_1, \gamma_2, \cdots) \in l_1$, we have

$$f(\cdot)(\gamma) = \sum \gamma_n \phi_n(\cdot).$$

Hence by Lemma 3.19, $f(\cdot)(\gamma) \in R([a, b])$ which implies that $f \in RG([a, b], l_\infty)$.

(b) First part follows from part (b) of Theorem 3.20, by taking $X = \mathbb{R}$ and $Z = l_\infty$, and the second part follows from [12, p. 312, Theorem 3.14]. □

Example 3.22. *Let us consider the function*

$$f(t) = (\sin t, \sin 2t, \cdots)$$

for $t \in [0, 2\pi]$. Then clearly $f \in (l_\infty)^{[0,2\pi]}$. Since $\{\sin nt\}$ is uniformly bounded and $\sin nt$ is Riemann integrable on $[0, 2\pi]$ for each n, f is Riemann-Gelfand integrable on $[0, 2\pi]$ by part (a) of Theorem 3.21.

Now it is easy to note that if the length of the smallest interval of a partition $\mathcal{P}$ of $[0, 2\pi]$ is equal to δ, then $\omega(\sin kt, \mathcal{P}) = 4\pi$ whenever $k > \frac{2\pi}{\delta}$ which implies that $\{\sin nt\}$ is not equi-Darboux integrable on $[0, 2\pi]$ and hence by part (b) of Theorem 3.21, f is not Riemann integrable on $[0, 2\pi]$.

Also we see that $\sin nt$ is continuous on $[0, 2\pi]$ for each n, but $\{\sin nt\}$ is equicontinuous at no point of $[0, 2\pi]$. Hence by part (d) of Theorem 3.20, f is continuous at no point of $[0, 2\pi]$.

Corollary 3.23. *Let $\{\phi_n\}$ be a sequence in $\mathbb{R}^{[a,b]}$ which converges pointwise on $[a,b]$ so that $(\phi_1(t), \phi_2(t), \cdots) \in c$ for each $t \in [a,b]$. Let*

$$f(t) = (\phi_1(t), \phi_2(t), \cdots)$$

for each $t \in [a,b]$. Then $f \in c^{[a,b]}$ and the following statements hold good:

(a) f is bounded on $[a,b]$ if and only if $\{\phi_n\}$ is uniformly bounded on $[a,b]$.

(b) $f \in RP([a,b], c)$ if and only if $\{\phi_n\}$ is uniformly bounded on $[a,b]$ and $\{\phi_n\} \subset R([a,b])$.

(c) $f \in R([a,b], c)$ if and only if $\{\phi_n\}$ is equi-Riemann integrable on $[a,b]$.

(d) $f \in D([a,b], c)$ if and only if $\{\phi_n\}$ is uniformly bounded and equicontinuous almost everywhere on $[a,b]$.

Proof. (a), (c) & (d) Follow from parts (a), (b) and (e) respectively of Theorem 3.20 by taking $X = \mathbb{R}$ and $Z = c$.

(b) Let $f \in RP([a,b], c)$. Then f is bounded and hence $\{\phi_n\}$ is uniformly bounded on $[a,b]$ by part (a). Next let $e_n, n = 1, 2, \cdots$, be the standard unit vectors of l_1, the dual of c. Then as in the proof of part (a) of Theorem 3.21, $\phi_n(\cdot) = e_n f(\cdot) \in R([a,b])$ for $n = 1, 2, \cdots$ and the necessary part follows.

Conversely, let $\{\phi_n\}$ be uniformly bounded on $[a,b]$ and let $\{\phi_n\} \subset R([a,b])$. Then as in part (a) of Theorem 3.21, for each $\gamma = (\gamma_1, \gamma_2, \cdots) \in l_1$,

$$\gamma(f) = \sum \gamma_n \phi_n \in R([a,b])$$

which implies that $f \in RD([a,b], c)$. As c is separable, the result follows from [6, p. 944, Theorem 29]. □

Corollary 3.24. *If $\{\phi_n\}$ is a uniformly convergent sequence in $R([a,b])$, then $(\phi_1, \phi_2, \cdots) \in D([a,b], c)$.*

Proof. According to hypothesis, each ϕ_n is bounded and continuous almost everywhere on $[a,b]$ which implies that $\{\phi_n\}$ is uniformly bounded and equicontinuous almost everywhere on $[a,b]$. Hence the result follows from part (d) of Corollary 3.23. □

Theorem 3.25. *Let $\{\phi_n\}$ be a sequence in $\mathbb{R}^{[a,b]}$ such that $\sum_n \phi_n$ is absolutely uniformly convergent on $[a,b]$ and let $f(t) = (\phi_1(t), \phi_2(t), \cdots)$ for each $t \in [a,b]$. Then $f \in (l_1)^{[a,b]}$ and the following statements are equivalent:*

(a) $f \in RD([a,b], l_1)$.

(b) $f \in D([a,b], l_1)$.

(c) $\{\phi_n\} \subset D([a,b])$.

(d) $\{\phi_n\}$ is equi-Darboux integrable on $[a,b]$.

(e) $\{\phi_n\}$ is equicontinuous almost everywhere and uniformly bounded on $[a,b]$.

Proof. $(a) \iff (b)$ Follows from [6, p. 946, Theorem 34] as l_1 has the Schur property as well as the property of Lebesgue [6, p. 939, Theorem 26].

$(b) \implies (d)$ It is easy to verify that

$$\omega_{\mathcal{P}}(\phi_n) \leq \omega_{\mathcal{P}}(f)$$

for all n and for all partitions $\mathcal{P}$ of $[a,b]$, whence the result follows.

$(d) \implies (c)$ Obvious.

$(c) \implies (e)$ Let (c) hold. Then ϕ_n is bounded and continuous almost everywhere on $[a,b]$ for each n. Let E_n be the set of points of discontinuity of ϕ_n in $[a,b]$. Then $\lambda(E_n) = 0$ for each n. Let $E = \bigcup_n E_n$. Then $\lambda(E) = 0$. Now it should be noted that $\{\phi_n\}$ is uniformly convergent on $[a,b]$. So $\{\phi_n\}$ is uniformly bounded on $[a,b]$, and equicontinuous on $[a,b] \setminus E$, and hence equicontinuous almost everywhere on $[a,b]$.

$(e) \Longrightarrow (c)$ Trivial.

$(c) \Longrightarrow (a)$ For any $\gamma = (\gamma_1, \gamma_2, \cdots) \in l_\infty = (l_1)^*$, the series $\sum_n \gamma_n \phi_n$ is clearly absolutely uniformly convergent and hence converges uniformly on $[a, b]$ with uniform limit $\gamma(f)$. Also $\gamma_n \phi_n \in D([a, b])$ for each n. Hence by Lemma 3.18, $\gamma(f) \in D([a, b])$ which implies that $f \in RD([a, b], l_1)$. $\square$

Following Klippert and Williams [9, p. 53, Definition 2.3], we have the following definitions:

Definition 3.26. *Let $\{f_\iota\}_{\iota \in I}$ be a net in Y^S and let $f \in Y^S$. Let $s \in S$. We say that $\{f_\iota\}_{\iota \in I}$*

(a) *is pointwise Cauchy (resp. converges pointwise to f) about or at s if for each $\varepsilon > 0$, there exists a neighbourhood V of s such that for each $t \in V$, there exists an $\iota_t \in I$ such that*

$$d(f_\iota(t), f_\kappa(t)) < \varepsilon \text{ for all } \iota, \kappa \in I \text{ with } \iota_t \leq \iota, \kappa$$

$$\left(\text{resp. } d(f_\iota(t), f(t)) < \varepsilon \text{ for all } \iota \in I \text{ with } \iota_t \leq \iota\right),$$

(b) *is uniformly Cauchy (resp. converges uniformly to f) about or at s if for each $\varepsilon > 0$, there exist an $\iota_0 \in I$ and a neighbourhood V of s such that*

$$d(f_\iota(t), f_\kappa(t)) < \varepsilon \text{ for all } \iota, \kappa \in I \text{ with } \iota_0 \leq \iota, \kappa$$

$$\left(\text{resp. } d(f_\iota(t), f(t)) < \varepsilon \text{ for all } \iota \in I \text{ with } \iota_0 \leq \iota\right)$$

and for all $t \in V$.

From the very definition it follows that if a net converges uniformly about a point, then it converges pointwise to the same limit about that point; also it follows that a uniformly Cauchy net about a point is pointwise Cauchy about that point. It is clear that a net in Y^S converges pointwise about every point of S if and only if it converges pointwise on S. Also it is clear that if a net in Y^S converges uniformly on S, then it converges uniformly about every point of S. But the converse is not necessarily true [9, p. 54, Example 2.2]. However we have the following result whose proof is straightforward and so omitted:

Lemma 3.27. *Let a net $\{f_\iota\}_{\iota \in I}$ in Y^S converge uniformly to $f \in Y^S$ about every point of S. If S is compact, then $\{f_\iota\}_{\iota \in I}$ converges uniformly to f on S.*

The following result also follows trivially:

Lemma 3.28. *If a net $\{f_\iota\}_{\iota \in I}$ in Y^S converges pointwise to $f \in Y^S$ about $s \in S$ and is uniformly Cauchy about s, then it converges uniformly to f about that point.*

Definition 3.29. (a) *A function $f \in Y^S$ is said to be regulated* at $s \in S$ if there exists an element $l \in Y$ such that for any $\varepsilon > 0$, there exists a neighbourhood U of s with the property*

$$d(f(t), l) < \varepsilon$$

for all $t \in U - \{s\}$.

(b) *A net $\{f_\iota\}_{\iota \in I}$ in Y^S is said to be partially equi-regulated* at $s \in S$ if there exists a net $\{l_\iota\}_{\iota \in I} \subset Y$ such that for any $\varepsilon > 0$, there exist an $\iota_0 \in I$ and a neighbourhood U of s with the property*

$$d(f_\iota(t), l_\iota) < \varepsilon$$

for all $\iota \in I$ with $\iota_0 \leq \iota$ and for all $t \in U - \{s\}$; the net $\{l_\iota\}_{\iota \in I}$ is said to be the associated net of $\{f_\iota\}_{\iota \in I}$ at s.

Theorem 3.30. *Let $\{f_\iota\}_{\iota \in I}$ be a partially equi-regulated* net in Y^S at $s \in S$ with Cauchy associated net. Then the following statements hold good:*

(a) If $\{f_\iota(s)\}_{\iota\in I}$ is Cauchy, then $\{f_\iota\}_{\iota\in I}$ is uniformly Cauchy about s.

(b) If $\{f_\iota\}_{\iota\in I}$ converges pointwise to $f \in Y^S$ about s, then it converges uniformly to f thereabout. If, moreover, the associated net is convergent, then f is regulated at s.*

Proof. Let $\{l_\iota\}_{\iota\in I}$ be the associated net of $\{f_\iota\}_{\iota\in I}$, which is Cauchy in Y.

Let $\varepsilon > 0$. Then there exists an $\iota_1 \in I$ such that

$$d(l_\iota, l_\kappa) < \frac{\varepsilon}{3}$$

for all $\iota, \kappa \in I$ with $\iota_1 \leq \iota, \kappa$.

Also there exist an $\iota_2 \in I$ and a neighbourhood U_1 of s such that

$$d(f_\iota(t), l_\iota) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_2 \leq \iota$ and for all $t \in U_1 - \{s\}$.

(a) Let $\iota_0 \in I$ be such that $\iota_1, \iota_2 \leq \iota_0$. Then for any $t \in U_1 - \{s\}$ and for all $\iota, \kappa \in I$ with $\iota_0 \leq \iota, \kappa$, we have

$$d(f_\iota(t), f_\kappa(t)) \leq d(f_\iota(t), l_\iota) + d(l_\iota, l_\kappa) + d(l_\kappa, f_\kappa(t)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This, along with the hypothesis that $\{f_\iota(s)\}_{\iota\in I}$ is Cauchy, implies that $\{f_\iota\}_{\iota\in I}$ is uniformly Cauchy about s.

(b) It follows from hypothesis and part (a) that $\{f_\iota\}_{\iota\in I}$ is uniformly Cauchy about s and hence, by Lemma 3.28, it converges uniformly to f thereabout.

For the second part, let $\{l_\iota\}_{\iota\in I}$ converge to $l \in Y$. Then there exists an $\iota_3 \in I$ such that

$$d(l_\iota, l) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_3 \leq \iota$.

Since $\{f_\iota\}_{\iota\in I}$ converges uniformly to f about s, there exist an $\iota_4 \in I$ and a neighbourhood U_2 of s such that

$$d(f_\iota(t), f(t)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_4 \leq \iota$ and for all $t \in U_2$.

Let $U = U_1 \cap U_2$. Then U is a neighbourhood of s. Let $\iota \in I$ be such that $\iota_2, \iota_3, \iota_4 \leq \iota$. Then for all $t \in U - \{s\}$, we have

$$d(f(t), l) \leq d(f(t), f_\iota(t)) + d(f_\iota(t), l_\iota) + d(l_\iota, l) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which shows that f is regulated* at s. □

Following [8, p. 1127, Definition 4.2.1] and [2, p. 69, Definition 3.11], we call a net $\{f_\iota\}_{\iota\in I}$ in Y^S weakly exhaustive at a point $s \in S$ if for any $\varepsilon > 0$, there exists a neighbourhood U of s such that for each $t \in U$, there exists an $\iota_t \in I$ such that for all $\iota \in I$ with $\iota_t \leq \iota$, we have $d(f_\iota(t), f_\iota(s)) < \varepsilon$.

Gregoriades and Papanastassiou showed that the pointwise limit of a weakly exhaustive sequence of functions which are not necessarily continuous is continuous [8, p. 1127, Theorem 4.2.3]. We present below the net version of the said result:

Lemma 3.31. *Let $\{f_\iota\}_{\iota\in I}$ be a net in Y^S which converges pointwise to $f \in Y^S$ about $s \in S$. Then f is continuous at s if and only if $\{f_\iota\}_{\iota\in I}$ is weakly exhaustive thereat.*

Proof. Let $\varepsilon > 0$. Then by hypothesis, there exists a neighbourhood V_s of s such that for each $t \in V_s$, there is an $\iota_t \in I$ with

$$d(f_\iota(t), f(t)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_t \leq \iota$.

Since $s \in V_s$, there exists an $\iota_s \in I$ such that

$$d(f_\iota(s), f(s)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_s \leq \iota$.

Now let f be continuous at s. Then there exists a neighbourhood U_1 of s such that for each $t \in U_1$, we have

$$d(f(t), f(s)) < \frac{\varepsilon}{3}.$$

Let $U = V_s \cap U_1$. Then U is a neighbourhood of s. Let $t \in U$. Then $t \in V_s$ as well as $t \in U_1$. So there exists an $\iota_t \in I$ such that

$$d(f_\iota(t), f(t)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_t \leq \iota$, and

$$d(f(t), f(s)) < \frac{\varepsilon}{3}.$$

Let $\iota'_t \in I$ be such that $\iota_t, \iota_s \leq \iota'_t$. Then for all $\iota \in I$ with $\iota'_t \leq \iota$, we have

$$\begin{aligned} d(f_\iota(t), f_\iota(s)) &\leq d(f_\iota(t), f(t)) + d(f(t), f(s)) + d(f(s), f_\iota(s)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

which implies that $\{f_\iota\}_{\iota \in I}$ is weakly exhaustive at s.

Conversely, let $\{f_\iota\}_{\iota \in I}$ be weakly exhaustive at s. Then there exists a neighbourhood V_1 of s such that for each $t \in V_1$, there exists an $\iota''_t \in I$ such that for all $\iota \in I$ with $\iota''_t \leq \iota$, we have

$$d(f_\iota(t), f_\iota(s)) < \frac{\varepsilon}{3}.$$

Let $V = V_s \cap V_1$. Then V is a neighbourhood of s. Let $t \in V$. Then $t \in V_s$ as well as $t \in V_1$. So there exist $\iota_t, \iota''_t \in I$ such that

$$d(f_\iota(t), f(t)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_t \leq \iota$, and

$$d(f_\iota(t), f_\iota(s)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota''_t \leq \iota$.

Let $\iota \in I$ be such that $\iota_t, \iota_s, \iota''_t \leq \iota$. Then we have

$$\begin{aligned} d(f(t), f(s)) &\leq d(f(t), f_\iota(t)) + d(f_\iota(t), f_\iota(s)) + d(f_\iota(s), f(s)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

which shows that f is continuous at s. □

For definition of an exhaustive net of functions we refer to [8, p. 1119, Definition 3.2.9]. The following result follows from the very definitions and so the proof is not presented:

Lemma 3.32. *Let $\{f_\iota\}_{\iota \in I}$ be a net in Y^S and let $s \in S$. Let us consider the following statements:*

(a) $\{f_\iota\}_{\iota \in I}$ is exhaustive at s.

(b) $\{f_\iota\}_{\iota \in I}$ is partially equi-regulated at s with associated net $\{f_\iota(s)\}_{\iota \in I}$.*

(c) $\{f_\iota\}_{\iota \in I}$ is weakly exhaustive at s.

Then (a) $\Longleftrightarrow$ *(b)* $\Longrightarrow$ *(c).*

The following result was established in global sense by Gregoriades and Papanastassiou [8, p. 1120, Theorem 3.2.12 & Theorem 3.2.14]. We present here the local version:

Theorem 3.33. *Let* $\{f_\iota\}_{\iota\in I}$ *be a net in* Y^S *and let* $f \in Y^S$*. Let* $s \in S$*. Then* $\{f_\iota\}_{\iota\in I}$ *is exhaustive at* s *and converges pointwise to* f *thereabout if and only if* f *is continuous at* s *and* $\{f_\iota\}_{\iota\in I}$ *converges uniformly to* f *thereabout.*

Proof. Let $\{f_\iota\}_{\iota\in I}$ be exhaustive at s and converge pointwise to f about s. Then it follows from Lemma 3.32 ((a) $\Longrightarrow$ (c)) and Lemma 3.31 that f is continuous at s.

Again it follows from Lemma 3.32 ((a) $\Longleftrightarrow$ (b)) that $\{f_\iota\}_{\iota\in I}$ is partially equi-regulated* at s with associated net $\{f_\iota(s)\}_{\iota\in I}$ which is clearly convergent and hence Cauchy. Therefore it follows from part (*b*) of Theorem 3.30 that $\{f_\iota\}_{\iota\in I}$ converges uniformly to f about s.

For the converse part, let f be continuous at s and let $\{f_\iota\}_{\iota\in I}$ converge uniformly to f about s.

Let $\varepsilon > 0$. Then there exists a neighbourhood V_1 of s such that

$$d(f(t), f(s)) < \frac{\varepsilon}{3}$$

for all $t \in V_1$.

Also, there exist an $\iota_0 \in I$ and a neighbourhood V_2 of s such that

$$d(f_\iota(t), f(t)) < \frac{\varepsilon}{3}$$

for all $\iota \in I$ with $\iota_0 \le \iota$ and for all $t \in V_2$.

Let $V = V_1 \cap V_2$. Then V is a neighbourhood of s, and for all $\iota \in I$ with $\iota_0 \le \iota$ and for all $t \in V$, we have

$$\begin{aligned} d(f_\iota(t), f_\iota(s)) &\le d(f_\iota(t), f(t)) + d(f(t), f(s)) + d(f(s), f_\iota(s)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

which implies that $\{f_\iota\}_{\iota\in I}$ is exhaustive at s.

That $\{f_\iota\}_{\iota\in I}$ converges pointwise to f about s is obvious. □

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