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Characterization of Mixed n-Jordan and Pseudo n-Jordan Homomorphisms

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Abstract. In this article, a new notion of n-Jordan homomorphism namely the mixed n-Jordan homomorphism is introduced. It is proved that how a mixed $(n+1)$-Jordan homomorphism can be a mixed n-Jordan homomorphism and vice versa. By means of some examples, it is shown that the mixed n-Jordan homomorphisms are different from the n-Jordan homomorphisms and the pseudo n-Jordan homomorphisms. As a consequence, it shown that every mixed Jordan homomorphism from Banach algebra $\mathcal{A}$ into commutative semisimple Banach algebra $\mathcal{B}$ is automatically continuous. Under some mild conditions, every unital pseudo 3-Jordan homomorphism is a homomorphism.

1. Introduction and Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then, φ is called an *n-homomorphism* if for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).$$

The concept of an n-homomorphism was studied for complex algebras in [7] and [11]. Moreover, the map φ is called an *n-Jordan homomorphism* if $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{A}$. This notion was introduced by Herstein in [12]. A 2-homomorphism (2-Jordan homomorphism) is called simply a homomorphism (Jordan homomorphism). It is clear that every n-homomorphism is an n-Jordan homomorphism, but the converse is not valid in general. Indeed, it was Ancochea [2] who firstly studied the connection of Jordan homomorphisms and homomorphisms. The results of Ancochea were generalized and extended in several ways in [13] and [14]. There are plenty of known examples of n-Jordan homomorphism which are not n-homomorphism. For $n = 2$, it is proved in [13] that some Jordan homomorphism on the polynomial rings can not be homomorphism. In addition, each homomorphism is an n-homomorphism for every $n \geq 2$, but the converse generally does not hold. For instance, if $h : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism, then $g := -h$ is a 3-homomorphism which is not a homomorphism [7]. However, it is easily checked that if $\mathcal{A}$ is a unital algebra

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and h is a 3-homomorphism then $g(a) := h(1)h(a)$ is a homomorphism. Furthermore, the second author and Peyvaste in [6, Theorem 2.4] showed that if $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a 3-homomorphism, then $\tilde{\varphi}(a) := \varphi(u^3 a)$ $(a \in \mathcal{A})$ is a homomorphism in which $\varphi(u) = 1$. This result can be generalized for n-homomorphisms [3]. Herstein [12] proved the following result.

Theorem 1.1. *If φ is a Jordan homomorphism of a ring $\mathcal{R}$ onto a prime ring $\mathcal{R}'$ of characteristic different from* 2 *and* 3, *then either φ is a homomorphism or an anti-homomorphism.*

Next, Zelazko in [18] presented the upcoming result (see also [17]).

Theorem 1.2. *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

This result has been proved by the third author in [20] for 3-Jordan homomorphism with the extra condition that the Banach algebra $\mathcal{A}$ is unital. In other words, he presented the following theorem.

Theorem 1.3. *Suppose that $\mathcal{A}$ is a unital Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a 3-homomorphism.*

After that, An [1] extended the above theorem for all $n \in \mathbb{N}$ in [1] and showed that for unital ring $\mathcal{A}$ and ring $\mathcal{B}$ with char($\mathcal{B}$)> n, every n-Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is an n-homomorphism (n-anti-homomorphism) provided that every Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism (anti-homomorphism). Recently, the second author and İnceboz extended Theorem 1.2 for $n \in \{3, 4\}$ in [5] without the Banach algebra $\mathcal{A}$ is that of being unital by considering an extra condition on the mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ as

$$\varphi(a^2 b) = \varphi(ba^2), \quad a, b \in \mathcal{A}.$$

Some significant results concerning Jordan homomorphisms and their automatic continuity on Banach algebras obtained by the third author in [19], [21] and [22]. For the commutative case, Lee in [15] and Gselmann in [10] every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism where n is an arbitrary and fixed positive integer. Later, this problem solved in [4] based on the property of the Vandermonde matrix, which is different from the methods that are used in [10] and [15].

Let $\mathcal{A}$ and $\mathcal{B}$ be rings (algebras), and $\mathcal{B}$ be a right [left] $\mathcal{A}$-module. Then, a linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be *pseudo n-Jordan homomorphism* if there exists an element $w \in \mathcal{A}$ such that

$$\varphi(a^n w) = \varphi(a)^n \cdot w, \ [\varphi(a^n w) = w \cdot \varphi(a)^n] \quad (a \in \mathcal{A}).$$

The element w is called Jordan coefficient of φ. The concept of pseudo n-Jordan homomorphism was introduced and studied by Ebadian et al., in [8]. They also investigated the automatic continuity such homomorphisms on commutative C^*-algebras and semisimple (non unital) Banach algebras.

In section 2, we introduce the notion of mixed n-Jordan homomorphism on algebras. We prove that every 3-Jordan homomorphism φ from algebra $\mathcal{A}$ into φ-commutative algebra $\mathcal{B}$ is a mixed Jordan homomorphism provided that $\varphi(ab - ba) = 0$. Furthermore, we discuss the automatic continuity of mixed Jordan homomorphisms. We show that under which conditions a mixed $(n+1)$-Jordan homomorphism is a mixed n-Jordan homomorphism and vice versa. We prove that every n-Jordan homomorphism on non-commutative Banach algebras is a n-homomorphism with different conditions as in [1, Theorem 2.4] and [4, Theorem 2.2]. It is of interest to know whether the converse of [8, Theorem 2.3] holds. In the last section, we answer to this question. In fact, we show that every unital pseudo $(n+1)$-Jordan homomorphism is pseudo n-Jordan homomorphism with the same Jordan coefficient.

2. Mixed n-Jordan homomorphisms

We start this section with the definition of mixed n-Jordan homomorphisms.

Definition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then, φ is called an mixed n-Jordan homomorphism if for all $a, b \in \mathcal{A}$,*

$$\varphi(a^n b) = \varphi(a)^n \varphi(b).$$

A mixed 2-Jordan homomorphism is said to be *mixed Jordan homomorphism*. It is clear that every n-homomorphism is an mixed $(n-1)$-Jordan homomorphism for $n \geq 3$, and every mixed n-Jordan homomorphism is $(n+1)$-Jordan homomorphism but the converse is not true in general. The following example illustrates this fact.

Example 2.2. *Let $\mathcal{A} = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} : \quad X, Y \in M_2(\mathbb{C}) \right\}$, and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\varphi\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & 0 \\ 0 & Y^T \end{bmatrix}$, where Y^T is the transpose of matrix Y. Then, for all $U \in \mathcal{A}$, we have*

$$\varphi(U^n) = \varphi(U)^n = \begin{bmatrix} X^n & 0 \\ 0 & (Y^n)^T \end{bmatrix}.$$

Thus, φ is n-Jordan homomorphism, but φ is not mixed $(n-1)$-Jordan homomorphism. In fact, for

$$U = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}, \quad V = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

we get $(Y^{(n-1)}B)^T \neq (Y^T)^{(n-1)}B^T$. Therefore, $\varphi(U^{(n-1)}V) \neq \varphi(U)^{(n-1)}\varphi(V)$.

A left ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is a modular left ideal if there exists $u \in \mathcal{A}$ such that $\mathcal{A}(e_{\mathcal{A}} - u) \subseteq \mathcal{I}$, where $\mathcal{A}(e_{\mathcal{A}} - u) = \{x - xu : x \in \mathcal{A}\}$. The Jacobson radical Rad $(\mathcal{A})$ of $\mathcal{A}$ is the intersection of all maximal modular left ideals of $\mathcal{A}$. An algebra $\mathcal{A}$ is called *semisimple* whenever its Jacobson radical Rad $(\mathcal{A})$ is trivial. For example, every C^*-algebra is semisimple.

Let n be an integer $n \geq 2$. Recall that an associative ring $\mathcal{R}$ is of characteristic not n if $na = 0$ implies $a = 0$ for every $a \in \mathcal{R}$, and $\mathcal{R}$ is of characteristic greater than n if $n!a = 0$ implies $a = 0$ for every $a \in \mathcal{R}$.

We bring some trivial observations are as follows:

• It is known that every mixed n-Jordan homomorphism is $(n+1)$-Jordan homomorphism, and so by [4, Theorem 2.2], every mixed n-Jordan homomorphism φ between commutative algebras $\mathcal{A}$ and $\mathcal{B}$ is $(n+1)$-homomorphism.

• Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{B}$ be a Banach algebra with char$(\mathcal{B})> n$. By [1, Theorem 2.4], every mixed n-Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is an $(n+1)$-homomorphism if every Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism. Hence, under such assumptions and that $\mathcal{B}$ is a semisimple commutative Banach algebra, every surjective mixed n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous by [1, Corollary 2.5].

Proposition 2.3. *Let φ be a mixed Jordan homomorphism between algebras $\mathcal{A}$ and $\mathcal{B}$. Then*

(i) *φ is mixed $(2n)$-Jordan homomorphism for all $n \in \mathbb{N}$.*
(ii) *if $\mathcal{A}$ is unital, then $\varphi(x)\varphi(e) = \varphi(e)\varphi(x)$, where e is the identity of $\mathcal{A}$.*
(iii) *the mapping $\psi(x) := \varphi(x)\varphi(e)$ is a homomorphism.*

Proof. Suppose that φ is mixed Jordan homomorphism. Then for all $a, b \in \mathcal{A}$,

$$\varphi(a^2 b) = \varphi(a)^2 \varphi(b). \tag{1}$$

Replacing b by $a^2 b$ in (1), gives

$$\varphi(a^4 b) = \varphi(a)^2 \varphi(a^2 b) = \varphi(a)^4 \varphi(b).$$

Thus, φ is mixed 4-Jordan homomorphism. This argument can be repeated to achieve the desired result of part (i).

For part (ii), replacing b by $b+e$ in (1), we have

$$2\varphi(ab) = [\varphi(a)\varphi(e) + \varphi(e)\varphi(a)]\varphi(b) \tag{2}$$

for all $a, b \in \mathcal{A}$. Switching b by e in (2), we find

$$2\varphi(a) = \varphi(a)\varphi(e)^2 + \varphi(e)\varphi(a)\varphi(e) \tag{3}$$

for all $a \in \mathcal{A}$. Multiplying $\varphi(e)$ from the left in (3), and using the equality $\varphi(e)^3 = \varphi(e)$, we get

$$2\varphi(a)\varphi(e) = \varphi(a)\varphi(e) + \varphi(e)\varphi(a)\varphi(e)^2 \tag{4}$$

for all $a \in \mathcal{A}$. Similarly,

$$2\varphi(e)\varphi(a) = \varphi(e)\varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a)\varphi(e) \tag{5}$$

for all $a \in \mathcal{A}$. Relations (4) and (5) necessitate that

$$2[\varphi(e)\varphi(a) - \varphi(a)\varphi(e)] = \varphi(e)^2\varphi(a)\varphi(e) - \varphi(a)\varphi(e) \tag{6}$$

for all $a \in \mathcal{A}$. Multiplying $\varphi(e)$ from the right in (6), we arrive at

$$2\varphi(e)[\varphi(e)\varphi(a) - \varphi(a)\varphi(e)] = \varphi(e)\varphi(a)\varphi(e) - \varphi(e)\varphi(a)\varphi(e) = 0$$

for all $a \in \mathcal{A}$. Thus,

$$\varphi(e)^2\varphi(a) = \varphi(e)\varphi(a)\varphi(e) \tag{7}$$

for all $a \in \mathcal{A}$. Once more, multiplying $\varphi(e)$ from the right in (7), we obtain

$$\varphi(e)\varphi(a) = \varphi(e)^2\varphi(a)\varphi(e). \tag{8}$$

It now follows from (6) and (8) that $\varphi(e)\varphi(a) = \varphi(a)\varphi(e)$, for all $a \in \mathcal{A}$. This completes the proof of (ii). By part (ii) and (2) we see that $\varphi(ab) = \varphi(a)\varphi(b)\varphi(e)$, for all $a, b \in \mathcal{A}$. It concludes from the last equality that the mapping ψ is a homomorphism. $\square$

For certain calculations, we use the notation $[a, b] = ab - ba$ which is called the *Lie product* of a and b. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map between Banach algebras. Then, we say that $\mathcal{B}$ is φ-commutative if for all $a, b \in \mathcal{A}$, $[\varphi(a), \varphi(b)] = 0$.

Note that every commutative Banach algebra is I-commutative, where I is the identity map.

Example 2.4. (i) *Consider the Banach algebras* $\mathcal{A} = \left\{\begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad u, a, b, c \in \mathbb{C}\right\}$, $\mathcal{B} = \left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \quad a, b \in \mathbb{C}\right\}$ *with the usual sum and product. Define the linear map* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *by*

$$\varphi\left(\begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, $\mathcal{B}$ *is non-commutative Banach algebra, but it is* φ*-commutative.*

(ii) *Let* $\mathcal{A}$ *as the part* (i). *Consider* $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ *defined via* $\varphi\left(\begin{bmatrix} a & b & c \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}\right) = u$. *Then,* $\varphi(XY) = \varphi(YX)$ *and* $[\varphi(X), \varphi(Y)] = 0$ *for all* $X, Y \in \mathcal{A}$. *We see that* φ *is not* 3*-Jordan homomorphism and so it is neither mixed Jordan homomorphism nor* 3*-homomorphism.*

Lemma 2.5. *Let φ be an n-Jordan homomorphism from unital Banach algebra $\mathcal{A}$ into φ-commutative Banach algebra $\mathcal{B}$. Then, for all $a \in \mathcal{A}$,*

$$\varphi(a) = \varphi(e)^{n-1}\varphi(a) = \varphi(a)\varphi(e)^{n-1}.$$

Proof. Let φ be a Jordan homomorphism. Then

$$\varphi((a+3)^2 - 2(a+2)^2 + a^2) = \varphi(a+3)^2 - 2\varphi(a+2)^2 + \varphi(a)^2 \tag{9}$$

for all $a \in \mathcal{A}$. Let e be the identity of $\mathcal{A}$. By assumption $\varphi(e) = \varphi(e)^2$, and so (9) gives

$$2\varphi(a) = \varphi(a)\varphi(e) + \varphi(e)\varphi(a) \tag{10}$$

for all $a \in \mathcal{A}$. On the other hand,

$$[\varphi(a), \varphi(e)] = \varphi(a)\varphi(e) - \varphi(e)\varphi(a) = 0, \quad (a \in \mathcal{A}). \tag{11}$$

It follows from (10) and (11) that $\varphi(a) = \varphi(e)\varphi(a) = \varphi(a)\varphi(e)$, for all $a \in \mathcal{A}$. Now, assume that $n = 3$. Then

$$\varphi((a+2)^3 - 2(a+1)^3 + a^3) = \varphi(a+2)^3 - 2\varphi(a+1)^3 + \varphi(a)^3 \tag{12}$$

for all $a \in \mathcal{A}$. Since $\varphi(e) = \varphi(e)^3$, equation (12) implies that

$$3\varphi(a) = \varphi(a)\varphi(e)^2 + \varphi(e)^2\varphi(a) + \varphi(e)\varphi(a)\varphi(e) \tag{13}$$

for all $a \in \mathcal{A}$. By the φ-commutativity of $\mathcal{B}$, we have

$$\varphi(a)\varphi(e) = \varphi(e)\varphi(a) \tag{14}$$

for all $a \in \mathcal{A}$. Plugging (13) into (14), we find $\varphi(a) = \varphi(e)^2\varphi(a) = \varphi(a)\varphi(e)^2$. Similarly, one can obtain the result for all $n \geq 4$. □

Lemma 2.6. *Let $\mathcal{A}$ be a unital Banach algebra with unit e, and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an n-Jordan homomorphism. Then*

$$\varphi(a^2) = \varphi(a)^2\varphi(e)^{n-2} \qquad (a \in \mathcal{A}).$$

Proof. Refer to the proof of Theorem 2.4 from [1]. □

As mentioned before every n-Jordan homomorphism between commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is an n-homomorphism [4, Theorem 2.2]. In view of the proof of this theorem, we see the same result holds by the weaker condition on $\mathcal{B}$, as φ-commutativity. However, in the next result, we show that under some conditions every n-Jordan homomorphism on non-commutative Banach algebras can be an n-homomorphism.

Theorem 2.7. *Every n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ from unital Banach algebra $\mathcal{A}$ into φ-commutative Banach algebra $\mathcal{B}$ satisfying the following condition is an n-homomorphism.*

$$\varphi(x^2) = 0 \Longrightarrow \varphi(x) = 0, \quad (x \in \mathcal{A}). \tag{15}$$

Proof. Define the mapping $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ through

$$\psi(a) := \varphi(a)\varphi(e)^{n-2}, \quad (a \in \mathcal{A}).$$

By Lemma 2.6, ψ is Jordan homomorphism and $\mathcal{B}$ is ψ-commutative. One the other hand, Lemma 2.5 necessitates that ψ satisfies condition (15). We wish to show that ψ is a homomorphism. For all $a, b \in \mathcal{A}$, we have

$$\psi(ab + ba) = \psi(a)\psi(b) + \psi(b)\psi(a). \tag{16}$$

The ψ-commutativity of $\mathcal{B}$ implies that equality (16) converts to

$$\psi(ab+ba) = 2\psi(a)\psi(b) \tag{17}$$

for all $a, b \in \mathcal{A}$. The mapping ψ is a Jordan homomorphism and hence

$$\psi([a,b]^2) = [\psi(a),\psi(b)]^2 = 0 \tag{18}$$

for all $a, b \in \mathcal{A}$. Since ψ satisfies condition (15), it concludes from (18) that

$$\psi(ab-ba) = 0. \tag{19}$$

for all $a, b \in \mathcal{A}$. Plugging (17) into (19), we have $\psi(ab) = \psi(a)\psi(b)$. By Lemma 2.5, we get

$$\psi(a)\varphi(e) = \varphi(a)\varphi(e)^{n-1} = \varphi(a). \tag{20}$$

for all $a \in \mathcal{A}$. It follows from Lemma 2.5 and relation (20) that

$$\begin{aligned}
\varphi(a_1a_2\cdots a_n) &= \psi(a_1a_2\cdots a_n)\varphi(e)\\
&= \psi(a_1)\psi(a_2)\cdots\psi(a_n)\varphi(e)\\
&= (\varphi(a_1)\varphi(e)^{n-2})(\varphi(a_2)\varphi(e)^{n-2})\cdots(\varphi(a_n)\varphi(e)^{n-2})\varphi(e)\\
&= \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)\varphi(e)^{(n-1)^2}\\
&= \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).
\end{aligned}$$

for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$. Therefore, φ is an n-homomorphism. □

Theorem 2.8. *Let φ be a 3-Jordan homomorphism from algebra $\mathcal{A}$ into φ-commutative algebra $\mathcal{B}$ such that $\varphi([a,b]) = 0$ for any $a, b \in \mathcal{A}$. Then, φ is mixed Jordan homomorphism.*

Proof. By assumption

$$\varphi(x^3) = \varphi(x)^3, \tag{21}$$

for all $x \in \mathcal{A}$. Replacing x by $a+b$ in (21), we obtain

$$\varphi(aba + ba^2 + a^2b + b^2a + ab^2 + bab) = 3[\varphi(a)^2\varphi(b) + \varphi(a)\varphi(b)^2] \tag{22}$$

for all $a, b \in \mathcal{A}$. Switching b by $-b$ in (22), we get

$$\varphi(-aba - ba^2 - a^2b + b^2a + ab^2 + bab) = 3[-\varphi(a)^2\varphi(b) + \varphi(a)\varphi(b)^2] \tag{23}$$

for all $a, b \in \mathcal{A}$. Relations (22) and (23) show that

$$\varphi(b^2a + ab^2 + bab) = 3\varphi(a)\varphi(b)^2, \quad (a, b \in \mathcal{A}). \tag{24}$$

Since $\varphi([a,b]) = 0$, we have

$$\varphi(b^2a) = \varphi(ab^2) = \varphi(bab), \tag{25}$$

for all $a, b \in \mathcal{A}$. It follows from (24) and (25) that $\varphi(b^2a) = \varphi(b)^2\varphi(a)$, for all $a, b \in \mathcal{A}$. Therefore, φ is mixed Jordan homomorphism. □

As a consequence of Theorem 2.8, we have the next result.

Corollary 2.9. *Let φ be a mapping from algebra $\mathcal{A}$ into φ-commutative Banach algebra $\mathcal{B}$, such that $\varphi([a,b]) = 0$ for all $a, b \in \mathcal{A}$. Suppose $\delta > 0$ and φ satisfy $|\varphi(a^2) - \varphi(a)^2| \leq \delta$. Then, φ is a mixed Jordan homomorphism.*

The following theorem is a well-known result, due to Šilov, concerning the automatic continuity of homomorphisms between Banach algebras.

Theorem 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{B}$ is commutative and semisimple. Then, every homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous.*

In 1967, B. E. Johnson proved that if $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective homomorphism between a Banach algebra $\mathcal{A}$ and a semisimple Banach algebra $\mathcal{B}$, then φ is automatically continuous and so the Johnson's result extended to n-homomorphism in [9]. One may refer to [7] for automatic continuity of 3-homomorphism.

We say that a linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a *co-Jordan homomorphism* if for all $a \in \mathcal{A}$, $\varphi(a^2) = -\varphi(a)^2$. For example, the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(a) = -a$ is a co-Jordan homomorphism. Here, we show that Theorem 2.10 holds for mixed Jordan homomorphisms.

Theorem 2.11. *Let φ be a mixed Jordan homomorphism from Banach algebra $\mathcal{A}$ into $\mathbb{C}$. Then, φ is automatically continuous.*

Proof. Suppose that there exist $x_0 \in \mathcal{A}$ such that $\|x_0\| < 1$ and $\varphi(x_0) = 1$. Take $y = \sum_{n=1}^{\infty} x_0^n$. Then, $x_0 + x_0^2 y = y - x_0^2$, and so

$$1 + \varphi(y) = \varphi(x_0) + \varphi(x_0)^2\varphi(y) = \varphi(x_0 + x_0^2 y) = \varphi(y) - \varphi(x_0^2).$$

Thus, $\varphi(x_0^2) = -1$. Since φ is mixed Jordan homomorphism, we have

$$\varphi(a^2 b) = \varphi(a)^2\varphi(b) \tag{26}$$

for all $a, b \in \mathcal{A}$. Replacing a by $u + v$ in (26), we get

$$\varphi((uv + vu)b) = 2\varphi(u)\varphi(v)\varphi(b) \tag{27}$$

for all $a, u, v \in \mathcal{A}$. Interchanging b by x_0^2 in (27), we obtain

$$\varphi((uv + vu)x_0^2) = 2\varphi(u)\varphi(v)\varphi(x_0^2) = -2\varphi(u)\varphi(v) \tag{28}$$

for all $a, u, v \in \mathcal{A}$. Substituting (u, v) into (u^2, x_0^2) in (28), we arrive at

$$\varphi(u^2 x_0^4) + \varphi(x_0^2 u^2 x_0^2) = -2\varphi(u^2)\varphi(x_0^2) = 2\varphi(u^2) \tag{29}$$

for all $u \in \mathcal{A}$. In addition, $\varphi(x_0^4) = \varphi(x_0)^2\varphi(x_0^2) = -1$, and

$$\varphi(x_0^2 u^2 x_0^2) = \varphi(x_0)^2\varphi(u^2 x_0^2) = \varphi(x_0)^2\varphi(u)^2\varphi(x_0^2) = -\varphi(u)^2,$$

for all $u \in \mathcal{A}$. By (29), we have $\varphi(u)^2 = -\varphi(u^2)$ for all $u \in \mathcal{A}$. Hence, φ is co-Jordan homomorphism and it is continuous by [22, Proposition 2.1]. If there is no $x_0 \in \mathcal{A}$ such that $\|x_0\| < 1$ and $\varphi(x_0) = 1$, then for all $x \in \mathcal{A}$ with $\|x\| < 1$ we have $|\varphi(x)| \leq 1$. Therefore, φ is continuous. □

The upcoming corollary is a direct consequence of Theorem 2.11.

Corollary 2.12. *Let φ be a mixed Jordan homomorphism from Banach algebras $\mathcal{A}$ into commutative semisimple Banach algebra $\mathcal{B}$. Then, φ is automatically continuous.*

Proof. Let $(a_n) \subseteq \mathcal{A}$, $a_n \longrightarrow 0$ and $\varphi(a_n) \longrightarrow b$. Suppose that $h \in \mathfrak{M}(\mathcal{B})$, where $\mathfrak{M}(\mathcal{B})$ is the maximal ideal space of $\mathcal{B}$. Then, $h \circ \varphi$ is a mixed Jordan homomorphism and so it is continuous by Theorem 2.11. Thus,

$$h(b) = \lim_n h(\varphi(a_n)) = \lim_n h \circ \varphi(a_n) = 0.$$

Since $\mathcal{B}$ is semisimple, we have $b = 0$, and thus φ is continuous by the close graph theorem. □

The next result is the same as Theorem 2.10 for 3-homomorphism.

Corollary 2.13. *Let φ be a 3-homomorphism from Banach algebras $\mathcal{A}$ into commutative semisimple Banach algebra $\mathcal{B}$. Then, φ is automatically continuous.*

A linear map φ between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called *unital* if $\varphi(e) = e'$, where e and e' are the unit element of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Theorem 2.14. *Every unital mixed $(n+1)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

Proof. Let φ be a mixed $(n+1)$-Jordan homomorphism. Then

$$\varphi(a^{n+1}b) = \varphi(a)^{n+1}\varphi(b) \tag{30}$$

for all $a, b \in \mathcal{A}$. Since φ is unital, by putting $b = e$ in (30) we get

$$\varphi(a^{n+1}) = \varphi(a)^{n+1} \tag{31}$$

for all $a \in \mathcal{A}$. Replacing a by $a + \lambda e$ in (31), where λ is a complex number, and compare powers of λ, we arrive at

$$\varphi(a^2) = \varphi(a^2) \tag{32}$$

for all $a \in \mathcal{A}$. Hence, φ is a Jordan homomorphism. Interchanging a by $a + \lambda e$ in (30), we obtain

$$\varphi(a^{n+1}b) = (\varphi(a) + \lambda\varphi(e))^{n+1}\varphi(b) \tag{33}$$

for all $a, b \in \mathcal{A}$. Comparing powers of λ^n in (33) and using $\varphi(e) = e'$, one deduce that φ is a homomorphism. □

The next corollary follows immediately from Theorem 2.14.

Corollary 2.15. *Every unital mixed $(n+1)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a mixed n-Jordan homomorphism.*

The following example shows the condition being unital for Banach algebras $\mathcal{A}$ and $\mathcal{B}$ in Theorem 2.14 is essential.

Example 2.16. *Let*

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\},$$

and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ via

$$\varphi\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $\varphi(X^2) \neq \varphi(X)^2$, for all $X \in \mathcal{A}$. Hence, φ is not Jordan homomorphism, and so it is not homomorphism. But for all $n \geq 3$ and all $X, Y \in \mathcal{A}$, we have $\varphi(X^nY) = \varphi(X)^n\varphi(Y)$. Therefore, φ is mixed n-Jordan homomorphism for all $n \geq 3$.

Corollary 2.17. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital mixed n-Jordan homomorphism. Then, φ is continuous under one of the following conditions.*

(i) *$\mathcal{B}$ is semisimple and commutative.*
(ii) *$\mathcal{B}$ is semisimple and φ is surjective.*
(iii) *$\mathcal{B}$ is C^*-algebra and φ is surjective.*

Proof. By Theorem 2.14, φ is a homomorphism and so it is n-Jordan homomorphism. Thus, the result follows from Corollaries 2.9, 2.10 and 2.11 of [22]. □

In the upcoming result, we prove the converse of Theorem 2.14 under some conditions. The idea of the proof is taken from [8, Theorem 2.6]. We include the proof for the sake of completeness.

Theorem 2.18. *Let $\mathcal{A}$, $\mathcal{B}$ be unital Banach algebras and $\varphi : A \longrightarrow B$ be a unital mixed n-Jordan honmomorphism. Suppose that there exists an idempotent p in $\mathcal{A}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b, x \in \mathcal{A}$ with $ab = px$. Then, $\varphi(a^{n+1}px) = \varphi(a)^{n+1}\varphi(px)$ for all $a, x \in \mathcal{A}$. In particular, φ is mixed $(n+1)$-Jordan honmomorphism.*

Proof. Let e be a until element of $\mathcal{A}$ and $a \in \mathcal{A}$. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$, $e - \lambda$ is invertible and $(e - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned}
\varphi(px) &= \varphi((e-\lambda a)(e-\lambda a)^{-1}px)\\
&= \varphi(e-\lambda a)\varphi((e-\lambda a)^{-1}px)\\
&= (\varphi(e) - \lambda\varphi(a))\varphi\left(\sum_{n=0}^{\infty}\lambda^n a^n px\right)\\
&= \varphi(e)\varphi(px) + \varphi\left(\sum_{n=1}^{\infty}\lambda^n a^n px\right) - \lambda\varphi(a)\varphi\left(\sum_{n=0}^{\infty}\lambda^n a^n px\right)\\
&= \varphi(px) + \sum_{n=1}^{\infty}\lambda^n\varphi(a^n px) - \lambda\varphi(a)\varphi\left(\sum_{n=0}^{\infty}\lambda^n a^n px\right)\\
&= \varphi(px) + \sum_{n=0}^{\infty}\lambda^{n+1}\varphi(a^{n+1}px) - \varphi(a)\sum_{n=0}^{\infty}\lambda^{n+1}\varphi(a^n px)
\end{aligned}$$

Hence, $\sum_{n=0}^{\infty}\lambda^{n+1}[\varphi(a^{n+1}px) - \varphi(a)\varphi(a^n px)] = 0$ for $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$. Thus, $\varphi(a^{n+1}px) = \varphi(a)\varphi(a^n px)$ for $n = 0, 1, 2, \cdots$. For $n \geq 1$, we have

$$\varphi(a)\varphi(a^n px) = \varphi(a)\varphi(a)^n\varphi(px) = \varphi(a)^{n+1}\varphi(px).$$

Therefore, $\varphi(a^{n+1}px) = \varphi(a)^{n+1}\varphi(px)$ for all $a, x \in \mathcal{A}$. If we take $p = e$, we get $\varphi(a^{n+1}x) = \varphi(a)^{n+1}\varphi(x)$. This finishes the proof. □

We should note that Theorem 2.18 is true if "mixed" removed from it. In the following result, under certain conditions, we prove that each mixed Jordan homomorphism is a mixed n-Jordan homomorphism.

Proposition 2.19. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital mixed Jordan homomorphism. Then, φ is mixed n-Jordan homomorphism.*

Proof. The proof follows from Theorem 2.14. □

Proposition 2.20. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map such that*

$$\|\varphi(a^n b) - \varphi(a)^n\varphi(b)\| \leqslant \delta\|a\|^n\|b\|, \tag{34}$$

for all $a, b \in \mathcal{A}$ and for some $\delta > 0$. If $\mathcal{B}$ is commutative and semisimple, then φ is continuous.

Proof. Replacing b by a in (34), we get

$$\|\varphi(a^{n+1}) - \varphi(a)^{n+1}\| \leqslant \delta\|a\|^{n+1}. \tag{35}$$

Thus, φ is almost n-Jordan homomorphism and so it is continuous by [22, Theorem 3.4]. □

Theorem 2.21. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map such that*

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| \pm \|b\|), \tag{36}$$

for all $a, b \in \mathcal{A}$ and for some $\delta > 0$. Then, φ is $(n+1)$-Jordan homomorphism.

Proof. At first, we consider the inequality

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| - \|b\|), \tag{37}$$

for all $a, b \in \mathcal{A}$. Putting $a = b$ in (37), we get $\varphi(a^{n+1}) = \varphi(a)^{n+1}$ and so φ is $(n+1)$-Jordan homomorphism. Now, assume that

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| + \|b\|) \tag{38}$$

for all $a, b \in \mathcal{A}$. Replacing b by a in (38), we find

$$\|\varphi(a^{n+1}) - \varphi(a)^{n+1}\| \leqslant 2\delta\|a\| \tag{39}$$

for all $a \in \mathcal{A}$. Setting $a = 2^m x$, we obtain

$$\|\varphi(x^{n+1}) - \varphi(x)^{n+1}\| \leqslant \frac{\delta 2^{m+1}}{2^{m(n+1)}}\|x\| \tag{40}$$

for all $x \in \mathcal{A}$. Letting $m \to \infty$, we obtain $\varphi(x^{n+1}) = \varphi(x)^{n+1}$ and hence φ is $(n+1)$-Jordan homomorphism. □

3. Pseudo *n*-Jordan homomorphisms

We commence this section with the concept of pointwise pseudo n-Jordan for homomorphisms which is different from pseudo n-Jordan.

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be rings (algebras), and $\mathcal{B}$ be a right [left] $\mathcal{A}$-module. We say that a linear mapping $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ is a *pointwise pseudo n-Jordan* homomorphism if for each $a \in \mathcal{A}$ there exists an element $\omega_a \in \mathcal{A}$ such that $\psi(a^n \omega_a) = \psi(a)^n \cdot \omega_a$ $[(\psi(\omega_a a^n) = \omega_a \cdot \psi(a)^n)]$. We say that ω_a is a Jordan coefficient of ψ depended on a.

It is obvious that every n-Jordan homomorphism from unital Banach algebra $\mathcal{A}$ into $\mathcal{B}$ which is unitary Banach $\mathcal{A}$-module is a pseudo n-Jordan homomorphism. Moreover, we can see that for a pseudo n-Jordan homomorphism, there are infinitely many Jordan coefficient. However, every pseudo n-Jordan homomorphism is a pointwise pseudo n-Jordan homomorphism. Now, let φ be a mixed n-Jordan homomorphism such that has a fixed point, say ω. Then, φ is a pseudo n-Jordan homomorphism with a Jordan coefficient ω. The following example indicates this fact that the converse is false in general.

Example 3.2. *Let*

$$U_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : \ a, b, c \in \mathbb{R} \right\}$$

be the algebra of 2×2 matrices with the usual sum and product. Let $\psi : U_2(\mathbb{R}) \longrightarrow U_2(\mathbb{R})$ be a linear map defined by

$$\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

For every $n \in \mathbb{N}$, we have

$$\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n\right) = \psi\left(\begin{bmatrix} a^n & \sum_{k=0}^{n-1} a^{n-k-1} b c^k \\ 0 & c^n \end{bmatrix}\right) = \begin{bmatrix} a^n & \sum_{k=0}^{n-1} a^{n-k-1} b c^k \\ 0 & 0 \end{bmatrix}$$

and

$$\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^n = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}.$$

Thus, ψ is not an n-Jordan homomorphism and thus it is not mixed $(n-1)$-Jordan homomorphism. Assume that $t, s \in \mathbb{R}$. Put $\omega = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix}$. Then,

$$\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \omega\right) = \psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^n \omega.$$

Therefore, ψ is a pseudo n-Jordan homomorphism. On the other hand, for each $a, b, c \in \mathbb{R}$, take

$$\omega_{a,b,c} = \begin{bmatrix} s & -sc^{1-n}\sum_{k=2}^{n-2} a^{n-k-2}bc^k \\ 0 & 0 \end{bmatrix}.$$

Then,

$$\psi\left(\omega_{a,b,c}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n\right) = \omega_{a,b,c}\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^n.$$

This means that ψ is a pointwise pseudo n-Jordan homomorphism. Note that

$$\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n \omega_{a,b,c}\right) \neq \psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^n \omega_{a,b,c} \text{ and } \psi\left(\omega\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^n\right) \neq \omega\psi\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^n.$$

Here, we remind that part (3) of [8, Example 2.2] is not true. Indeed, it is corrected in Example 3.2.

It is known that every Jordan homomorphism is n-Jordan homomorphism [21]. The next example shows that the same result is false for mixed Jordan homomorphisms and pseudo n-Jordan homomorphisms.

Example 3.3. *Let $\mathcal{A}$ be a Banach algebra and $f : \mathcal{A} \longrightarrow \mathcal{A}$ be a homomorphism. Define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\varphi(x) = -f(x)$. Then, φ is a 3-homomorphism. Thus,*

$$\varphi(x^2a) = \varphi(x)^2\varphi(a),$$

and so φ is a mixed Jordan homomorphism, but φ is not a mixed 3-Jordan homomorphism. Suppose that φ has a fixed point, say a. Hence $\varphi(a) = a$. Thus,

$$\varphi(x^2a) = \varphi(x)^2\varphi(a) = \varphi(x)^2a.$$

Therefore, φ is a pseudo Jordan homomorphism with a Jordan coefficient a, but φ is not a pseudo 3-Jordan homomorphism. Note that for all $n \in \mathbb{N}$, φ is a pseudo $(2n)$-Jordan homomorphism with a Jordan coefficient a, but φ is not a pseudo $(2n+1)$-Jordan homomorphism for all $n \in \mathbb{N}$.

In the sequel, $\begin{pmatrix} n \\ k \end{pmatrix}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}$ with $n \geq k$ by $n!/(k!(n-k)!)$.

Here and subsequently, let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{B}$ be a right $\mathcal{A}$-module. Besides, it is assumed that φ between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is unital. The following result is the converse of [8, Theorem 2.3].

Theorem 3.4. *Every unital pseudo $(n+1)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ with a Jordan coefficient w is a pseudo n-Jordan homomorphism.*

Proof. We firstly have

$$\varphi((a+le)^{n+1}w) = (\varphi(a+le))^{n+1} \cdot w, \tag{41}$$

for all $a, b \in \mathcal{A}$, where l is an integer with $1 \le l \le n$. It follows from equality (41) and assumption that

$$\sum_{i=1}^{n} l^i \begin{pmatrix} n+1 \\ i \end{pmatrix} [\varphi(a^i w) - \varphi(a)^i \cdot w] = 0, \qquad (1 \le l \le n), \tag{42}$$

for all $a \in \mathcal{A}$. We can rewrite the equalities in (42) as follows

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^n \\ 3 & 3^2 & \cdots & 3^n \\ \cdots & \cdots & \cdots & \cdots \\ n & n^2 & \cdots & n^n \end{bmatrix} \begin{bmatrix} \Gamma_1(a,w) \\ \Gamma_2(a,w) \\ \Gamma_3(a,w) \\ \cdots \\ \Gamma_n(a,w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

for all $a \in \mathcal{A}$, where $\Gamma_i(a,w) = \begin{pmatrix} n+1 \\ i \end{pmatrix} [\varphi(a^i w) - \varphi(a)^i \cdot w]$ for all $1 \le i \le n$. It is shown in [4, Lemma 2.1] that the above square matrix is invertible. This implies that $\Gamma_i(a,w) = 0$ for all $1 \le i \le n$ and all $a \in \mathcal{A}$. In particular, $\Gamma_n(a,w) = 0$. This means that φ is a pseudo n-Jordan homomorphism. □

The next corollary is a consequence of Theorem 3.4.

Corollary 3.5. *Every unital $(n+1)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-Jordan homomorphism.*

Example 3.6. *Let*

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\},$$

and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\varphi\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $X \in \mathcal{A}$, $\varphi(X^2) \neq \varphi(X)^2$. Hence, φ is not Jordan homomorphism, but for all $n \ge 3$ and all $X \in \mathcal{A}$, we have $\varphi(X^n) = \varphi(X)^n$. Therefore, φ is n-Jordan homomorphism for all $n \ge 3$. Assume that $s, t, r \in \mathbb{R}$ is arbitrary. Put

$$w = \begin{bmatrix} 0 & s & t \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}.$$

For all $n \in \mathbb{N}$ and $X \in \mathcal{A}$, we get $\varphi(X^n w) = \varphi(X)^n w$. Therefore, φ is a pseudo n-Jordan homomorphism. In other words, we showed the condition that being unital for Banach algebras $\mathcal{A}$ and $\mathcal{B}$ in Corollary 3.5 is essential.

Suppose that $\mathcal{A}$ is a Banach algebra and M is an $\mathcal{A}$-module. Let $w \in \mathcal{A}$. Then, w is called a left (right) *separating point* of M if the condition $wx = 0$ ($xw = 0$) for $x \in M$ implies that $x = 0$ [16].

In the following result, under mild conditions, we prove that each pseudo Jordan homomorphism is n-Jordan homomorphism.

Theorem 3.7. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital pseudo Jordan homomorphism with a Jordan coefficient w such that w is a right separating point of $\mathcal{B}$. Then, φ is n-Jordan homomorphism.*

Proof. Assume that φ is a pseudo Jordan homomorphism. Then

$$\varphi(a^2 w) = \varphi(a)^2 \cdot w \tag{43}$$

for all $a \in \mathcal{A}$. Replacing a by $a+b$ in (43), we get

$$\varphi[(ab+ba)w] = [\varphi(a)\varphi(b) + \varphi(b)\varphi(a)] \cdot w \tag{44}$$

for all $a, b \in \mathcal{A}$. Since φ is unital, one can show that

$$\varphi[(ab+ba)w] = \varphi(ab+ba) \cdot w \tag{45}$$

for all $a, b \in \mathcal{A}$. It follows from (44) and (45) that

$$\Big(\varphi(ab+ba) - [\varphi(a)\varphi(b) + \varphi(b)\varphi(a)]\Big) \cdot w = 0 \tag{46}$$

Since w is a right separating point of $\mathcal{B}$, we get

$$\varphi(ab+ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$$

for all $a, b \in \mathcal{A}$. Thus, φ is a Jordan homomorphism. Now, by Lemma 2.6 of [22], φ is n-Jordan homomorphism. □

From Theorem 3.7, [4, Theorem 2.3] and [1, Corollary 2.5], we have the following trivial consequence.

Corollary 3.8. *By hypotheses of Theorem 3.7, φ is n-homomorphism if either*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are commutative, or*
(ii) *$\mathcal{B}$ is commutative and semisimple.*

Corollary 3.9. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital pseudo n-Jordan homomorphism with a Jordan coefficient w such that w is a right separating point of $\mathcal{B}$. Then, φ is continuous with each of the following conditions.*

(i) *$\mathcal{B}$ is semisimple and commutative.*
(ii) *$\mathcal{B}$ is semisimple and φ is surjective.*
(iii) *$\mathcal{B}$ is C^*-algebra and φ is surjective.*

Proof. By Theorem 3.4, φ is pseudo Jordan homomorphism and by Theorem 3.7 it is n-Jordan homomorphism. Thus, the result follows from Corollaries 2.9, 2.10 and 2.11 from [22]. □

Theorem 3.10. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital pseudo 3-Jordan homomorphism with a Jordan coefficient w such that w is a right separating point of $\mathcal{B}$. Suppose that $\mathcal{B}$ is commutative and*

$$\varphi(abcw) = \varphi(acbw), \quad (a, b, c \in \mathcal{A}). \tag{47}$$

Then, φ is a homomorphism.

Proof. Assume that e is an unit element of $\mathcal{A}$. Letting $a = e$ in (47), we get

$$\varphi(bcw - cbw) = 0$$

for all $b, c \in \mathcal{A}$. Thus, $\varphi((ab)cw) = \varphi(c(ab)w) = \varphi(c(ba)w)$ and hence

$$\varphi(a(bc)w) = \varphi((bc)aw) = \varphi(b(ca)w) = \varphi(b(ac)w)$$

for all $a, b, c \in \mathcal{A}$. That is

$$\varphi(abcw) = \varphi(xyzw), \tag{48}$$

whenever (x, y, z) is a permutation of (a, b, c). Since φ is a pseudo 3-Jordan homomorphism, $\varphi(a^3w) = \varphi(a)^3 \cdot w$, for all $a \in \mathcal{A}$. Replacing a by $a+b$, we get

$$\varphi[(ab^2 + b^2a + a^2b + ba^2 + aba + bab)w] = [3\varphi(a)\varphi(b)^2 + 3\varphi(a)^2\varphi(b)] \cdot w \tag{49}$$

for all $a, b \in \mathcal{A}$. Interchanging b by $-b$ in (49), we obtain

$$\varphi[(ab^2 + b^2a - a^2b - ba^2 - aba + bab)w] = [3\varphi(a)\varphi(b)^2 - 3\varphi(a)^2\varphi(b)] \cdot w \tag{50}$$

for all $a, b \in \mathcal{A}$. Relations (49), and (50) imply that

$$\varphi[(ab^2 + b^2a + bab)w] = [3\varphi(a)\varphi(b)^2] \cdot w \tag{51}$$

for all $a, b \in \mathcal{A}$. Replacing b by $b - c$ in (51), we deduce

$$\varphi[(abc + acb + bac + bca + cab + cba)w] = [6\varphi(a)\varphi(b)\varphi(c)] \cdot w \tag{52}$$

for all $a, b, c \in \mathcal{A}$. It follows from (48) and (52) that

$$\varphi(abcw) = [\varphi(a)\varphi(b)\varphi(c)] \cdot w \tag{53}$$

for all $a, b, c \in \mathcal{A}$. Put $c = e$ in (53), we arrive at

$$(\varphi(ab) - [\varphi(a)\varphi(b)]) \cdot w = 0$$

for all $a, b \in \mathcal{A}$. Since w is a right separating point of $\mathcal{B}$, we get

$$\varphi(ab) = \varphi(a)\varphi(b), \quad a, b \in \mathcal{A}.$$

Thus, φ is a homomorphism. $\square$

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