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Hybrid Viscosity Approximation Methods for Systems of Variational Inequalities and Hierarchical Fixed Point Problems

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Abstract. In this paper, we propose an implicit iterative method and an explicit iterative method for solving a general system of variational inequalities with a hierarchical fixed point problem constraint for an infinite family of nonexpansive mappings. We show that the proposed algorithms converge strongly to a solution of the general system of variational inequalities, which is a unique solution of the hierarchical fixed point problem.

1. Introduction

Let X be a smooth Banach space. Let $C \subset X$ be a nonempty closed convex set. Let $T : C \to X$ be a nonlinear mapping. Use Fix(T) to denote the set of fixed points of T. A mapping $T : C \to X$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Tu - Tv\| \leq L\|u - v\|, \quad \forall u, v \in C.$$

If $L = 1$, T is called nonexpansiv. If $L \in [0, 1)$, T is called contractive.

Let $A, B : C \to X$ be two nonlinear mappings. Recall that the general system of variational inequalities (GSVI) is to find $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \langle Av^* + u^* - v^*, j(u - u^*)\rangle \geq 0, & \forall u \in C, \\ \langle Bu^* + v^* - u^*, j(u - v^*)\rangle \geq 0, & \forall u \in C. \end{cases} \tag{1}$$

In [11], the authors proved the equivalence between the GSVI (1) and the fixed point problem. Ceng *et al.* [12] introduced a pair of implicit and explicit iterative methods for solving GSVI (1). It is worth mentioning that the system of variational inequalities plays an important role in game theory and economics, see e.g., [3, 4, 14, 22] and the references therein.

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Note that the GSVI (1) was introduced and studied by Ceng, Wang and Yao [13] in a real Hilbert space. Problem (1) can be reduced to the following classical variational inequality (VI) of finding $u^* \in C$ such that

$$\langle Au^*, u - u^* \rangle \geq 0, \quad \forall u \in C. \tag{2}$$

This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics; see e.g., [1, 2, 15, 18–21, 24, 27, 28, 33–35] and the references therein.

In case of Banach space setting and $A = B$ and $u^* = v^*$, then the VI is defined as

$$\langle Au^*, j(u - u^*) \rangle \geq 0, \quad \forall u \in C. \tag{3}$$

Aoyama, Iiduka and Takahashi [5] proposed an iterative scheme to find the approximate solution of (3) and proved the weak convergence of the sequences generated by the proposed scheme. In [16, 17], Kikkawa and Takahashi studied an implicit iteration scheme that converges strongly to a solution of the stated problem. Recently, in [29], Wang, Yu and Guo proposed a new implicit iteration method, which converges strongly to a common fixed point, for solving some variational inequality in a Banach space. Buong and Phong [10] introduced two new implicit iterative algorithms, which converge strongly in Banach spaces.

The purpose of this paper is to find a solution of a general system of variational inequalities (GSVI) with a hierarchical fixed point problem (HFPP) constraint for an infinite family of nonexpansive mappings in a real strictly convex and 2-uniformly smooth Banach space X. We propose an implicit iterative method and an explicit iterative method. We show that the proposed algorithms converge strongly to a solution of the GSVI, which is a unique solution of the HFPP. Our results improve and extend the corresponding results in the literature.

2. Preliminaries and Algorithms

Let X be a real Banach space with its dual space X^*. Let $U := \{u \in X : \|u\| = 1\}$ be the unit sphere. A Banach space X is said to be strictly convex if for $u, v \in U$ with $u \neq v$, we have $\|(1-\delta)u + \delta v\| < 1, \ \forall \delta \in (0, 1)$. X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $u, v \in U$, $\|u - v\| \geq \epsilon$ implies $\|u + v\|/2 \geq 1 - \delta$. Define a function $\rho : [0, \infty) \to [0, \infty)$ as follows:

$$\rho(\tau) = \sup\{\frac{1}{2}(\|u + v\| + \|u - v\|) - 1 : u, v \in X, \|u\| = 1, \|v\| = \tau\}.$$

X is said to be uniformly smooth if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let $1 < q \leq 2$. X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined as

$$J(u) := \{\varphi \in X^* : \langle u, \varphi \rangle = \|u\|^2 = \|\varphi\|^2\}, \ \forall u \in X, \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is known that J is single-valued if and only if X is smooth. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Recall that a mapping $S : X \to X$ is said to be

(i) accretive if for each $u, v \in X$, there exists $j(u - v) \in J(u - v)$ such that

$$\langle S(u) - S(v), j(u - v) \rangle \geq 0.$$

(ii) δ-strongly accretive if for each $u, v \in X$, there exist $j(u - v) \in J(u - v)$ and $\delta \in (0, 1)$ such that

$$\langle S(u) - S(v), j(u - v) \rangle \geq \delta \|u - v\|^2.$$

(iii) ν-inverse-strongly accretive if for each $u, v \in X$, there exist $j(u - v) \in J(u - v)$ and $\nu \in (0, 1)$ such that

$$\langle S(u) - S(v), j(u - v) \rangle \geq \nu \|S(u) - S(v)\|^2.$$

(iv) ζ-strictly pseudocontractive [8] if for each $u, v \in X$, there exist $j(u-v) \in J(u-v)$ and $\zeta \in (0,1)$ such that

$$\langle S(u) - S(v), j(u-v)\rangle \leq \|u-v\|^2 - \zeta\|u - v - (S(u) - S(v))\|^2.$$

Lemma 2.1. *([30]) Let $1 < q \leq 2$ and X be a q-uniformly smooth Banach space. Then*

$$\|u+v\|^q \leq \|u\|^q + q\langle v, J_q(u)\rangle + 2\|\kappa v\|^q, \quad \forall u, v \in X,$$

where κ is the q-uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(u) = \{\varphi \in X^* : \langle u, \varphi\rangle = \|u\|^q, \ \|\varphi\| = \|u\|^{q-1}\}, \quad \forall u \in X.$$

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(u) + \varpi(u - \Pi(u))] = \Pi(u),$$

whenever $\Pi(u) + \varpi(u - \Pi(u)) \in C$ for $u \in C$ and $\varpi \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(u) = u$ for each $u \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.2. *([23]) Let C be a nonempty closed convex subset of a smooth Banach space X and D be a nonempty subset of C and Π be a retraction of C onto D. Then the following are equivalent*

(i) Π is sunny and nonexpansive;
(ii) $\|\Pi(u) - \Pi(v)\|^2 \leq \langle u - v, j(\Pi(u) - \Pi(v))\rangle, \forall u, v \in C$;
(iii) $\langle u - \Pi(u), j(v - \Pi(u))\rangle \leq 0, \forall u \in C, v \in D$.

Lemma 2.3. *([11]) Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. For given $u^*, v^* \in C$, (u^*, v^*) is a solution of the GSVI (1)) if and only if $u^* \in \mathrm{GSVI}(C, A, B)$ where $\mathrm{GSVI}(C, A, B)$ is the set of fixed points of the mapping $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$ and $v^* = \Pi_C(u^* - \varrho Bu^*)$.*

Proposition 2.4. *([11]) Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $A : C \to X$ be ν-inverse-strongly accretive. Then,*

$$\|(I - \varpi A)u - (I - \varpi A)v\|^2 \leq \|u - v\|^2 + 2\varpi(\kappa^2\varpi - \nu)\|Au - Av\|^2,$$

In particular, if $0 \leq \varpi \leq \frac{\nu}{\kappa^2}$, then $I - \varpi A$ is nonexpansive.

Lemma 2.5. *([11]) Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$. If $0 \leq \varpi \leq \frac{\nu}{\kappa^2}$ and $0 \leq \varrho \leq \frac{\vartheta}{\kappa^2}$, then $G : C \to C$ is nonexpansive.*

Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let Ξ_C be the set of all contractive self-mappings on C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let $f \in \Xi_C$ with coefficient $\rho \in (0, 1)$ and $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$. Assume that $\varpi \in (0, \frac{\nu}{\kappa^2}]$ and $\varrho \in (0, \frac{\vartheta}{\kappa^2}]$ where κ is the 2-uniformly smooth constant of X (see Lemma 2.1). Very recently, in order to solve GSVI (1), Ceng *et al.* [12] proved the following result.

Algorithm 2.6. *([12]) Let $\theta_t \in [0, 1), \forall t \in (0, 1)$ such that $\lim_{t\to 0^+} \theta_t/t = 0$. Define the net $\{x_t\}$ by*

$$x_t = tf(x_t) + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \varpi A)\Pi_C(I - \varrho B)x_t, \quad \forall t \in (0, 1).$$

It was proven in [12] that the net $\{x_t\}$ converges as $t \to 0^+$ strongly to the unique solution $p^* \in \text{GSVI}(C, A, B)$ to the follow VI:

$$\langle (I - f)p^*, j(x - p^*)\rangle \geq 0, \quad \forall x \in \text{GSVI}(C, A, B). \tag{5}$$

Recently, Buong and Anh [9] proposed the following implicit iteration method:

$$x_t = T^t x_t, \ T^t := T_0^t T_N^t \cdots T_1^t, \quad t \in (0, 1), \tag{6}$$

where $\{T_i^t\}_{i=0}^N$ are defined by

$$T_i^t x := (1 - \vartheta_t^i)x + \vartheta_t^i T_i x, \ i = 1, ..., N, \ T_0^t y := (I - \varpi_t \varrho F)y. \tag{7}$$

Takahashi [26] introduced a W-mapping, generated by $T_k, T_{k-1}, ..., T_1$ and real numbers $\nu_k, \nu_{k-1}, ..., \nu_1$ as follows:

$$\begin{cases} U_{k,k+1} = I, \\ U_{k,k} = \nu_k T_k U_{k,k+1} + (1 - \nu_k)I, \\ U_{k,k-1} = \nu_{k-1} T_{k-1} U_{k,k} + (1 - \nu_{k-1})I, \\ \cdots \\ U_{k,2} = \nu_2 T_2 U_{k,3} + (1 - \nu_2)I, \\ W_k = U_{k,1} = \nu_1 T_1 U_{k,2} + (1 - \nu_1)I. \end{cases} \tag{8}$$

Kikkawa and Takahashi [16] proved strong convergence of a sequence $\{x_k\}_{k=1}^\infty$, defined by the following implicit iterative scheme: $x_k = \iota_k f(x_k) + (1 - \iota_k)W_k x_k$. In [9], they considered the following strongly convergent implicit method:

$$S_k x = (1 - \frac{1}{k})Ux + \frac{1}{k} f(x), \quad \text{and} \quad Ux = \lim_{k\to\infty} W_k x = \lim_{k\to\infty} U_{k,1} x. \tag{9}$$

Note that the method (9) contains the limit mapping U, and hence, it is quite difficult to realize.

In [10], Buong and Phuong introducd a mapping V_k, defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1} \cdots T^k, \quad T^i = (1 - \nu_i)I + \nu_i T_i, \ i = 1, 2, ..., k. \tag{10}$$

Buong and Phuong presented the following iterations

$$x_k = V_k(I - \varpi_k F)x_k, \quad \forall k \geq 1, \tag{11}$$

and

$$x_k = \iota_k(I - \varpi_k F)x_k + (I - \iota_k)V_k x_k, \quad \forall k \geq 1, \tag{12}$$

where ϖ_k and ι_k are the positive parameters, satisfying some additional conditions. The authors [10] proved the strong convergence theorems for the methods (11) and (12).

We will make use of the following well known results.

Lemma 2.7. *Let X be a real Banach space. Then for all $u, v \in X$,*

(i) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, j(u + v)\rangle$;
(ii) $\|u + v\|^2 \geq \|u\|^2 + 2\langle v, j(u)\rangle$.

Lemma 2.8. *([6, 7]) Let X be a uniformly convex Banach space or a reflexive Banach space satisfying Opial's condition, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a nonexpansive mapping with* $\text{Fix}(T) \neq \emptyset$. *Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, if $\{x_k\}$ is a sequence of C such that $x_k \rightharpoonup x$ and $(I - T)x_k \to y$, then $(I - T)x = y$.*

Let LIM be a continuous linear functional on l^∞ and $(a_1, a_2, ...) \in l^\infty$. We write $\mathrm{LIM}_k a_k$ instead of $\mathrm{LIM}((a_1, a_2, ...))$. LIM is called a Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}_k 1 = 1$ and $\mathrm{LIM}_k a_{k+1} = \mathrm{LIM}_k a_k$ for all $(a_1, a_2, ...) \in l^\infty$. It is well known that for the Banach limit LIM, the following hold:

(i) for all $k \geq 1$, $a_k \leq c_k$ implies $\mathrm{LIM}_k a_k \leq \mathrm{LIM}_k c_k$;
(ii) $\mathrm{LIM}_k a_{k+N} = \mathrm{LIM}_k a_k$ for any fixed positive integer N;
(iii) $\liminf_{k\to\infty} a_k \leq \mathrm{LIM}_k a_k \leq \limsup_{k\to\infty} a_k$ for all $(a_1, a_2, ...) \in l^\infty$.

Lemma 2.9. *([32]) Let $(a_1, a_2, ...) \in l^\infty$. If $\mathrm{LIM}_k a_k = 0$, then there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that $a_{k_i} \to 0$ as $i \to \infty$.*

Lemma 2.10. *([11]) Let C be a nonempty closed convex subset of a real smooth Banach space X. Assume that the mapping $F : C \to X$ is accretive and weakly continuous along segments (that is, $F(x + ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$\text{find } x^* \in C : \quad \langle F(x^*), j(x - x^*)\rangle \geq 0, \quad \forall x \in C,$$

is equivalent to the dual variational inequality

$$\text{find } x^* \in C : \quad \langle F(x), j(x - x^*)\rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.11. *([11]) Let C be a nonempty closed convex subset of a real smooth Banach space X, and let $F : C \to X$ be a mapping.*

(i) If F is ζ-strictly pseudocontractive, then F is Lipschitz continuous with constant $1 + \frac{1}{\zeta}$.
(ii) If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$, then $I - F$ is Lipschitzian with constant $\sqrt{\frac{1-\delta}{\zeta}} \in (0, 1]$.
(iii) If F is δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$, then for any fixed number $\varpi \in [0, 1)$, $I - \varpi F$ is Lipschitzian with constant $1 - \varpi(1 - \sqrt{\frac{1-\delta}{\zeta}}) \in (0, 1]$.

3. Implicit Iterative Methods

In this section, we propose implicit iterative algorithms for solving a general system of variational inequalities (GSVI) with a hierarchical fixed point problem (HFPP) constraint for an infinite family of nonexpansive mappings, and derive the strong convergence of the sequences generated by the proposed algorithms to a unique solution of the HFPP.

The following lemmas and proposition will be used to prove our main results in the sequel.

Lemma 3.1. *([10]) Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^k$, $k \geq 1$, be k nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^k \mathrm{Fix}(T_i) \neq \emptyset$. Let a, b and ν_i, $i = 1, 2, ..., k$, be real numbers such that $0 < a \leq \nu_i \leq b < 1$, and let V_k be a mapping, defined by* (10) *for all $k \geq 1$. Then, $\mathrm{Fix}(V_k) = \mathcal{F}$.*

Lemma 3.2. *([10]) Let C be a nonempty closed convex subset of a Banach space X and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$. Let V_k be a mapping, defined by* (10)*. Then, for each $x \in C$ and $i \geq 1$, $\lim_{k\to\infty} V_k^i x$ exists.*

Remark 3.3. *(i) We can define the mappings*

$$V_\infty^i x := \lim_{k\to\infty} V_k^i x \text{ and } Vx := V_\infty^1 x = \lim_{k\to\infty} V_k x, \ \forall x \in C. \tag{13}$$

(ii) It can be readily seen from the proof of Lemma 3.2 that if D is a nonempty and bounded subset of C, then the following holds:

$$\lim_{k\to\infty} \sup_{x\in D} \|V_k^i x - V_\infty^i x\| = 0, \quad \forall i \geq 1.$$

Lemma 3.4. *([10]) Let C be a nonempty closed convex subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that the set of common fixed points $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Then, $\mathrm{Fix}(V) = \mathcal{F}$.*

Inspired by Lemma 3.4, we present the following proposition.

Proposition 3.5. *Let C be a nonempty closed convex subset of a real strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let the mapping $G : C \to C \subset X$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$ where $0 < \varpi \leq \frac{\nu}{\kappa^2}$ and $0 < \varrho \leq \frac{\vartheta}{\kappa^2}$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Then, $\mathrm{Fix}(V \circ G) = \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$. Then it is obvious that $Gp = p$ and $V_k^i p = p$ for all integers $i, k \geq 1$ with $k \geq i$. So, we have $V_\infty^i Gp = p$ for all integer $i \geq 1$. In particular, we have $(V \circ G)p = V_\infty^1 Gp$ and hence $\mathcal{F} \subset \mathrm{Fix}(V \circ G)$. Next, we prove that $\mathrm{Fix}(V \circ G) \subset \mathcal{F}$. Now, let $x \in \mathrm{Fix}(V \circ G)$ and $y \in \mathcal{F}$. Then,

$$\begin{aligned}\|V_kGx - V_kGy\| = \|V_k^1Gx - V_k^1Gy\| &= \|(1-\nu_1)(V_k^2Gx - V_k^2Gy) + \nu_1(T_1V_k^2Gx - T_1V_k^2Gy)\| \\ &\leq (1-\nu_1)\|V_k^2Gx - V_k^2Gy\| + \nu_1\|V_k^2Gx - V_k^2Gy\| \\ &= \|V_k^2Gx - V_k^2Gy\| \leq \|V_k^{i+1}Gx - V_k^{i+1}Gy\| \leq \|V_k^kGx - V_k^kGy\| \\ &\leq \|Gx - Gy\| \leq \|x - y\|,\end{aligned}$$

which together with $\|(V \circ G)x - (V \circ G)y\| = \|x - y\|$ implies that

$$\|V_\infty^iGx - V_\infty^iGy\| = \|V_\infty^{i+1}Gx - V_\infty^{i+1}Gy\| = \|Gx - y\|.$$

Therefore, we have

$$\|(1-\nu_i)(V_\infty^{i+1}Gx - V_\infty^{i+1}Gy) + \nu_i(T_iV_\infty^{i+1}Gx - T_iV_\infty^{i+1}Gy)\| = \|V_\infty^{i+1}Gx - V_\infty^{i+1}Gy\| = \|Gx - y\|,$$

for every $i \geq 1$. Since X is strictly convex, $0 < \nu_i < 1$, and $y \in \mathcal{F}$, we have $Gx - y = T_iV_\infty^{i+1}Gx - T_iV_\infty^{i+1}Gy = T_iV_\infty^{i+1}Gx - y$ and $Gx - y = V_\infty^{i+1}Gx - V_\infty^{i+1}Gy = V_\infty^{i+1}Gx - y$, and hance, $Gx = T_iV_\infty^{i+1}Gx$ and $Gx = V_\infty^{i+1}Gx$ for every $i \geq 1$. Consequently, for every $i \geq 1$, we have $Gx = T_iGx$. In particular, when $i = 1$, we have that $Gx = T_1V_\infty^2Gx$ and $Gx = V_\infty^2Gx$. So, it follows that $x = (V \circ G)x = (1-\nu_1)V_\infty^2Gx + \nu_1T_1V_\infty^2Gx = Gx$, which together with $Gx = T_iGx, \forall i \geq 1$, implies that for every $i \geq 1$, we have $x = T_ix$. It means that $x \in \mathcal{F}$. □

Lemma 3.6. *([25]) Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\nu_k\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{k\to\infty} \nu_k \leq \limsup_{k\to\infty} \nu_k < 1$. Suppose that $x_{k+1} = \nu_kx_k + (1-\nu_k)z_k, \forall k \geq 1$, and $\limsup_{k\to\infty}(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|) \leq 0$. Then $\lim_{k\to\infty} \|z_k - x_k\| = 0$.*

Lemma 3.7. *([31]) Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{k+1} \leq (1-\vartheta_k)a_k + \vartheta_k\iota_k + \delta_k, \quad \forall k \geq 1,$$

where $\{\vartheta_k\}$, $\{\iota_k\}$ and $\{\delta_k\}$ satisfy the conditions:

(i) $\{\vartheta_k\} \subset [0,1]$, $\sum_{k=1}^{\infty} \vartheta_k = \infty$, *or equivalently,* $\prod_{k=1}^{\infty}(1-\vartheta_k) = 0$;
(ii) $\limsup_{k\to\infty} \iota_k \leq 0$;
(iii) $\{\delta_k\} \subset [0,\infty)$, $\sum_{k=1}^{\infty} \delta_k < \infty$.

Then $\lim_{k\to\infty} a_k = 0$.

Throughout this paper, we use Ξ_C to denote the set of all contractive self-mappings on C. Now, we are in a position to prove the following main results.

Theorem 3.8. *Let C be a nonempty closed convex subset of a real strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let $f \in \Xi_C$ with coefficient $\rho \in (0,1)$, and $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \geq 1$. Assume that $\varpi \in (0, \frac{\nu}{\kappa^2}]$ and $\varrho \in (0, \frac{\vartheta}{\kappa^2}]$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (10). Let $\{x_k\}_{k=1}^{\infty}$ be generated in the implicit manner*

$$x_k = \iota_k f(x_k) + (1 - \iota_k)\Pi_C(I - \varpi_k F)V_k \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)x_k, \quad \forall k \geq 1, \tag{14}$$

where $\{\varpi_k\}_{k=1}^{\infty} \subset [0,1)$ and $\{\iota_k\}_{k=1}^{\infty} \subset (0,1)$ such that $\lim_{k\to\infty} \iota_k = 0$, $\lim_{k\to\infty} \varpi_k/\iota_k = 0$. Then $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$ if and only if $\{x_k\}_{k=1}^{\infty}$ is bounded, and in this case, $\{x_k\}_{k=1}^{\infty}$ converges as $k \to \infty$ strongly to an element of $\mathcal{F}$. In addition, if we define $Q : \Xi_C \to \mathcal{F}$ by

$$Q(f) := s-\lim_{k\to\infty} x_k, \quad \forall f \in \Xi_C,$$

then $Q(f)$ solves the following VI

$$\langle (I - f)(Q(f)), j(Q(f) - x)\rangle \leq 0, \quad \forall f \in \Xi_C, x \in \mathcal{F}. \tag{15}$$

In particular, if $f = u \in C$ is a constant, then the above mapping $Q : \Xi_C \to \mathcal{F}$ reduces to the sunny nonexpansive retraction of Reich from C onto $\mathcal{F}$,

$$\langle Q(u) - u, j(Q(u) - x)\rangle \leq 0, \quad \forall u \in C, x \in \mathcal{F}.$$

Proof. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$ where $0 < \varpi \leq \frac{\nu}{\kappa^2}$ and $0 < \varrho \leq \frac{\vartheta}{\kappa^2}$. In terms of Lemma 2.5 we know that G is a nonexpansive mapping on C. Then, the implicit iterative scheme (14) can be rewritten as

$$x_k = \iota_k f(x_k) + (1 - \iota_k)\Pi_C(I - \varpi_k F)V_k G x_k, \quad \forall k \geq 1. \tag{16}$$

Consider the mapping $U_k x = \iota_k f(x) + (1 - \iota_k)\Pi_C(I - \varpi_k F)V_k G x, \forall x \in C$. From Lemma 2.11, it follows that for each $x, y \in C$,

$$\begin{aligned}
\|U_k x - U_k y\| &= \|\iota_k(f(x) - f(y)) + (1 - \iota_k)[\Pi_C(I - \varpi_k F)V_k Gx - \Pi_C(I - \varpi_k F)V_k Gy]\| \\
&\leq \iota_k\|f(x) - f(y)\| + (1 - \iota_k)\|\Pi_C(I - \varpi_k F)V_k Gx - \Pi_C(I - \varpi_k F)V_k Gy\| \\
&\leq \iota_k \rho\|x - y\| + (1 - \iota_k)\|(I - \varpi_k F)V_k Gx - (I - \varpi_k F)V_k Gy\| \\
&\leq \iota_k \rho\|x - y\| + (1 - \iota_k)(1 - \varpi_k \tau)\|V_k Gx - V_k Gy\| \\
&\leq \iota_k \rho\|x - y\| + (1 - \iota_k)\|x - y\| \\
&= (1 - \iota_k(1 - \rho))\|x - y\|,
\end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}} \in [0,1)$. So, $U_k : C \to C$ is contraction. Therefore, the Banach contraction principle guarantees that U_k has a unique fixed point in C, satisfying (16).

Next, we show that $\mathcal{F} \neq \emptyset$ if and only if $\{x_k\}_{k=1}^{\infty}$ is bounded. Indeed, assume $\mathcal{F} \neq \emptyset$. Take an arbitrarily given $p \in \mathcal{F}$. Then we get $V_k p = p$ and $Gp = p$. So, it follows from (16) that for each $k \geq 1$,

$$\begin{aligned}
\|x_k - p\| &\leq \iota_k\|f(x_k) - p\| + (1 - \iota_k)\|\Pi_C(I - \varpi_k F)V_k G x_k - p\| \\
&\leq \iota_k(\|f(x_k) - f(p)\| + \|f(p) - p\|) + (1 - \iota_k)[\|\Pi_C(I - \varpi_k F)V_k G x_k - \Pi_C(I - \varpi_k F)p\| + \|\Pi_C(I - \varpi_k F)p - p\|] \\
&\leq \iota_k(\rho\|x_k - p\| + \|f(p) - p\|) + (1 - \iota_k)[\|(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)p\| + \|(I - \varpi_k F)p - p\|] \\
&\leq \iota_k(\rho\|x_k - p\| + \|f(p) - p\|) + (1 - \iota_k)[(1 - \varpi_k \tau)\|V_k G x_k - p\| + \varpi_k\|F(p)\|] \\
&\leq \iota_k(\rho\|x_k - p\| + \|f(p) - p\|) + (1 - \iota_k)[(1 - \varpi_k \tau)\|x_k - p\| + \varpi_k\|F(p)\|] \\
&\leq \iota_k(\rho\|x_k - p\| + \|f(p) - p\|) + (1 - \iota_k)[\|x_k - p\| + \varpi_k\|F(p)\|] \\
&= (1 - \iota_k(1 - \rho))\|x_k - p\| + \iota_k\|f(p) - p\| + (1 - \iota_k)\varpi_k\|F(p)\| \\
&\leq \iota_k\|f(p) - p\| + (1 - \iota_k(1 - \rho))\|x_k - p\| + \varpi_k\|F(p)\|,
\end{aligned}$$

which implies that

$$\|x_k - p\| \le \frac{1}{1-\rho}\|f(p) - p\| + \frac{\varpi_k}{\iota_k} \cdot \frac{\|F(p)\|}{1-\rho}.$$

Because $\lim_{k\to\infty} \varpi_k/\iota_k = 0$, we deduce that

$$\limsup_{k\to\infty} \|x_k\| \le \|p\| + \frac{1}{1-\rho}\|f(p) - p\| < \infty.$$

This shows that $\{x_k\}_{k=1}^{\infty}$ is bounded.

Conversely, assume $\{x_k\}_{k=1}^{\infty}$ is bounded. Then $\{f(x_k)\}, \{Gx_k\}, \{V_kGx_k\}$ and $\{F(VGx_k)\}$ are bounded. In terms of (16),

$$x_k - V_kGx_k = \frac{\iota_k}{1-\iota_k}(f(x_k) - x_k) + \Pi_C(I - \varpi_k F)V_kGx_k - V_kGx_k, \tag{17}$$

we obtain

$$\begin{aligned} \|x_k - V_kGx_k\| &\le \frac{\iota_k}{1-\iota_k}\|f(x_k) - x_k\| + \|\Pi_C(I - \varpi_k F)V_kGx_k - V_kGx_k\| \\ &\le \frac{\iota_k}{1-\iota_k}\|f(x_k) - x_k\| + \|(I - \varpi_k F)V_kGx_k - V_kGx_k\| \\ &= \frac{\iota_k}{1-\iota_k}\|f(x_k) - x_k\| + \varpi_k\|F(V_kGx_k)\|, \end{aligned}$$

which together with $\lim_{k\to\infty} \iota_k = 0$, yields

$$\lim_{k\to\infty} \|x_k - V_kGx_k\| = 0. \tag{18}$$

Furthermore, from Remark 3.3 (ii), we deduce that if D is a nonempty and bounded subset of C, then, for $\varepsilon > 0$, there exists $k_0 > i$ such that for all $k > k_0$

$$\sup_{x\in D} \|V_k^i x - V_\infty^i x\| \le \varepsilon. \tag{19}$$

Taking $D = \{Gx_k : k \ge 1\}$ and setting $i = 1$, from (19) we have

$$\|V_kGx_k - VGx_k\| \le \sup_{x\in D} \|V_kGx - VGx\| \le \varepsilon,$$

which immediately imply that

$$\lim_{k\to\infty} \|V_kGx_k - VGx_k\| = 0. \tag{20}$$

Noticing that $\|x_k - VGx_k\| \le \|x_k - V_kGx_k\| + \|V_kGx_k - VGx_k\|$, from (18) and (20) we get

$$\lim_{k\to\infty} \|x_k - (V \circ G)x_k\| = 0. \tag{21}$$

Now, define $g : C \to [0, \infty)$ by

$$g(x) = \mathrm{LIM}_k\|x_k - x\|^2, \quad \forall x \in C,$$

where LIM is a Banach limit on l^∞. Let

$$K = \{x \in C : g(x) = \min_{y\in C} \mathrm{LIM}_k\|x_k - y\|^2\}.$$

It is clear that the nonexpansivity of V_k implies the one of V, which together with the one of G, leads to the one of VG. Also, it is easily seen that K is a nonempty closed convex bounded subset of C. Since (note that $\|x_k - VGx_k\| \to 0$)

$$\begin{aligned} g(VGx) &= \mathrm{LIM}_k\|x_k - VGx\|^2 \\ &= \mathrm{LIM}_k\|VGx_k - VGx\|^2 \\ &\le \mathrm{LIM}_k\|x_k - x\|^2 = g(x), \end{aligned}$$

it follows that $(V \circ G)(K) \subset K$; that is, K is invariant under $V \circ G$. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, $V \circ G$ has a fixed point, say z, in K. By Proposition 3.5, we get $z \in (\mathrm{Fix})(\mathrm{V} \circ \mathrm{G}) = \mathcal{F}$. Since z is also a minimizer of g over C, it follows that for $t \in (0, 1)$ and $x \in C$,

$$0 \le \frac{g(z + t(x - z)) - g(z)}{t} = \mathrm{LIM}_k \frac{\|(x_k - z) + t(z - x)\|^2 - \|x_k - z\|^2}{t}.$$

The uniform smoothness of X implies that the duality map $j(\cdot)$ is norm-to-norm uniformly continuous on bounded sets of X. Letting $t \to 0$, we find that the two limits above can be interchanged and obtain

$$\mathrm{LIM}_k\langle x - z, j(x_k - z)\rangle \le 0, \quad \forall x \in C. \tag{22}$$

Since $x_k - z = \iota_k(f(x_k) - z) + (1 - \iota_k)(\Pi_C(I - \varpi_k F)V_k Gx_k - z)$, we have

$$\begin{aligned} \|x_k - z\|^2 &= \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)\langle \Pi_C(I - \varpi_k F)V_k Gx_k - z, j(x_k - z)\rangle \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)\|\Pi_C(I - \varpi_k F)V_k Gx_k - z\|\|x_k - z\| \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)(\|\Pi_C(I - \varpi_k F)V_k Gx_k - \Pi_C(I - \varpi_k F)z\| + \|\Pi_C(I - \varpi_k F)z - z\|)\|x_k - z\| \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)(\|(I - \varpi_k F)V_k Gx_k - (I - \varpi_k F)z\| + \|(I - \varpi_k F)z - z\|)\|x_k - z\| \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)((1 - \varpi_k\tau)\|V_k Gx_k - z\| + \varpi_k\|F(z)\|)\|x_k - z\| \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)(\|x_k - z\| + \varpi_k\|F(z)\|)\|x_k - z\| \\ &\le \iota_k\langle f(x_k) - z, j(x_k - z)\rangle + (1 - \iota_k)\|x_k - z\|^2 + \varpi_k\|F(z)\|\|x_k - z\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_k - z\|^2 &\le \langle f(x_k) - z, j(x_k - z)\rangle + \frac{\varpi_k}{\iota_k}\|F(z)\|\|x_k - z\| \\ &= \langle f(x_k) - x, j(x_k - z)\rangle + \langle x - z, j(x_k - z)\rangle + \frac{\varpi_k}{\iota_k}\|F(z)\|\|x_k - z\|. \end{aligned} \tag{23}$$

So by (22), for $x \in C$,

$$\begin{aligned} \mathrm{LIM}_k\|x_k - z\|^2 &\le \mathrm{LIM}_k\langle f(x_k) - x, j(x_k - z)\rangle + \mathrm{LIM}_k\langle x - z, j(x_k - z)\rangle \\ &\le \mathrm{LIM}_k\langle f(x_k) - x, j(x_k - z)\rangle \\ &\le \mathrm{LIM}_k\|f(x_k) - x\|\|x_k - z\|. \end{aligned}$$

In particular,

$$\mathrm{LIM}_k\|x_k - z\|^2 \le \mathrm{LIM}_k\|f(x_k) - f(z)\|\|x_k - z\| \le \rho\|x_k - z\|^2.$$

thus,

$$\mathrm{LIM}_k\|x_k - z\|^2 = 0,$$

and there exists a subsequence which is denoted by $\{x_{k_i}\}$ such that $x_{k_i} \to z$ as $i \to \infty$.

Now, assume that there exists another subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that $x_{k_m} \to \bar{z} \in \mathrm{Fix}(V \circ G) = \mathcal{F}$ (due to (21) and Proposition 3.5). It follows from (23) that

$$\|\bar{z} - z\|^2 \le \langle f(\bar{z}) - z, j(\bar{z} - z)\rangle. \tag{24}$$

Interchange $\bar{z}$ and z to obtain

$$\|z - \bar{z}\|^2 \le \langle f(z) - \bar{z}, j(z - \bar{z})\rangle. \tag{25}$$

Adding up (24) and (25) yields

$$\begin{aligned} 2\|\bar{z} - z\|^2 &\le \langle f(\bar{z}) - f(z), j(\bar{z} - z)\rangle + \langle \bar{z} - z, j(\bar{z} - z)\rangle \\ &\le (1 + \rho)\|\bar{z} - z\|^2. \end{aligned}$$

Since $\rho \in (0, 1)$, this implies that $\bar{z} = z$. Therefore, $x_k \to z$ as $k \to \infty$.

Define $Q : \Xi_C \to \mathcal{F}$ by

$$Q(f) := s\text{-}\lim_{k\to\infty} x_k. \tag{26}$$

Since $x_k = \iota_k f(x_k) + (1 - \iota_k)\Pi_C(I - \varpi_k F)V_k G x_k$, we have

$$(I - f)x_k = -\frac{1 - \iota_k}{\iota_k}(x_k - V_k G x_k + V_k G x_k - \Pi_C(I - \varpi_k F)V_k G x_k). \tag{27}$$

Hence, for $p \in \mathcal{F}$,

$$\begin{aligned} \langle (I - f)x_k, j(x_k - p)\rangle &= -\frac{1 - \iota_k}{\iota_k}\langle x_k - V_k G x_k, j(x_k - p)\rangle - \frac{1 - \iota_k}{\iota_k}\langle V_k G x_k - \Pi_C(I - \varpi_k F)V_k G x_k, j(x_k - p)\rangle \\ &\le \frac{1 - \iota_k}{\iota_k}\|V_k G x_k - \Pi_C(I - \varpi_k F)V_k G x_k\|\|x_k - p\| \\ &\le \frac{1 - \iota_k}{\iota_k}\|V_k G x_k - (I - \varpi_k F)V_k G x_k\|\|x_k - p\| \\ &= \frac{\varpi_k}{\iota_k}F(V_k G x_k)\|\|x_k - p\|. \end{aligned} \tag{28}$$

Because $\varpi_k/\iota_k \to 0$ and $x_k \to Q(f)$ as $k \to \infty$, taking the limit as $k \to \infty$ in (28), we obtain that

$$\langle (I - f)Q(f), j(Q(f) - p)\rangle \le 0. \tag{29}$$

If $f(x) = u \quad (\forall x \in C)$ is a constant, then

$$\langle Q(u) - u, j(Q(u) - p)\rangle \le 0. \tag{30}$$

Hence, Q reduces to the sunny nonexpansive retraction from C to $\mathcal{F}$. □

Theorem 3.9. *Let C be a nonempty closed convex subset of a real strictly convex and 2-uniformly smooth Banach space X with weakly sequentially continuous duality mapping j. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $0 \le \varpi \le \frac{\nu}{\kappa^2}$ and $0 \le \varrho \le \frac{\vartheta}{\kappa^2}$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (10). Let $\{x_k\}_{k=1}^\infty$ be generated in the implicit manner*

$$x_k = \Pi_C(I - \varpi_k F)V_k \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)x_k, \quad \forall k \ge 1, \tag{31}$$

where $\{\varpi_k\}_{k=1}^\infty \subset (0, 1)$ such that $\lim_{k\to\infty} \varpi_k = 0$. Then $\{x_k\}_{k=1}^\infty$ converges in norm, as $k \to \infty$, to the unique solution of the following VI

$$\text{find } \tilde{x} \in \mathcal{F} : \langle F(\tilde{x}), j(x - \tilde{x})\rangle \le 0, \quad \forall x \in \mathcal{F}. \tag{32}$$

Proof. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$ where $0 < \varpi \le \frac{\nu}{\kappa^2}$ and $0 < \varrho \le \frac{\vartheta}{\kappa^2}$. Note that G is a nonexpansive mapping on C. Then, the implicit iterative scheme (31) can be rewritten as

$$x_k = \Pi_C(I - \varpi_k F)V_k G x_k, \quad \forall k \ge 1. \tag{33}$$

Consider the mapping $U_k x = \Pi_C(I - \varpi_k F)V_k G x, \forall x \in C$. By Proposition 2.4 and Lemma 8, we know that $\Pi_C(I - \varpi A)$ and $\Pi_C(I - \varrho B)$ are nonexpansive, and $I - \varpi_k F$ is contractive with coefficient $1 - \iota_k(1 - \sqrt{\frac{1-\delta}{\zeta}})$. Hence,

$$\|U_k x - U_k y\| \le \|(I - \varpi_k F)V_k G x - (I - \varpi_k F)V_k G y\| \le (1 - \varpi_k \tau)\|x - y\|,$$

for all $x, y \in C$, where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$. This means that U_k is a contraction. Therefore, the Banach contraction principle guarantees that U_k has a unique fixed point in C, which we denote by x_k. This shows that the implicit scheme (33) is well defined.

Now, we show that $\{x_k\}_{k=1}^{\infty}$ is bounded. As a matter of fact, take an arbitrarily given $p \in \mathcal{F}$. Then $V_k p = p$ and $Gp = p$. So, it follows from (33) and Lemma 2.11 that

$$\begin{aligned}
\|x_k - p\| &= \|\Pi_C(I - \varpi_k F)V_k G x_k - \Pi_C(I - \varpi_k F)p + \Pi_C(I - \varpi_k F)p - p\| \\
&\le \|\Pi_C(I - \varpi_k F)V_k G x_k - \Pi_C(I - \varpi_k F)p\| + \|\Pi_C(I - \varpi_k F)p - p\| \\
&\le \|(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)p\| + \|(I - \varpi_k F)p - p\| \\
&\le (1 - \varpi_k \tau)\|V_k G x_k - p\| + \varpi_k\|F(p)\| \\
&\le (1 - \varpi_k \tau)\|x_k - p\| + \varpi_k\|F(p)\|,
\end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$. Thus, it immediately follows that

$$\|x_k - p\| \le \frac{1}{\tau}\|F(p)\|.$$

Therefore, $\{x_k\}_{k=1}^{\infty}$ is bounded and so are the sequences $\{Gx_k\}_{k=1}^{\infty}$, $\{V_k G x_k\}_{k=1}^{\infty}$ and $\{F(V_k G x_k)\}_{k=1}^{\infty}$. Furthermore, by the nonexpansivity of V_k and G, we know that $V_k G : C \to C$ is nonexpansive. Thus,

$$\|x_k - V_k G x_k\| = \|\Pi_C(I - \varpi_k F)V_k G x_k - V_k G x_k\| \le \varpi_k\|F(V_k G x_k) \to 0$$

as $k \to \infty$. That is,

$$\lim_{k\to\infty} \|x_k - V_k G x_k\| = 0. \tag{34}$$

Repeating the same arguments as those of (20) in the proof of Theorem 3.8, we have

$$\lim_{k\to\infty} \|V_k G x_k - V G x_k\| = 0.$$

Noticing that $\|x_k - VGx_k\| \le \|x_k - V_k G x_k\| + \|V_k G x_k - VGx_k\|$, from (34) we get

$$\lim_{k\to\infty} \|x_k - (V \circ G)x_k\| = 0. \tag{35}$$

We can rewrite (33) as

$$x_k = \Pi_C(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)V_k G x_k + (I - \varpi_k F)V_k G x_k.$$

for any $p \in \mathcal{F} \subset C$, by Lemma 2.2(iii), we have

$$\langle x_k - (I - \varpi_k F)V_k G x_k, j(x_k - p)\rangle = \langle \Pi_C(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)V_k G x_k, j(\Pi_C(I - \varpi_k F)V_k G x_k - p)\rangle \le 0.$$

According to this fact, we deduce that

$$\begin{aligned}\|x_k - p\|^2 &= \langle x_k - p, j(x_k - p)\rangle \\ &= \langle x_k - (I - \varpi_k F)V_k G x_k, j(x_k - p)\rangle + \langle (I - \varpi_k F)V_k G x_k - p, j(x_k - p)\rangle \\ &\le \langle (I - \varpi_k F)V_k G x_k - p, j(x_k - p)\rangle \\ &= \langle (I - \varpi_k F)V_k G x_k - (I - \varpi_k F)p, j(x_k - p)\rangle - \varpi_k \langle F(p), j(x_k - p)\rangle \\ &\le (1 - \varpi_k \tau)\|V_k G x_k - p\|\|x_k - p\| - \varpi_k \langle F(p), j(x_k - p)\rangle \\ &\le (1 - \varpi_k \tau)\|x_k - p\|^2 - \varpi_k \langle F(p), j(x_k - p)\rangle.\end{aligned}$$

It turns out that

$$\|x_k - p\|^2 \le \frac{1}{\tau}\langle F(p), j(p - x_k)\rangle, \quad \forall p \in \mathcal{F}. \tag{36}$$

Since X is reflexive and $\{x_k\}_{k=1}^{\infty}$ is bounded, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup \tilde{x} \in C$. Noticing (35), we can use Lemma 2.8 to get $\tilde{x} \in \mathrm{Fix}(V \circ G) = \mathcal{F}$. Therefore, we can substitute $\tilde{x}$ for p in (36) to get

$$\|x_k - \tilde{x}\|^2 \le \frac{1}{\tau}\langle F(\tilde{x}), j(\tilde{x} - x_k)\rangle, \quad \forall p \in \mathcal{F}, \tag{37}$$

which together with the weakly sequential continuity of j implies that

$$\lim_{i\to\infty} \|x_{k_i} - \tilde{x}\| = 0.$$

We also show that $\tilde{x}$ solves the VI (32). From (33), we have

$$\begin{aligned} & x_k = \Pi_C(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)V_k G x_k + (I - \varpi_k F)V_k G x_k \\ & \Rightarrow \quad x_k = \Pi_C(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)V_k G x_k - ((I - \varpi_k F)x_k - (I - \varpi_k F)V_k G x_k) + x_k - \varpi_k F(x_k) \\ & \Rightarrow \quad F(x_k) = \frac{1}{\varpi_k}[\Pi_C(I - \varpi_k F)V_k G x_k - (I - \varpi_k F)V_k G x_k - ((I - \varpi_k F)x_k - (I - \varpi_k F)V_k G x_k)].\end{aligned}$$

For any $z \in \mathcal{F}$, utilizing the nonexpansivity of $V_k G$, we obtain that

$$\langle x_k - V_k G x_k, j(x_k - z)\rangle = \langle (I - V_k G)x_k - (I - V_k G)z, j(x_k - z)\rangle \ge 0,$$

and hence,

$$\begin{aligned}\langle F(x_k), j(x_k - z)\rangle &\le -\frac{1}{\varpi_k}\langle (I - \varpi_k F)x_k - (I - \varpi_k F)V_k G x_k, j(x_k - z)\rangle \\ &= -\frac{1}{\varpi_k}\langle x_k - V_k G x_k, j(x_k - z)\rangle + \langle F(x_k) - F(V_k G x_k), j(x_k - z)\rangle \\ &\le \langle F(x_k) - F(V_k G x_k), j(x_k - z)\rangle.\end{aligned}$$

Therefore,

$$\langle F(x_k), j(x_k - z)\rangle \le \langle F(x_k) - F(V_k G x_k), j(x_k - z)\rangle. \tag{38}$$

Since F is δ-strongly accretive, we have

$$0 \le \delta\|x_k - z\|^2 \le \langle F(x_k) - F(z), j(x_k - z)\rangle.$$

It follows that

$$\langle F(z), j(x_k - z) \le \langle F(x_k), j(x_k - z)\rangle. \tag{39}$$

Combining (38) and (39), we get

$$\langle F(z), j(x_k - z) \le \langle F(x_k) - F(V_k G x_k), j(x_k - z)\rangle. \tag{40}$$

Now, replacing k in (40) with k_i and $i \to \infty$, noticing that $x_{k_i} \to \tilde{x}$ and $x_{k_i} - V_{k_i} G x_{k_i} \to 0$ (due to (34)) as $i \to \infty$, we derive

$$\langle F(z), j(\tilde{x} - z)\rangle \le 0, \quad \forall z \in \mathcal{F},$$

which is equivalent to its dual variational inequality (due to Lemma 2.10)

$$\langle F(\tilde{x}), j(\tilde{x} - z) \le 0, \quad \forall z \in \mathcal{F}. \tag{41}$$

That is, $\tilde{x} \in \mathcal{F}$ is a solution of VI (32). Now, we show that the solution set of VI (32) is a singleton. As a matter of fact, we assume that $\bar{x} \in \mathcal{F}$ is another solution of VI (32). Then, we have

$$\langle F(\bar{x}), j(\bar{x} - \tilde{x})\rangle \le 0.$$

From (41), we have

$$\langle F(\tilde{x}), j(\tilde{x} - \bar{x})\rangle \le 0.$$

So,

$$\langle F(\bar{x}), j(\bar{x} - \tilde{x})\rangle + \langle F(\tilde{x}), j(\tilde{x} - \bar{x})\rangle \le 0.$$

It follows that $\delta\|\bar{x} - \tilde{x}\|^2 \le 0$. Therefore, $\bar{x} = \tilde{x}$. In summary, we have shown that each (strong) cluster point of the sequence $\{x_k\}_{k=1}^\infty$ (as $k \to \infty$) equals to $\tilde{x}$. Therefore, $x_k \to \tilde{x}$ as $k \to \infty$. This completes the proof. □

4. Explicit Iterative Methods

In this section, we propose explicit iterative algorithms for solving a general system of variational inequalities (GSVI) with a hierarchical fixed point problem (HFPP) constraint for an infinite family of nonexpansive mappings, and derive the strong convergence of the sequences generated by the proposed algorithms to a unique solution of the HFPP.

Algorithm 4.1. *Let C be a nonempty closed convex subset of a real strictly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let $f \in \Xi_C$ with coefficient $\rho \in (0, 1)$, and let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta \ge 1$. Assume that $\varpi \in (0, \frac{\nu}{\kappa^2}]$ and $\varrho \in (0, \frac{\vartheta}{\kappa^2}]$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \ne \emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (10). For an arbitrarily given $z_1 \in C$, let the sequence $\{z_k\}_{k=1}^\infty$ be generated iteratively by*

$$z_{k+1} = \vartheta_k f(z_k) + (1 - \vartheta_k)\Pi_C(I - \varpi_k F)V_k \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)z_k, \quad \forall k \ge 1, \tag{42}$$

where $\{\varpi_k\}_{k=1}^\infty \subset [0, 1)$ and $\{\vartheta_k\}_{k=1}^\infty \subset (0, 1)$ and ϖ, ϱ are two positive numbers.

Theorem 4.2. *Assume that*

(i) $\lim_{k\to\infty} \vartheta_k = 0$ *and* $\sum_{k=1}^\infty \vartheta_k = \infty$;
(ii) $\lim_{k\to\infty} \varpi_k/\vartheta_k = 0$;
(iii) $\sum_{k=1}^\infty |\vartheta_{k+1} - \vartheta_k| < \infty$ *or* $\lim_{k\to\infty} \vartheta_k/\vartheta_{k+1} = 1$;
(iv) $\sum_{k=1}^\infty |\varpi_{k+1} - \varpi_k| < \infty$ *or* $\lim_{k\to\infty} |\varpi_{k+1} - \varpi_k|/\vartheta_{k+1} = 1$.

Then the sequence $\{z_k\}_{k=1}^{\infty}$ *generated by scheme* (42) *converges strongly to* $\tilde{x} \in \mathcal{F}$, *where* $\tilde{x} = Q(f)$ *is given in* (26), *provided* $\lim_{n\to\infty}(\sum_{k=n}^{\infty} \nu_{k+1})/\iota_n = 0$.

Proof. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$ where $0 < \varpi \le \frac{\nu}{\kappa^2}$ and $0 < \varrho \le \frac{\vartheta}{\kappa^2}$. In terms of Lemma 2.5 we know that G is a nonexpansive mapping on C. Then the scheme (42) is rewritten as

$$z_{k+1} = \vartheta_k f(z_k) + (1 - \vartheta_k)\Pi_C(I - \varpi_k F)V_k G z_k, \quad \forall k \ge 1. \tag{43}$$

It can be readily seen that $\Pi_C(I - \varpi_k F)$ and V_k both are nonexpansive mappings on C. Take an arbitrarily given $p \in \mathcal{F}$. Then we have $Gp = p$ and $V_k p = p$ for all $k \ge 1$. Thus, we deduce that

$$\begin{aligned}
\|z_{k+1} - p\| &= \|\vartheta_k(f(z_k) - p) + (1 - \vartheta_k)(\Pi_C(I - \varpi_k F)V_k G z_k - p)\| \\
&\le \vartheta_k\|f(z_k) - p\| + (1 - \vartheta_k)\|\Pi_C(I - \varpi_k F)V_k G z_k - p\| \\
&\le (1 - \vartheta_k)(\|\Pi_C(I - \varpi_k F)V_k G z_k - \Pi_C(I - \varpi_k F)p\| + \|\Pi_C(I - \varpi_k F)p - p\|) \\
&\quad + \vartheta_k(\|f(z_k) - f(p)\| + \|f(p) - p\|) \\
&\le \vartheta_k\rho\|z_k - p\| + \vartheta_k\|f(p) - p\|) + (1 - \vartheta_k)(\|z_k - p\| + \|(I - \varpi_k F)p - p\|) \\
&\le (1 - \vartheta_k(1 - \rho))\|z_k - p\| + \vartheta_k\|f(p) - p\| + \varpi_k\|F(p)\|.
\end{aligned} \tag{44}$$

Because $\lim_{k\to\infty} \varpi_k/\vartheta_k = 0$, we may assume without loss of generality that $\varpi_k \le \vartheta_k$ for all $k \ge 1$. Hence, from (44), we get

$$\|z_{k+1} - p\| \le (1 - \vartheta_k(1 - \rho))\|z_k - p\| + \vartheta_k(\|f(p) - p\| + \|F(p)\|), \quad \forall k \ge 1.$$

By induction, we conclude that

$$\|z_k - p\| \le \max\{\|z_1 - p\|, \frac{\|f(p) - p\| + \|F(p)\|}{1 - \rho}\}, \quad \forall k \ge 1. \tag{45}$$

Therefore, $\{z_k\}$ is bounded, so are the sequences $\{f(z_k)\}, \{Gz_k\}, \{VGz_k\}$ and $\{F(VGz_k)\}$. Also, from (43), we have

$$\begin{aligned}
\|z_{k+1} - V_k G z_k\| &\le \vartheta_k\|f(z_k) - V_k G z_k\| + (1 - \vartheta_k)\|\Pi_C(I - \varpi_k F)V_k G z_k - V_k G z_k\| \\
&\le \vartheta_k\|f(z_k) - V_k G z_k\| + (1 - \vartheta_k)\|(I - \varpi_k F)V_k G z_k - V_k G z_k\| \\
&= \vartheta_k\|f(z_k) - V_k G z_k\| + (1 - \vartheta_k)\varpi_k\|F(V_k G z_k)\| \\
&\le \vartheta_k\|f(z_k) - V_k G z_k\| + \varpi_k\|F(V_k G z_k)\|,
\end{aligned}$$

which together with $\vartheta_k \to 0$ and $\varpi_k \to 0$, implies that

$$\lim_{k\to\infty} \|z_{k+1} - V_k G z_k\| = 0. \tag{46}$$

Now, we note that

$$\begin{aligned}
\|\Pi_C(I - \varpi_k F)V_k G z_{k-1} - \Pi_C(I - \varpi_{k-1} F)V_{k-1} G z_{k-1}\| &\le \|\Pi_C(I - \varpi_k F)V_k G z_{k-1} - \Pi_C(I - \varpi_k F)V_{k-1} G z_{k-1}\| \\
&\quad + \|\Pi_C(I - \varpi_k F)V_{k-1} G z_{k-1} - \Pi_C(I - \varpi_{k-1} F)V_{k-1} G z_{k-1}\| \\
&\le \|V_k G z_{k-1} - V_{k-1} G z_{k-1}\| + |\varpi_k - \varpi_{k-1}|\|F(V_{k-1} G z_{k-1})\| \\
&\le \nu_k\|G z_{k-1} - T_k G z_{k-1}\| + |\varpi_k - \varpi_{k-1}|\|F(V_{k-1} G z_{k-1})\|.
\end{aligned}$$

Simple calculation shows that

$$\begin{aligned}
z_{k+1} - z_k &= \vartheta_k f(z_k) + (1 - \vartheta_k)\Pi_C(I - \varpi_k F)V_k G z_k - \vartheta_{k-1} f(z_{k-1}) - (1 - \vartheta_{k-1})\Pi_C(I - \varpi_{k-1} F)V_{k-1} G z_{k-1} \\
&= (\vartheta_k - \vartheta_{k-1})(f(z_{k-1}) - \Pi_C(I - \varpi_{k-1} F)V_{k-1} G z_{k-1}) + \vartheta_k(f(z_k) - f(z_{k-1})) \\
&\quad + (1 - \vartheta_k)(\Pi_C(I - \varpi_k F)V_k G z_k - \Pi_C(I - \varpi_k F)V_k G z_{k-1}) \\
&\quad + (1 - \vartheta_k)(\Pi_C(I - \varpi_k F)V_k G z_{k-1} - \Pi_C(I - \varpi_{k-1} F)V_{k-1} G z_{k-1}),
\end{aligned}$$

which together with the last inequality, implies that

$$\begin{aligned}\|z_{k+1}-z_k\| &\le |\vartheta_k-\vartheta_{k-1}|\|f(z_{k-1})-\Pi_C(I-\varpi_{k-1}F)V_{k-1}Gz_{k-1}\|+\vartheta_k\|f(z_k)-f(z_{k-1})\| \\ &\quad+(1-\vartheta_k)\|\Pi_C(I-\varpi_kF)V_kGz_k-\Pi_C(I-\varpi_kF)V_kGz_{k-1}\| \\ &\quad+(1-\vartheta_k)\|\Pi_C(I-\varpi_kF)V_kGz_{k-1}-\Pi_C(I-\varpi_{k-1}F)V_{k-1}Gz_{k-1}\| \\ &\le |\vartheta_k-\vartheta_{k-1}|\|f(x_{k-1})-\Pi_C(I-\varpi_{k-1}F)V_{k-1}Gz_{k-1}\|+\vartheta_k\rho\|z_k-z_{k-1}\| \\ &\quad+(1-\vartheta_k)\|z_k-z_{k-1}\|+\|\Pi_C(I-\varpi_kF)V_kGz_{k-1}-\Pi_C(I-\varpi_{k-1}F)V_{k-1}Gz_{k-1}\| \\ &\le |\vartheta_k-\vartheta_{k-1}|\|f(z_{k-1})-\Pi_C(I-\varpi_{k-1}F)V_{k-1}Gz_{k-1}\|+\vartheta_k\rho\|z_k-z_{k-1}\| \\ &\quad+(1-\vartheta_k)\|z_k-z_{k-1}\|+\nu_k\|Gz_{k-1}-T_kGz_{k-1}\|+|\varpi_k-\varpi_{k-1}|\|F(V_{k-1}Gz_{k-1})\| \\ &\le |\vartheta_k-\vartheta_{k-1}|(\|f(z_{k-1})\|+\|V_{k-1}Gz_{k-1}\|+\|F(V_{k-1}Gz_{k-1})\|)+\vartheta_k\rho\|z_k-z_{k-1}\| \\ &\quad+(1-\vartheta_k)\|z_k-z_{k-1}\|+\nu_k(\|Gz_{k-1}\|+\|T_kGz_{k-1}\|)+|\varpi_k-\varpi_{k-1}|\|F(V_{k-1}Gz_{k-1})\| \\ &\le |\vartheta_k-\vartheta_{k-1}|M+\vartheta_k\rho\|z_k-z_{k-1}\|+(1-\vartheta_k)\|z_k-z_{k-1}\|+\nu_kM+\||\varpi_k-\varpi_{k-1}|M \\ &= (1-\vartheta_k(1-\rho))\|z_k-z_{k-1}\|+M(|\vartheta_k-\vartheta_{k-1}|+|\varpi_k-\varpi_{k-1}|)+\nu_kM,\end{aligned}$$

where $\sup_{k\ge1}\{\|f(z_k)\|+\|V_kGz_k\|+\|F(V_kGz_k)\|+\|Gz_k\|+\|T_{k+1}Gz_k\|\}\le M$ for some $M>0$. So, utilizing Lemma 3.7, we obtain that

$$\lim_{k\to\infty}\|z_{k+1}-z_k\|=0.$$

This together with (46) implies that

$$\lim_{k\to\infty}\|z_k-V_kGz_k\|=0.$$

From (19) in the proof of Theorem 3.8, we deduce that for any nonempty and bounded subset D of C,

$$\lim_{k\to\infty}\sup_{x\in D}\|V_kx-Vx\|=0.$$

Taking $D=\{Gz_k:k\ge1\}$, we have

$$\|V_kGz_k-VGz_k\|\le\sup_{x\in D}\|V_kx-Vx\| \quad\text{and}\quad \|V_nGz_k-VGz_k\|\le\sup_{x\in D}\|V_nx-Vx\|,\quad \forall k,n\ge1.$$

So, it follows that

$$\lim_{k\to\infty}\|V_kGz_k-VGz_k\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|V_nGz_k-VGz_k\|=0,\quad \forall k\ge1. \tag{47}$$

Noticing that $\|z_k-VGz_k\|\le\|z_k-V_kGz_k\|+\|V_kGz_k-VGz_k\|$, we get

$$\lim_{k\to\infty}\|z_k-(V\circ G)z_k\|=0. \tag{48}$$

Let us show that

$$\limsup_{k\to\infty}\langle\tilde{x}-f(\tilde{x}),j(\tilde{x}-z_k)\rangle\le0, \tag{49}$$

where $\tilde{x}=Q(f)$. Indeed, in terms of (16), we can write

$$x_n-z_k=\iota_n(f(x_n)-z_k)+(1-\iota_n)(\Pi_C(I-\varpi_nF)V_nGx_n-z_k).$$

Putting

$$a_k(n)=(\|V_nGz_k-z_k\|+\varpi_n\|F(V_nGz_k)\|)[2\|x_n-z_k\|+\|V_nGz_k-z_k\|+\varpi_n\|F(V_nGz_k)\|],$$

and using Lemma 2.7 (i), we obtain

$$\begin{aligned}\|x_n - z_k\|^2 &\le (1-\iota_n)^2\|\Pi_C(I-\varpi_n F)V_nGx_n - z_k\|^2 + 2\iota_n\langle f(x_n) - z_k, j(x_n - z_k)\rangle \\ &\le (1-\iota_n)^2(\|\Pi_C(I-\varpi_n F)V_nGx_n - \Pi_C(I-\varpi_n F)V_nGz_k\| \\ &\quad + \|\Pi_C(I-\varpi_n F)V_nGz_k - z_k\|)^2 + 2\iota_n\langle f(x_n) - x_n, j(x_n - z_k)\rangle + 2\iota_n\|x_n - z_k\|^2 \\ &\le (1-\iota_n)^2(\|x_n - z_k\| + \|\Pi_C(I-\varpi_n F)V_nGx_z - z_k\|)^2 \\ &\quad + 2\iota_n\langle f(x_n) - x_n, j(x_n - z_k)\rangle + 2\iota_n\|x_n - z_k\|^2 \\ &\le (1-\iota_n)^2(\|x_n - z_k\| + \|V_nGz_k - z_k\| + \varpi_n\|F(V_nGz_k)\|)^2 \\ &\quad + 2\iota_n\langle f(x_n) - x_n, j(x_n - z_k)\rangle + 2\iota_n\|x_n - z_k\|^2 \\ &\le (1-\iota_n)^2\|x_n - z_k\|^2 + a_k(n) + 2\iota_n\langle f(x_n) - x_n, j(x_n - z_k)\rangle + 2\iota_n\|x_n - z_k\|^2.\end{aligned}$$

The last inequality implies that

$$\langle x_n - f(x_n), j(x_n - z_k)\rangle \le \frac{\iota_n}{2}\|x_n - z_k\|^2 + \frac{1}{2\iota_n}a_k(n). \tag{50}$$

In terms of (48), we get

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{2\iota_n}a_k(n) &= \limsup_{k\to\infty}\frac{1}{2\iota_n}(\|V_nGz_k - z_k\| + \varpi_n\|F(V_nGz_k)\|)[2\|x_n - z_k\| + \|V_nGz_k - z_k\| + \varpi_n\|F(V_nGz_k)\|] \\ &\le \limsup_{k\to\infty}\frac{1}{2\iota_n}(\|V_nGz_k - VGz_k\| + \|VGz_k - z_k\| + \varpi_n\|F(V_nGz_k)\|)[2\|x_n - z_k\| \\ &\quad + \|V_nGz_k - VGz_k\| + \|VGz_k - z_k\| + \|F(V_nGz_k)\|] \\ &= \limsup_{k\to\infty}\frac{1}{2\iota_n}(\|V_nGz_k - VGz_k\| + \varpi_n\|F(V_nGz_k)\|)[2\|x_n - z_k\| \\ &\quad + \|V_nGz_k - VGz_k\| + \|F(V_nGz_k)\|].\end{aligned} \tag{51}$$

Since for $p \in \mathcal{F}$,

$$\begin{aligned}\|V_nGz_k - VGz_k\| &\le \sum_{l=n}^{\infty}\|V_{l+1}Gz_k - V_lGz_k\| \\ &\le \sum_{l=n}^{\infty}(2\nu_{l+1}\|Gz_k - p\|) \\ &\le 2\|z_k - p\|\sum_{l=n}^{\infty}\nu_{l+1},\end{aligned}$$

we conclude from (51) that

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{2\iota_n}a_k(n) &\le \limsup_{k\to\infty}\frac{1}{2\iota_n}(\|V_nGz_k - VGz_k\| + \varpi_n\|F(V_nGz_k)\|)[2\|x_n - z_k\| \\ &\quad + \|V_nGz_k - VGz_k\| + \|F(V_nGz_k)\|] \\ &\le \limsup_{k\to\infty}\frac{1}{2\iota_n}(2\|z_k - p\|\sum_{l=n}^{\infty}\nu_{l+1} + \varpi_n\|F(V_nGz_k)\|)[2\|x_n - p\| \\ &\quad + 2\|z_k - p\| + \|V_nGz_k - VGz_k\| + \|F(V_nGz_k)\|] \\ &\le \frac{1}{2\iota_n}(\sum_{l=n}^{\infty}\nu_{l+1} + \varpi_n)M_0^2,\end{aligned}$$

where $\sup_{k,n\geq 1}\{2\|x_n - p\| + 2\|z_k - p\| + \|V_n G z_k - VGz_k\| + \|F(V_n G z_k)\|\} \leq M_0$ for some $M_0 > 0$. So, it follows from (50) that

$$\limsup_{k\to\infty}\langle x_n - f(x_n), j(x_n - z_k)\rangle \leq \frac{\iota_n}{2}M_0^2 + \frac{1}{2\iota_n}(\sum_{l=n}^{\infty}\nu_{l+1} + \varpi_n)M_0^2. \tag{52}$$

Since $\lim_{n\to\infty}(\sum_{k=n}^{\infty}\nu_{k+1})/\iota_n = 0$ and $\lim_{n\to\infty}\varpi_n/\iota_n = 0$ (see the assumptions on the parameter sequences of Theorem 3.8, taking the lim sup as $n\to\infty$ in (52) and noticing the fact that the two limits are interchangeable due to the fact that the duality map J is norm-to-norm uniformly continuous on bounded sets of X, we obtain (49).

Finally, we show that $z_k \to \tilde{x}$ as $k\to\infty$. Indeed, we write

$$z_{k+1} - \tilde{x} = \vartheta_k(f(z_k) - \tilde{x}) + (1-\vartheta_k)(\Pi_C(I-\varpi_k F)V_k G z_k - \tilde{x}),$$

and apply Lemma 2.7 to get

$$\begin{aligned}\|z_{k+1}-\tilde{x}\|^2 &\leq (1-\vartheta_k)^2\|\Pi_C(I-\varpi_k F)V_k G z_k - \tilde{x}\|^2 + 2\vartheta_k\langle f(z_k)-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle\\
&\leq (1-\vartheta_k)^2(\|\Pi_C(I-\varpi_k F)V_k G z_k - \Pi_C(I-\varpi_k F)V_k G\tilde{x}\|\\
&\quad + \|\Pi_C(I-\varpi_k F)V_k G\tilde{x} - \tilde{x}\|)^2 + 2\vartheta_k\langle f(z_k)-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle\\
&\leq (1-\vartheta_k)^2(\|z_k-\tilde{x}\| + \|\Pi_C(I-\varpi_k F)\tilde{x}-\tilde{x}\|)^2 + 2\vartheta_k\langle f(z_k)-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle\\
&\leq (1-\vartheta_k)^2(\|z_k-\tilde{x}\| + \varpi_k\|F(\tilde{x})\|)^2 + 2\vartheta_k\langle f(z_k)-f(\tilde{x}), j(z_{k+1}-\tilde{x})\rangle + 2\vartheta_k\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle\\
&\leq (1-\vartheta_k)^2\|z_k-\tilde{x}\|^2 + \varpi_k\|F(\tilde{x})\|(2\|z_k-\tilde{x}\| + \varpi_k\|F(\tilde{x})\|)\\
&\quad + 2\vartheta_k\rho\|z_k-\tilde{x}\|\|z_{k+1}-\tilde{x}\| + 2\vartheta_k\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle\\
&\leq (1-\vartheta_k)^2\|z_k-\tilde{x}\|^2 + \varpi_k\|F(\tilde{x})\|(2\|z_k-\tilde{x}\| + \|F(\tilde{x})\|)\\
&\quad + \rho\vartheta_k(\|z_k-\tilde{x}\|^2 + \|z_{k+1}-\tilde{x}\|^2) + 2\vartheta_k\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle.\end{aligned}$$

It then follows that

$$\begin{aligned}\|z_{k+1}-\tilde{x}\|^2 &\leq \frac{1-(2-\rho)\vartheta_k+\vartheta_k^2}{1-\rho\vartheta_k}\|z_k-\tilde{x}\|^2 + \frac{\vartheta_k}{1-\rho\vartheta_k}[\frac{\varpi_k}{\vartheta_k}\|F(\tilde{x})\|(2\|z_k-\tilde{x}\| + \|F(\tilde{x})\|) + 2\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle]\\
&= (1-\frac{2(1-\rho)\vartheta_k}{1-\rho\vartheta_k})\|z_k-\tilde{x}\|^2 + \frac{2(1-\rho)\vartheta_k}{1-\rho\vartheta_k}\cdot\frac{1}{2(1-\rho)}[\frac{\varpi_k}{\vartheta_k}\|F(\tilde{x})\|(2\|z_k-\tilde{x}\| + \|F(\tilde{x})\|)\\
&\quad + \vartheta_k\|z_k-\tilde{x}\|^2 + 2\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle].\end{aligned}$$

Put $\tilde{\vartheta}_k = \frac{2(1-\rho)\vartheta_k}{1-\rho\vartheta_k}$ and

$$\tilde{\iota}_k = \frac{1}{2(1-\rho)}[\frac{\varpi_k}{\vartheta_k}\|F(\tilde{x})\|(2\|z_k-\tilde{x}\| + \|F(\tilde{x})\|) + \vartheta_k\|z_k-\tilde{x}\|^2 + 2\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle].$$

It follows that

$$\|z_{k+1}-\tilde{x}\|^2 \leq (1-\tilde{\vartheta}_k)\|z_k-\tilde{x}\|^2 + \tilde{\vartheta}_k\tilde{\iota}_k. \tag{53}$$

Observe that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x})\rangle &= \limsup_{k\to\infty}(\langle f(\tilde{x})-\tilde{x}, j(z_k-\tilde{x})\rangle + \langle f(\tilde{x})-\tilde{x}, j(z_{k+1}-\tilde{x}) - j(z_k-\tilde{x})\rangle)\\
&= \limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x}, j(z_k-\tilde{x})\rangle \leq 0\end{aligned}$$

due to (49). It is easily seen from conditions (i), (ii) that

$$\tilde{\vartheta}_k \to 0, \quad \sum_{k=1}^{\infty}\tilde{\vartheta}_k = \infty \quad \text{and} \quad \limsup_{k\to\infty}\tilde{\iota}_k \leq 0.$$

Finally, apply Lemma 3.7 to (53) to conclude that $z_k \to \tilde{x}$ as $k\to\infty$. □

Algorithm 4.3. *Let C be a nonempty closed convex subset of a real strictly convex and 2-uniformly smooth Banach space X with weakly sequentially continuous duality mapping j. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be ν-inverse-strongly accretive and ϑ-inverse-strongly accretive, respectively. Let $F : C \to X$ be δ-strongly accretive and ζ-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $0 < \varpi \le \frac{\nu}{\kappa^2}$ and $0 < \varrho \le \frac{\vartheta}{\kappa^2}$ where κ is the 2-uniformly smooth constant of X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, A, B) \neq \emptyset$. Let $\{V_k\}_{k=1}^{\infty}$ be defined by (10). For an arbitrarily given $z_1 \in C$, let the sequence $\{z_k\}_{k=1}^{\infty}$ be generated iteratively by*

$$z_{k+1} = \vartheta_k z_k + (1 - \vartheta_k)\Pi_C(I - \varpi_k F)V_k \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)z_k, \quad \forall k \ge 1, \tag{54}$$

where $\{\varpi_k\}_{k=1}^{\infty}$ and $\{\vartheta_k\}_{k=1}^{\infty}$ are two sequences in $[0, 1]$ and ϖ, ϱ are two positive numbers.

Theorem 4.4. *Assume that the sequences $\{\varpi_k\}_{k=1}^{\infty}$ and $\{\vartheta_k\}_{k=1}^{\infty}$ satisfy the following conditions:*

(i) $\lim_{k\to\infty} \varpi_k = 0$ *and* $\sum_{k=1}^{\infty} \varpi_k = \infty$;

(ii) $0 < \liminf_{k\to\infty} \vartheta_k \le \limsup_{k\to\infty} \vartheta_k < 1$.

Then the sequence $\{z_k\}_{k=1}^{\infty}$ defined by (54) converges strongly to the unique solution $\tilde{x} \in \mathcal{F}$ of VI (32).

Proof. Let the mapping $G : C \to C$ be defined as $G := \Pi_C(I - \varpi A)\Pi_C(I - \varrho B)$, where $0 < \varpi \le \frac{\nu}{\kappa^2}$ and $0 < \varrho \le \frac{\vartheta}{\kappa^2}$. In terms of Lemma 2.5 we know that G is a nonexpansive mapping on C. Take an arbitrarily given $p \in \mathcal{F}$. Then $Gp = p$ and $V_k p = p$ for all $k \ge 1$. By Lemma 2.11, we have

$$\begin{aligned}
\|z_{k+1} - p\| &= \|\vartheta_k(z_k - p) + (1 - \vartheta_k)(\Pi_C(I - \varpi_k F)V_k G z_k - p)\| \\
&\le \vartheta_k\|z_k - p\| + (1 - \vartheta_k)\|(I - \varpi_k F)V_k G z_k - p\| \\
&= \vartheta_k\|z_k - p\| + (1 - \vartheta_k)\|(I - \varpi_k F)V_k G z_k - (I - \varpi_k F)p + (I - \varpi_k F)p - p\| \\
&\le \vartheta_k\|z_k - p\| + (1 - \vartheta_k)[\|(I - \varpi_k F)V_k G z_k - (I - \varpi_k F)p\| + \|(I - \varpi_k F)p - p\|] \\
&\le \vartheta_k\|z_k - p\| + (1 - \vartheta_k)[(1 - \varpi_k\tau)\|z_k - p\| + \varpi_k\|F(p)\|] \\
&= (1 - (1 - \vartheta_k)\varpi_k\tau)\|z_k - p\| + (1 - \vartheta_k)\varpi_k\tau\frac{\|F(p)\|}{\tau},
\end{aligned}$$

where $\tau = 1 - \sqrt{\frac{1-\delta}{\zeta}}$. By induction, we deduce that

$$\|z_k - p\| \le \max\{\|z_1 - p\|, \frac{\|F(p)\|}{\tau}\}, \quad \forall k \ge 1.$$

Therefore, $\{z_k\}_{k=1}^{\infty}$ is bounded. Hence, $\{Gz_k\}_{k=1}^{\infty}, \{V_k G z_k\}_{k=1}^{\infty}$ and $\{F(V_k G z_k)\}_{k=1}^{\infty}$ are also bounded. Now, set $v_k = \Pi_C(I - \varpi_k F)V_k G z_k$ for all $k \ge 1$,then $z_{k+1} = \vartheta_k z_k + (1 - \vartheta_k)v_k$ for all $k \ge 1$. Hence, it follows that

$$\begin{aligned}
\|v_{k+1} - v_k\| &= \|\Pi_C(I - \varpi_{k+1}F)V_{k+1}Gz_{k+1} - \Pi_C(I - \varpi_k F)V_k G z_k\| \\
&\le \|(I - \varpi_{k+1}F)V_{k+1}Gz_{k+1} - (I - \varpi_k F)V_k G z_k\| \\
&= \|V_{k+1}Gz_{k+1} - V_k G z_k - \varpi_{k+1}F(V_{k+1}Gz_{k+1}) + \varpi_k F(V_k G z_k)\| \\
&\le \|V_{k+1}Gz_{k+1} - V_k G z_k\| + \varpi_{k+1}\|F(V_{k+1}Gz_{k+1})\| + \varpi_k\|F(V_k G z_k)\| \\
&\le \|V_{k+1}Gz_{k+1} - V_{k+1}Gz_k\| + \|V_{k+1}Gz_k - V_k G z_k\| + \varpi_{k+1}\|F(V_{k+1}Gz_{k+1})\| + \varpi_k\|F(V_k G z_k)\| \\
&\le \|z_{k+1} - z_k\| + \nu_{k+1}\|T_{k+1}Gz_k - Gz_k\| + \varpi_{k+1}\|F(V_{k+1}Gz_{k+1})\| + \varpi_k\|F(V_k G z_k)\|,
\end{aligned}$$

which together with $\nu_k \to 0, \quad \varpi_k \to 0$ and the boundedness of $\{Gz_k\}, \{T_{k+1}Gz_k\}$ and $\{F(V_k G z_k)\}$ implies that

$$\limsup_{k\to\infty}(\|v_{k+1} - v_k\| - \|z_{k+1} - z_k\|) \le 0.$$

So, by Lemma 3.6 we get

$$\lim_{k\to\infty}\|v_k - z_k\| = 0.$$

Consequently,

$$\lim_{k\to\infty} \|z_{k+1} - z_k\| = \lim_{k\to\infty}(1 - \vartheta_k)\|v_k - z_k\| = 0.$$

At the same time, we note that

$$\begin{aligned}\|v_k - V_kGz_k\| &= \|\Pi_C(I - \varpi_k F)V_kGz_k - V_kGz_k\| \\ &\le \|(I - \varpi_k F)V_kGz_k - V_kGz_k\| \\ &= \varpi_k\|F(V_kGz_k)\| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

It follows from $\|v_k - z_k\| \to 0$ that

$$\lim_{k\to\infty} \|z_k - V_kGz_k\| = 0.$$

Repeating the same arguments as those of (20) in the proof of Theorem 3.8, we have

$$\lim_{k\to\infty} \|V_kGz_k - VGz_k\| = 0.$$

Noticing that $\|z_k - VGz_k\| \le \|z_k - V_kGz_k\| + \|V_kGz_k - VGz_k\|$, we get

$$\lim_{k\to\infty} \|z_k - (V \circ G)z_k\| = 0. \tag{55}$$

It is clear that, V is a nonexpansive self-mapping on C because V_k is a nonexpansive self-mapping on C for all $k \ge 1$. Taking into account $v_k = \Pi_C(I - \varpi_k F)V_kGz_k$ for all $k \ge 1$, from Lemma 2.5 and (55) we have

$$\begin{aligned}\|v_k - VGv_k\| &\le \|v_k - z_k\| + \|z_k - VGz_k\| + \|VGz_k - VGv_k\| \\ &\le \|v_k - z_k\| + \|z_k - VGz_k\| + \|z_k - v_k\| \\ &= 2\|v_k - z_k\| + \|z_k - VGz_k\| \to 0 \quad \text{as } k \to \infty.\end{aligned} \tag{56}$$

Next, we show that

$$\limsup_{k\to\infty} \langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle \le 0, \tag{57}$$

where $\tilde{x} \in \mathcal{F}$ is the unique solution of VI (32).

To see this, we choose a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that

$$\limsup_{k\to\infty} \langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle = \lim_{i\to\infty}\langle F(\tilde{x}), j(\tilde{x} - v_{k_i})\rangle.$$

We may also assume that $v_{k_i} \rightharpoonup z \in C$. Note that $z \in \mathrm{Fix}(V \circ G) = \mathcal{F}$ in terms of Propositions 2.4 and 3.5 and (56). Therefore, it follows from VI (32) and the weakly sequential continuity of J that

$$\limsup_{k\to\infty} \langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle = \lim_{i\to\infty}\langle F(\tilde{x}), j(\tilde{x} - v_{k_i})\rangle = \langle F(\tilde{x}), j(\tilde{x} - z)\rangle \le 0.$$

Since $v_k = \Pi_C(I - \varpi_k F)V_kGz_k$ for all $k \ge 1$, according to Lemma 2.2 (iii), we have

$$\langle (I - \varpi_k F)V_kGz_k - \Pi_C(I - \varpi_k F)V_kGz_k, j(\tilde{x} - v_k)\rangle \le 0. \tag{58}$$

From (58), we have

$$\begin{aligned}\|v_k - \tilde{x}\|^2 &= \langle \Pi_C(I - \varpi_k F)V_kGz_k - \tilde{x}, j(v_k - \tilde{x})\rangle \\ &= \langle \Pi_C(I - \varpi_k F)V_kGz_k - (I - \varpi_k F)V_kGz_k, j(v_k - \tilde{x})\rangle + \langle (I - \varpi_k F)V_kGz_k - \tilde{x}, j(v_k - \tilde{x})\rangle \\ &\le \langle (I - \varpi_k F)V_kGz_k - \tilde{x}, j(v_k - \tilde{x})\rangle \\ &= \langle (I - \varpi_k F)V_kGz_k - (I - \varpi_k F)\tilde{x}, j(v_k - \tilde{x})\rangle + \langle (I - \varpi_k F)\tilde{x} - \tilde{x}, j(v_k - \tilde{x})\rangle \\ &\le (1 - \varpi_k\tau)\|z_k - \tilde{x}\|\|v_k - \tilde{x}\| + \varpi_k\langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle \\ &\le \frac{(1 - \varpi_k\tau)^2}{2}\|z_k - \tilde{x}\|^2 + \frac{(1}{2}\|v_k - \tilde{x}\|^2 + \varpi_k\langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned} \|v_k - \tilde{x}\|^2 &\leq (1 - \varpi_k \tau)^2 \|z_k - \tilde{x}\|^2 + 2\varpi_k \langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle \\ &\leq (1 - \varpi_k \tau)\|z_k - \tilde{x}\|^2 + 2\varpi_k \langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle. \end{aligned} \tag{59}$$

Finally, we prove that $z_k \to \tilde{x}$ as $k \to \infty$. Indeed, from (54) and (59), we find that

$$\begin{aligned} \|z_{k+1} - \tilde{x}\|^2 &\leq \vartheta_k \|z_k - \tilde{x}\|^2 + (1 - \vartheta_k)\|v_k - \tilde{x}\|^2 \\ &\leq \vartheta_k \|z_k - \tilde{x}\|^2 + (1 - \vartheta_k)(1 - \varpi_k \tau)\|z_k - \tilde{x}\|^2 + 2\varpi_k (1 - \vartheta_k)\langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle \\ &= [1 - \varpi_k (1 - \vartheta_k)\tau]\|z_k - \tilde{x}\|^2 + \varpi_k (1 - \vartheta_k)\tau(\frac{2}{\tau}\langle F(\tilde{x}), j(\tilde{x} - v_k)\rangle). \end{aligned} \tag{60}$$

We apply Lemma 3.7 to the relation (60) and conclude that $z_k \to \tilde{x}$ as $k \to \infty$. This completes the proof. □

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References

[1] M. A. Alghamdi, N. Shahzad, H. Zegeye, *A scheme for a solution of a variational inequality for a monotone mapping and a fixed point of a pseudocontractive mapping*, J. Inequal. Appl., **2015**(2015), Art. ID. 292.
[2] M. A. Alghamdi, N. Shahzad, H. Zegeye, *Construction of a common solution of a finite family of variational inequality problems for monotone mappings*, J. Nonlinear Sci. Appl., **9**(2016), 1645–1657.
[3] Q. H. Ansari, J. C. Yao, *A fixed point theorem and its applications to the system of variational inequalities*, Bull. Austral. Math. Soc., **59** (1999), 433–442.
[4] Q. H. Ansari, J. C. Yao, *Systems of generalized variational inequalities and their applications*, Appl. Anal., **76**(2000), 203–217.
[5] K. Aoyama, H. Iiduka, W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl., **2006**(2006:35390), 13 pages.
[6] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space, Proceedings of the National Academy of Sciences of the United States of America*, **54**(1965),1041–1044.
[7] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100**(1967), 201–225.
[8] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20**(1967), 197–228.
[9] N. Buong, N.T.Q. Anh, *An implicit iteration method for variational inequalities over the set of common fixed points of a finite family of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2011**(2011:276859), 10 pages.
[10] N. Buong, N. T. H. Phuong, *Strong convergence to solutions for a class of variational inequalities in Banach spaces by implicit iteration methods*, J. Optim. Theory Appl., **159**(2013), 399–411.
[11] L. C. Ceng, Q. H. Ansari, J. C. Yao, *Mann-type steepest-descent and modified steepest-descent methods for variational inequalities in Banach spaces*, Numer. Funct. Anal. Optim., **29**(2013), 987–1033.
[12] L. C. Ceng, H. Gupta, Q. H. Ansari, *Implicit and explicit algorithms for a system of nonlinear variational inequalities in Banach spaces*, J. Nonlinear Convex Anal.,**16(5)**(2015), 1–20.
[13] L. C. Ceng, C. Y. Wang, J. C. Yao, *Strong convergence theorems by a related extragradient method for a general system of variational inequalities*, Math. Meth. Oper. Res., **67**(2008), 375–390.
[14] F. Facchinei, J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems I*, Springer-Verlag, New York, (2003).
[15] R. Glowinski, J. L. Lions, R. Tremolieres, *Numerical Analysis and Variational Inequalities*, North-Holland, Amsterdam, (1981).
[16] M. Kikkawa, W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Hilbert spaces*, RIMS Kokyuroku **1484**(2006), 105–113.
[17] M. Kikkawa, W. Takahashi, *Strong convergence theorems by the viscosity approximation methods for a countable family of nonexpansive mappings*, Taiwan. J. Math.,**12(3)**(2008), 583–598.
[18] N. Kikuchi, J. T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, (1988).
[19] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications, Academic Press*, New York, (1980).
[20] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities, Lecture Notes in economics and Mathematical Systems*, Vol. 495, Springer-Verlag, Berlin, Heiderberg, New York, (2001).

[21] I. V. Konnov, *Equilibrium Models and Variational Inequalities*, Elsevier B. V., Amsterdam, (2007).
[22] L. J. Lin, Q. H. Ansari, *Collective fixed points and maximal elements with applications to abstract economics*, J. Math. Anal. Appl. **296**(2004), 455–472.
[23] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **67**(1979), 274–276.
[24] N. Shahzad, H. Zegeye, A. Alotaibi, *Convergence results for a common solution of a finite family of variational inequality problems for monotone mappings with Bregman distance function*, Fixed Point Theory Appl., **2013**(2013), Art. ID. 343.
[25] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227–239.
[26] W. Takahashi, *Weak and strong convergence theorems for families of nonexpansive mappings and their applications*, Ann. Univ. Mariae Curie-Sklodowska (Sect. A),**51**(1997), 277–292.
[27] R. U. Verma, *On a new system of nonlinear variational inequalities and associated iterative algorithms*, Math. Sci. Res. Hot-line **3**(1999), 65–68.
[28] R. U. Verma, *Projection methods, algorithms and a new system of nonlinear variational inequalities*, Comput. Math. Appl., **41**(2001), 1025–1031.
[29] S. Wang, L. Yu, B. Guo, *An implicit iterative scheme for an infinite countable family of asymptotically nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2008**(2008), 10 pages.
[30] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16**(1991), 1127-1138.
[31] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240–256.
[32] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279–291.
[33] H. Zegeye, N. Shahzad, M. A. Alghamdi, *Strong convergence theorems for a common point of solution of variational inequality, solutions of equilibrium and fixed point problems*, Fixed Point Theory Appl., **2012**(2012), Art. ID. 119.
[34] H. Zegeye, N. Shahzad, *Approximating common solution of variational inequality problems for two monotone mappings in Banach spaces*, Optim. Lett., **5**(2011), 691–704.
[35] L. C. Zeng, J. C. Yao, *Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings*, Nonlinear Anal., **64**(2006), 2507–2515.