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Meromorphic Solutions of Difference Equations Originated From Schwarzian Differential Equation

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Abstract. Let $f(z)$ be a meromorphic functions with finite order , $R(z)$ be a nonconstant rational function and k be a positive integer. In this paper, we consider the difference equation originated from Schwarzian differential equation, which is of form

$$\left[\Delta^3 f(z)\Delta f(z) - \frac{3}{2}(\Delta^2 f(z))^2\right]^k = R(z)(\Delta f(z))^{2k}.$$

We investigate the uniqueness of meromorphic solution f of difference Schwarzian equation if f shares three values with any meromrphic function. The exact forms of meromorphic solutions f of difference Schwarzian equation are also presented.

1. Introduction and main results

In this paper, we use the basic notions of Nevanlinna's theory, see [12, 28]. In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$. Let $S(r, f)$ denote any quantity satisfying $S(r, f) = o\,(T(r, f))$ for all r outside of a set with finite logarithmic measure.

Let $f(z)$ and $g(z)$ be two meromorphic functions, a be a small function relative to both f and g. We say that f and g share a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities, f and g are said to share a IM if $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Nevanlinna's four values theorem (see [26]) says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Mӧbius transformation of g. The condition 'f and g share four values CM' has been weakened to 'f and g share two values CM and two values IM' by Gundersen [9, 10], as well as by Mues [25].

For Schwarzian differential equation

$$\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = R(z, f) = \frac{P(z, f)}{Q(z, f)}, \tag{1}$$

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Ishizaki [18] showed that if the Schwarzian equation (1) possesses an admissible solution, then $d + 2k\sum_{j=1}^{l}\delta(\alpha_j f) \leq 4k$, where a_j are distinct complex constants, and $d = \deg R(z, f) = \max\{\deg P(z, f), \deg Q(z, f)\}$. In particular, when $R(z, f)$ is independent of z, it is shown that if (1) possesses an admissible solution f, then by some Möbius transformation $w = (af + b)/(cf + d)(ad - bc \neq 0)$, $R(z, f)$ can be reduced to some special forms, see [18, Theorem 3]. Liao and Ye[23] considered differential equation, which is a special type of the Schwarzian differential equation,

$$\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = R(z), \tag{2}$$

and gave the order of meromorphic solutions as follows.

Theorem 1.1. *[23, Theorem 3] Let $P(z)$ and $Q(z)$ be polynomials with $\deg P = m$ and $\deg Q = n$, and let $R(z) = P(z)/Q(z)$. If f is a transcendental meromorphic solution of (2), then $m-n+2k > 0$ and the order $\sigma(f) = (m-n+2k)/2k$.*

For every positive integer n, the forward differences $\Delta^n f(z)$ are defined as

$$\Delta f(z) = f(z + c) - f(z),\ \Delta^{n+1} f(z) = \Delta^n f(z + c) - \Delta^n f(z).$$

We know that $\Delta f(z)$ is considered as difference counterpart of f'. Recently, a number of papers focus on unicity of meromorphic functions sharing values with their shifts or difference operators, see, e.g. [1, 2, 5–8, 13–17, 22, 24, 27, 30]. Some papers studied uniqueness of meromorphic functions concerning meromorphic solutions of difference equations, see, e.g. [8, 15, 27]. Others considered the value distribution and the growth of order of meromorphic solutions of difference equations, see, e.g.[3, 4, 11, 19–21].

Chen and Li[4], Lan and Chen[20] considered the difference counterpart of form

$$\left[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right]^k = R(z, f), \tag{3}$$

which is originated from the Schwarzian differential equation (1), they obtained that the value distribution of meromorphic solutions of (3). Furthermore, Lan and Chen[21] considered the difference equation

$$\left[\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2\right]^k = R(z), \tag{4}$$

which is a special type of equation (3), where k is a positive integer and $R(z)$ is a nonconstant rational function. They obtain

Theorem 1.2. *[21, Theorem 1.3] Let $R(z) = \frac{P(z)}{Q(z)}$ be an irreducible rational function with $\deg P(z) = p$ and $\deg Q(z) = q$. Then*

(i) every transcendental meromorphic solution of (4) satisfies $\sigma(f) \geq 1$; if $p - q + 2k > 0$, then (4) has no rational solutions;

(ii) if $f(z)$ is a meromorphic solution of (4) with finite order, then $\frac{\Delta^2 f(z)}{\Delta f(z)}$ and $\frac{\Delta^3 f(z)}{\Delta f(z)}$ in (4) are nonconstant rational functions;

(iii) every transcendental meromorphic solution $f(z)$ with finite order has at most one Borel exceptional value unless

$$f(z) = b + R_0(z)e^{az},$$

where a, b are complex numbers with $a \neq 0$ and $R_0(z)$ is a nonzero rational function.

(iv) if $p - q + 2k > 0, \sigma(f) < \infty$, then $\Delta f(z)$ has at most one Borel exceptional value unless

$$\Delta f(z) = R_1(z)e^{az},$$

where a is complex number with $a \neq i2k_1\pi$ for any $k_1 \in \mathbb{Z}$, and $R_1(z)$ is a nonzero rational function.

Remark 1.3. *¿From Theorem 1.2, we see if $f(z)$ is a transcendental meromorphic solution of* (4) *with finite order, then $f(z)$ cannot have two finite Borel exceptional values.*

We note that $\Delta f(z)$ lies in the denominator in (4), and so $\Delta f(z) \not\equiv 0$. Thus, $f(z)$ cannot be a merommorphic function with period c. If we remove this restriction, we investigate the properties of meromorphic solutions of equation

$$\left[\Delta^3 f(z)\Delta f(z) - \frac{3}{2}(\Delta^2 f(z))^2\right]^k = R(z)(\Delta f(z))^{2k}, \tag{5}$$

and obtain

Theorem 1.4. *Let $f(z)$ be a transcendental meromorphic solution of equation* (5) *with finite order, where $R(z)$ is a nonconstant rational function. Let $g(z)$ be a meromorphic function and a, b be two distinct constants. If $f(z)$ and $g(z)$ share a, b, ∞ CM, then one of the following statements holds:*

(i) $f(z) \equiv g(z)$;

(ii) $f(z) = Ae^{mz} + B, g(z) = L(f)$, where $A(\neq 0), B$ are constants, $mc = 2k_1\pi i$ for some nonzero integer k_1, $L(f)$ is a Möbius transformation of f;

(iii) $f(z) = a + (b - a)\frac{Ae^{nz}-1}{Be^{mz}-1}$, $g = b + \frac{(b-a)}{A}\frac{A-Be^{(m-n)z}}{Be^{mz}-1}$, where A, B are nonzero constants, $\frac{n}{m}(\neq 1)$ means a rational constant, $mc = 2k_1\pi i$ for some nonzero integer k_1.

2. Lemmas

We now give some preparations.

Lemma 2.1. [3, 11] *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f), \sigma < \infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$,*

$$T(r, f(z + \eta)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).$$

Lemma 2.2. [3] *Let $A_0(z), \ldots, A_n(z)$ be entire functions such that there exists an integer $l, 0 \leq l \leq n$, such that*

$$\sigma(A_l) > \max_{\substack{1\leq j\leq n\\ j\neq l}}\{\sigma(A_j)\}.$$

If $f(z)$ is a meromorphic solution to

$$A_n(z)y(z + n) + \cdots + A_1(z)y(z + 1) + A_0(z)y(z) = 0,$$

then we have $\sigma(f) \geq \sigma(A_l) + 1$.

Lemma 2.3. [29] *Suppose that $n \geq 2$, and let $f_j(z)(j = 1, \ldots, n)$ be meromorphic functions and $g_j(z)(j = 1, \ldots, n)$ be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(ii) when $1 \leq j < k \leq n, g_j(z) - g_k(z)$ is not a constant;

(iii) when $1 \le j \le n, 1 \le h < k \le n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \ \ (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ *is of finite logarithmic measure.*

Then $f_j(z) \equiv 0$. $(j = 1, \dots, n)$

Lemma 2.4. *Let* $f(z)$ *be a finite order meromorphic solution of equation* (4)*, then* $\Delta f(z)$ *is a meromorphic solution of equation*

$$w(z + c) = Q(z)w(z),$$

where $Q(z)$ *is a nonconstant rational function.*

Proof. Set

$$Q(z) = \frac{\Delta f(z + c)}{\Delta f(z)}. \tag{6}$$

We then prove that $Q(z)$ is a nonconstant rational function.

Since $f(z)$ is of finite order, (6) shows $Q(z)$ is also of finite order and

$$\Delta f(z + c) = Q(z)\Delta f(z), \ \ \Delta f(z + 2c) = Q(z + c)\Delta f(z + c) = Q(z + c)Q(z)\Delta f(z).$$

Hence,

$$\begin{cases} \Delta^2 f(z) = \Delta f(z + c) - \Delta f(z) = (Q(z) - 1)\Delta f(z), \\ \Delta^3 f(z) = \Delta^2(\Delta f(z)) = \Delta f(z + 2c) - 2\Delta f(z + c) + \Delta f(z) = (Q(z + c)Q(z) - 2Q(z) + 1)\Delta f(z). \end{cases} \tag{7}$$

We see from (4) that

$$\frac{\Delta^3 f(z)}{\Delta f(z)} - \frac{3}{2}\left(\frac{\Delta^2 f(z)}{\Delta f(z)}\right)^2 = R_1(z), \tag{8}$$

where $R_1(z)$ is some nonconstant rational function. Thus, (7) and (8) show that

$$Q(z + c)Q(z) - 2Q(z) + 1 - \frac{3}{2}(Q(z) - 1)^2 = R_1(z), \tag{9}$$

that is,

$$Q(z + c) = \frac{\frac{3}{2}Q^2(z) - Q(z) + R_1(z) + \frac{1}{2}}{Q(z)}. \tag{10}$$

Since $R_1(z)$ is a nonconstant rational function, we deduce from (9) that $Q(z)$ cannot be a constant. If $Q(z)$ is transcendental, noting that $\frac{3}{2}Q^2(z) - Q(z) + R_1(z) + \frac{1}{2}$ and $Q(z)$ are irreducible, then we apply Valiron-Mohon'ko Theorem to (10), and deduce

$$T(r, Q(z + c)) = 2T(r, Q(z)) + S(r, Q),$$

which contradicts to Lemma 2.1. Hence, $Q(z)$ is a nonconstant rational function. □

Lemma 2.5. *Let* a, b *be two distinct constants,* β, γ *be nonconstant polynomials with* $\deg \beta \neq \deg \gamma$*, and*

$$f(z) = a + (b - a)\frac{e^\beta - 1}{e^\gamma - 1}. \tag{11}$$

Then $f(z)$ *cannot be a meromorphic solution of equation* (4).

Proof. Assume that f is a meromorphic solution of equation (4). Lemma 2.4 shows

$$\Delta f(z+c) = Q(z)\Delta f(z). \tag{12}$$

Without loss of generality, we assume $Q(z)$ is a nonconstant polynomial. Otherwise, we just multiply the dominator of $Q(z)$ of both sides of (12). We now divide our proof into two cases.

Case 2.1. $\deg\beta > \deg\gamma$. Rewriting (11) as

$$f(z) = a_{01}(z)e^{\beta(z)} + a_{00}(z), \tag{13}$$

where

$$a_{01}(z) = \frac{b-a}{e^{\gamma}-1}, \;\; a_{00}(z) = a - \frac{b-a}{e^{\gamma}-1}.$$

Obviously,

$$\sigma(a_{01}) = \sigma(a_{00}) = \deg\gamma < \deg\beta. \tag{14}$$

Since e^{β} is of regular growth order $\deg\beta$, we see a_{01}, a_{00} are small functions of e^{β}. We conclude from (13) that

$$\begin{aligned}\Delta f(z) =& a_{01}(z+c)e^{\beta(z+c)} + a_{00}(z+c) - a_{01}(z)e^{\beta(z)} - a_{00}(z)\\ =&(a_{01}(z+c)e^{\beta(z+c)-\beta(z)} - a_{01}(z))e^{\beta(z)} + a_{00}(z+c) - a_{00}(z)\\ =&a_{11}(z)e^{\beta(z)} + a_{10}(z),\end{aligned} \tag{15}$$

where

$$\begin{cases} a_{11}(z) = a_{01}(z+c)e^{\beta(z+c)-\beta(z)} - a_{01}(z),\\ a_{10}(z) = a_{00}(z+c) - a_{00}(z).\end{cases} \tag{16}$$

We deduce from (14), (16), Lemma 2.1 and $\deg(\beta(z+c)-\beta(z)) = \deg\beta - 1$ that

$$\sigma(a_{11}) \le \max\{\sigma(a_{01}), \deg\beta - 1\} < \deg\beta, \;\; \sigma(a_{10}) \le \sigma(a_{00}) < \deg\beta. \tag{17}$$

We assert that $a_{11}(z) \not\equiv 0$. Otherwise, (16) shows

$$a_{01}(z+c)e^{\beta(z+c)-\beta(z)} - a_{01}(z) = 0. \tag{18}$$

Applying Lemma 2.2 to equation (18), we have

$$\sigma(a_{01}) \ge \sigma(e^{\beta(z+c)-\beta(z)}) + 1 = (\deg\beta - 1) + 1 = \deg\beta,$$

which contradicts with (14).

Substituting (15) into (12), we obtain

$$(a_{11}(z+c)e^{\beta(z+c)-\beta(z)} - Q(z)a_{11}(z))e^{\beta(z)} + a_{10}(z+c) - Q(z)a_{10}(z) = 0.$$

By (17) and $\deg(\beta(z+c)-\beta(z)) = \deg\beta - 1$, applying Lemma 2.3 to the last equality, we have

$$a_{11}(z+c)e^{\beta(z+c)-\beta(z)} - Q(z)a_{11}(z) = 0. \tag{19}$$

Applying Lemma 2.2 to equation (19), we get

$$\sigma(a_{11}) \ge \sigma(e^{\beta(z+c)-\beta(z)}) + 1 = (\deg\beta - 1) + 1 = \deg\beta,$$

which contradicts with (17).

Case 2.2. $\deg\beta < \deg\gamma$. Rewriting (11) as

$$f(z) = a + \frac{b_{00}(z)}{e^{\gamma(z)} - 1}, \tag{20}$$

where

$$b_{00}(z) = (b-a)(e^{\beta(z)} - 1). \tag{21}$$

Thus, we conclude from (20) that

$$\begin{aligned}\Delta f(z) =& \frac{b_{00}(z+c)}{e^{\gamma(z+c)} - 1} - \frac{b_{00}(z)}{e^{\gamma(z)} - 1} = \frac{b_{00}(z+c)e^{\gamma(z)} - b_{00}(z)e^{\gamma(z+c)} - b_{00}(z+c) + b_{00}(z)}{(e^{\gamma(z+c)} - 1)(e^{\gamma(z)} - 1)}\\ =& \frac{b_{11}(z)e^{\gamma(z)} + b_{10}(z)}{(e^{\gamma(z+c)} - 1)(e^{\gamma(z)} - 1)},\end{aligned} \tag{22}$$

where

$$\begin{cases} b_{10}(z) = -b_{00}(z+c) + b_{00}(z) \\ b_{11}(z) = b_{00}(z+c) - b_{00}(z)e^{\gamma(z+c)-\gamma(z)} \end{cases} \tag{23}$$

By (21), (23) and Lemma 2.1, we have

$$\begin{cases} \sigma(b_{10}) \le \sigma(b_{00}) = \deg\beta < \deg\gamma \\ \sigma(b_{11}) \le \max\{\sigma(b_{00}), \sigma(e^{\gamma(z+c)-\gamma(z)})\} = \max\{\deg\beta, \deg\gamma - 1\} < \deg\gamma. \end{cases} \tag{24}$$

We again assert that $b_{11}(z) \not\equiv 0$. Otherwise, (23) shows

$$b_{00}(z+c) - e^{\gamma(z+c)-\gamma(z)}b_{00}(z) = 0. \tag{25}$$

Applying Lemma 2.2 to equation (25), we have

$$\sigma(b_{00}) \ge \sigma(e^{\gamma(z+c)-\gamma(z)}) + 1 = (\deg\gamma - 1) + 1 = \deg\gamma,$$

a contradiction. Substituting (22) into (12), we have

$$\frac{b_{11}(z+c)e^{\gamma(z+c)} + b_{10}(z+c)}{(e^{\gamma(z+2c)} - 1)(e^{\gamma(z+c)} - 1)} = Q(z)\frac{b_{11}(z)e^{\gamma(z)} + b_{10}(z)}{(e^{\gamma(z+c)} - 1)(e^{\gamma(z)} - 1)},$$

or

$$\frac{b_{11}(z+c)e^{\gamma(z+c)} + b_{10}(z+c)}{e^{\gamma(z+2c)} - 1} = Q(z)\frac{b_{11}(z)e^{\gamma(z)} + b_{10}(z)}{e^{\gamma(z)} - 1},$$

or

$$\begin{aligned}& b_{11}(z+c)e^{\gamma(z+c)+\gamma(z)} - Q(z)b_{11}(z)e^{\gamma(z+2c)+\gamma(z)} - Q(z)b_{10}(z)e^{\gamma(z+2c)}\\ & - b_{11}(z+c)e^{\gamma(z+c)} + (Q(z)b_{11}(z) + b_{10}(z+c))e^{\gamma(z)} + Q(z)b_{10}(z) - b_{10}(z+c) = 0.\end{aligned}$$

That is,

$$A_2(z)e^{2\gamma(z)} + A_1(z)e^{\gamma(z)} + A_0(z)e^0 = 0, \tag{26}$$

where

$$\begin{cases} A_0(z) = Q(z)b_{10}(z) - b_{10}(z+c) \\ A_1(z) = -Q(z)b_{10}(z)e^{\gamma(z+2c)-\gamma(z)} - b_{11}(z+c)e^{\gamma(z+c)-\gamma(z)} + Q(z)b_{11}(z) + b_{10}(z+c), \\ A_2(z) = b_{11}(z+c)e^{\gamma(z+c)-\gamma(z)} - Q(z)b_{11}(z)e^{\gamma(z+2c)-\gamma(z)}. \end{cases} \tag{27}$$

By (24), (27) and Lemma 2.1, we have

$$\begin{cases}\sigma(A_0) \le \sigma(b_{10}) < \deg \gamma \\ \sigma(A_1) \le \max\{\sigma(b_{10}), \sigma(b_{11}), \sigma(e^{\gamma(z+2c)-\gamma(z)}), \sigma(e^{\gamma(z+c)-\gamma(z)})\} = \max\{\sigma(b_{10}), \sigma(b_{11}), \deg \gamma - 1\} < \deg \gamma, \\ \sigma(A_2) \le \max\{\sigma(b_{11}), \sigma(e^{\gamma(z+c)-\gamma(z)}), \sigma(e^{\gamma(z+2c)-\gamma(z)})\} = \max\{\sigma(b_{11}), \deg \gamma - 1\}\} < \deg \gamma.\end{cases}$$

Thus, $\sigma(A_j) < \deg \gamma$ ($j = 0, 1, 2$). Since e^{γ} is of regular growth order $\deg \gamma$, we obtain

$$T(r, A_j) = o\{T(r, e^{\gamma})\} = o\{T(r, e^{2\gamma})\}, \quad j = 0, 1, 2.$$

Applying Lemma 2.3 to (26), we have

$$A_2(z) \equiv 0, \quad A_1(z) \equiv 0, \quad A_0(z) \equiv 0.$$

By $A_2(z) \equiv 0$ and (27), we obtain

$$b_{11}(z + c)e^{\gamma(z+c)-\gamma(z)} - Q(z)b_{11}(z)e^{\gamma(z+2c)-\gamma(z)} \equiv 0,$$

or

$$b_{11}(z + c) - Q(z)e^{\gamma(z+2c)-\gamma(z+c)}b_{11}(z) \equiv 0, \tag{28}$$

Applying Lemma 2.2 to equation (28), we have

$$\sigma(b_{11}) \ge \sigma(e^{\gamma(z+2c)-\gamma(z+c)}) + 1 = (\deg \gamma - 1) + 1 = \deg \gamma.$$

which contradicts with (24).

Thus, $f(z)$ of the form (12) cannot be a meromorphic solution of equation (4). □

Lemma 2.6. [19] *Let $A_0(z), \ldots, A_n(z)$ be entire functions of finite order such that among those coefficitets having the maximal order $\sigma = \max\{\sigma(A_k), 0 \le k \le n\}$, exactly one has its type strictly greater than the others. If $f(z) \not\equiv 0$ is a meromorphic solution of equation*

$$A_n(z)f(z + \omega_n) + \cdots + A_1(z)f(z + \omega_1) + A_0(z)f(z) = 0, \tag{29}$$

then $\sigma(f) \ge \sigma + 1$.

Lemma 2.7. [11, 19] *Let w be a transcendental meromorphic solution with finite order of difference equation*

$$P(z, w) = 0,$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function a, where a is a small function with respect to w, then

$$m\left(r, \frac{1}{w - a}\right) = S(r, w).$$

3. Proof of Theorem 1.4

Proof. **(i)** We first support that $\Delta f(z) \not\equiv 0$. Then equation (5) can be changed into equation (4).

Since $f(z)$ and $g(z)$ share a, b, ∞ CM, we have

$$N\left(r, \frac{1}{f - a}\right) = N\left(r, \frac{1}{g - a}\right), \quad N\left(r, \frac{1}{f - b}\right) = N\left(r, \frac{1}{g - b}\right), \quad N(r, f) = N(r, g).$$

By the second fundamental Nevanlinna Theorem, we have

$$\begin{aligned} T(r,g) &\leq N(r,g) + N\left(r, \frac{1}{g-a}\right) + N\left(r, \frac{1}{g-b}\right) + S(r,g) \\ &= N(r,f) + N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) + S(r,g) \\ &\leq 3T(r,f) + S(r,g). \end{aligned}$$

Thus, $g(z)$ is of finite order.

Since $f(z)$ and $g(z)$ share a, b, ∞ CM, we see again that

$$\frac{f(z)-a}{g(z)-a} = e^{\alpha(z)}, \tag{30}$$

and

$$\frac{f(z)-b}{g(z)-b} = e^{\beta(z)}, \tag{31}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials.

Assume, to the contrary, that $f(z) \not\equiv g(z)$. Then from (30) and (31), we obtain

$$e^{\alpha} \not\equiv 1, \;\; e^{\beta} \not\equiv 1, \;\; e^{\alpha} \not\equiv e^{\beta}, \;\; \alpha(z) \not\equiv \beta(z).$$

Again by (30) and (31), we get

$$f(z) = a + (b-a)\frac{e^{\beta}-1}{e^{\beta-\alpha}-1}, \tag{32}$$

or

$$f(z) = a + (b-a)\frac{e^{\beta}-1}{e^{\gamma}-1}, \tag{33}$$

where $\gamma = \beta - \alpha$ is a nonzero polynomial.

If β and γ are both constants, then f is a constant from (33), a contradiction.

If β is a constant and denoting $A = e^{\beta}$, then $A \neq 1$. (32) shows

$$f(z) = a + (b-a)\frac{A-1}{Ae^{-\alpha}-1}.$$

Hence, $f(z)$ has two distinct finite Borel exceptional values a and $a + (b-a)(1-A)$, which contradicts with Remark 1.3.

If α is a constant and denoting $B = e^{-\alpha}$, then $B \neq 1$. (32) shows

$$f(z) = a + (b-a)\frac{e^{\beta}-1}{Be^{\beta}-1}.$$

Thus, $f(z)$ has two distinct finite Borel exceptional values b and $a + \frac{b-a}{B}$, which contradicts with Remark 1.3 again.

If γ is a constant and denoting $A = \frac{b-a}{e^{\gamma}-1}$, $B = a - A$, then A, B are constants. By (33), we have

$$f(z) = a + Ae^{\beta} - A = Ae^{\beta} + B.$$

It is easy to see that $f(z)$ has two Borel values B and ∞. Theorem 1.2 (iii) shows $\deg \beta = 1$. Without loss of generality, we assume $\beta(z) = mz$, then $f(z) = Ae^{mz} + B$, where m is a nonzero constant. Thus,

$$\Delta f(z) = A(e^{mc}-1)e^{mz}, \quad \Delta f(z+c) = Ae^{mc}(e^{mc}-1)e^{mz}. \tag{34}$$

We note that $\Delta f(z) \not\equiv 0$ from (4). Thus, $e^{mc} - 1 \neq 0$ and $\Delta f(z+c) = e^{mc}\Delta f(z)$, which contradicts with Lemma 2.4.

We deduce from (33) and Lemma 2.5 that $\deg\beta = \deg\gamma$, and

$$\Delta f(z) = (b-a)\left(\frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1} - \frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\right). \tag{35}$$

Without loss of generality, we assume $Q(z)$ is a nonconstant polynomial in Lemma 2.4. By (35) and Lemma 2.4, we conclude that

$$\frac{e^{\beta(z+2c)}-1}{e^{\gamma(z+2c)}-1} - \frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1} = Q(z)\left(\frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1} - \frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1}\right),$$

or

$$\frac{e^{\beta(z+2c)}-1}{e^{\gamma(z+2c)}-1} + Q(z)\frac{e^{\beta(z)}-1}{e^{\gamma(z)}-1} = (Q(z)+1)\frac{e^{\beta(z+c)}-1}{e^{\gamma(z+c)}-1},$$

that is,

$$\begin{aligned}
&e^{\beta(z+2c)+\gamma(z+c)+\gamma(z)} + Q(z)e^{\beta(z)+\gamma(z+2c)+\gamma(z+c)} - (Q(z)+1)e^{\beta(z+c)+\gamma(z+2c)+\gamma(z)}\\
&- e^{\beta(z+2c)+\gamma(z+c)} - Q(z)e^{\beta(z)+\gamma(z+c)} - e^{\beta(z+2c)+\gamma(z)} - Q(z)e^{\beta(z)+\gamma(z+2c)}\\
&+ (Q(z)+1)e^{\beta(z+c)+\gamma(z+2c)} + (Q(z)+1)e^{\beta(z+c)+\gamma(z)} - e^{\gamma(z+c)+\gamma(z)}\\
&- Q(z)e^{\gamma(z+2c)+\gamma(z+c)} + (Q(z)+1)e^{\gamma(z+2c)+\gamma(z)} + e^{\beta(z+2c)} - (Q(z)+1)e^{\beta(z+c)}\\
&+ Q(z)e^{\beta(z)} - e^{\gamma(z+2c)} + (Q(z)+1)e^{\gamma(z+c)} - Qe^{\gamma(z)} = 0.
\end{aligned}$$

Rewriting the above equality as

$$A_4(z)e^{\beta(z)+2\gamma(z)} + A_3(z)e^{\beta(z)+\gamma(z)} + A_2(z)e^{2\gamma(z)} + A_1(z)e^{\beta(z)} + A_0(z)e^{\gamma(z)} = 0, \tag{36}$$

where

$$\begin{aligned}
A_4(z) =& e^{\beta(z+2c)-\beta(z)+\gamma(z+c)-\gamma(z)} + Q(z)e^{\gamma(z+2c)+\gamma(z+c)-2\gamma(z)} - (Q(z)+1)e^{\beta(z+c)-\beta(z)+\gamma(z+2c)-\gamma(z)},\\
A_3(z) =& -e^{\beta(z+2c)-\beta(z)+\gamma(z+c)-\gamma(z)} - Q(z)e^{\gamma(z+c)-\gamma(z)} - e^{\beta(z+2c)-\beta(z)} - Q(z)e^{\gamma(z+2c)-\gamma(z)}\\
&\quad + (Q(z)+1)e^{\beta(z+c)-\beta(z)+\gamma(z+2c)-\gamma(z)} + (Q(z)+1)e^{\beta(z+c)-\beta(z)},\\
A_2(z) =& -e^{\gamma(z+c)-\gamma(z)} - Q(z)e^{\gamma(z+2c)+\gamma(z+c)-2\gamma(z)} + (Q(z)+1)e^{\gamma(z+2c)-\gamma(z)}, && (37)\\
A_1(z) =& e^{\beta(z+2c)-\beta(z)} - (Q(z)+1)e^{\beta(z+c)-\beta(z)} + Q(z), && (38)\\
A_0(z) =& -e^{\gamma(z+2c)-\gamma(z)} + (Q(z)+1)e^{\gamma(z+c)-\gamma(z)} - Q(z). && (39)
\end{aligned}$$

Obviously,

$$\begin{cases}\sigma(A_4) \le \max\{\deg\beta - 1, \deg\gamma - 1\}, & \sigma(A_3) \le \max\{\deg\beta - 1, \deg\gamma - 1\},\\ \sigma(A_2) \le \deg\gamma - 1, \quad \sigma(A_1) \le \deg\beta - 1, & \sigma(A_0) \le \deg\gamma - 1.\end{cases}$$

That is,

$$\sigma(A_j) < \deg\beta = \deg\gamma, \qquad (j = 0,1,2,3,4). \tag{40}$$

Thus, equation (36) can be rewritten as

$$A_4(z)e^{\beta(z)+\gamma(z)} + A_3(z)e^{\beta(z)} + A_2(z)e^{\gamma(z)} + A_1(z)e^{\beta(z)-\gamma(z)} + A_0(z) = 0. \tag{41}$$

In the following, we divide our proof into four cases.

Case 3.1. $\deg(\beta+\gamma) < \deg\gamma$. Combining this with $\deg\beta = \deg\gamma$, we get

$$\deg(\beta-\gamma) = \deg\gamma, \;\; \deg(\beta-2\gamma) = \deg\gamma.$$

Thus, $e^{\beta}, e^{\gamma}, e^{\beta-\gamma}, e^{\beta-2\gamma}$ are of regular growth order $\deg\gamma$.

Equation (41) shows that

$$A_3(z)e^{\beta(z)} + A_2(z)e^{\gamma(z)} + A_1(z)e^{\beta(z)-\gamma(z)} + B_0(z) = 0, \tag{42}$$

where

$$B_0(z) = A_4(z)e^{\beta(z)+\gamma(z)} + A_0(z).$$

By this and (40), we obtain $\sigma(B_0) \leq \max\{\sigma(A_4), \sigma(A_0), \deg(\beta+\gamma)\} < \deg\gamma = \deg\beta$. Then

$$\begin{cases} T(r,A_j) = o\{T(r,e^{\beta})\} = o\{T(r,e^{\gamma})\} = oT(r,e^{\beta-\gamma}) = o\{T(r,e^{\beta-2\gamma})\} \quad (j=1,2,3) \\ T(r,B_0) = o\{T(r,e^{\beta})\} = o\{T(r,e^{\gamma})\} = o\{T(r,e^{\beta-\gamma})\} = o\{T(r,e^{\beta-2\gamma})\} \end{cases}$$

Together with (42) and Lemma 2.3, we have

$$B_0(z) \equiv 0, \quad A_j(z) \equiv 0, \quad j = 1,2,3.$$

By $A_2(z) \equiv 0$ and (37), we have

$$-e^{\gamma(z+c)-\gamma(z)} - Q(z)e^{\gamma(z+2c)+\gamma(z+c)-2\gamma(z)} + (Q(z)+1)e^{\gamma(z+2c)-\gamma(z)} \equiv 0.$$

or

$$-Q(z)e^{\gamma(z+2c)-\gamma(z)} + (Q(z)+1)e^{\gamma(z+2c)-\gamma(z+c)} - 1 \equiv 0. \tag{43}$$

In **Case 3.1**, we again split two subcases.

Subcase 3.1.1. $\deg\gamma \geq 2$. Let $H(z) = e^{\gamma(z+c)-\gamma(z)}$, then

$$e^{\gamma(z+2c)-\gamma(z)} = e^{\gamma(z+2c)-\gamma(z+c)+\gamma(z+c)-\gamma(z)} = H(z+c)H(z).$$

Thus, equation (43) can be written as

$$-Q(z)H(z+c)H(z) + (Q(z)+1)H(z+c) - 1 = 0.$$

For any given meromorphic function $w(z)$, set

$$P(z,w) = -Q(z)w(z+c)w(z) + (Q(z)+1)w(z+c) - 1.$$

Then $P(z,H(z)) \equiv 0$. Moreover, $P(z,0) = -1 \not\equiv 0$. By this and Lemma 2.7, we have $m\left(r, \frac{1}{H}\right) = S(r,H)$. But

$$m\left(r, \frac{1}{H}\right) = m(r, e^{\gamma(z)-\gamma(z+c)}) = T\left(r, \frac{1}{H}\right) = T(r,H) + O(1).$$

Thus, $T(r,H) = S(r,H)$, a contradiction.

Subcase 3.1.2. $\deg\gamma = 1$. Let $\gamma(z) = mz + n_1$, where $m \neq 0, n_1$ are complex constants. Then $\gamma(z+2c) - \gamma(z+c) = mc, \gamma(z+2c) - \gamma(z) = 2mc$. Substituting these into (43), we have

$$(e^{mc} - 1)(e^{mc}Q(z) - 1) = 0.$$

Since $Q(z)$ is a nonconstant polynomial, we have $e^{mc}=1$. Then $e^{\gamma(z+c)}=e^{\gamma(z)}$. By $\deg\beta=\deg\gamma$, $\deg(\beta+\gamma)<\deg\beta$, we may assume $\beta(z)=-mz+n_2$, where n_2 is a complex constant. So, $e^{\beta(z+c)}=e^{\beta(z)}$. By $e^{\beta(z+c)}=e^{\beta(z)}, e^{\gamma(z+c)}=e^{\gamma(z)}$ and (32), we see $f(z+c)=f(z)$. Thus, $\Delta f(z)=0$. This contradicts with $\Delta f(z)\not\equiv 0$.

Case 3.2. $\deg(\beta-\gamma)<\deg\gamma$. Equation (41) shows that

$$\left(A_4(z)e^{\beta-\gamma}\right)e^{2\gamma}+\left(A_3(z)e^{\beta-\gamma}+A_2(z)\right)e^{\gamma}+\left(A_1(z)e^{\beta-\gamma}+A_0(z)\right)e^{0}=0, \tag{44}$$

By (40), (44), $\deg(\beta-\gamma)<\deg\gamma$ and Lemma 2.3, we obtain

$$A_4(z)e^{\beta-\gamma}\equiv 0,\ \ A_3(z)e^{\beta-\gamma}+A_2(z)\equiv 0,\ \ A_1(z)e^{\beta-\gamma}+A_0(z)\equiv 0.$$

Substituting (38), (39) and $\beta(z)=\alpha(z)+\gamma(z)$ into the last equality $A_1(z)e^{\beta-\gamma}+A_0(z)\equiv 0$, we have

$$e^{\gamma(z+2c)-\gamma(z)}\left(e^{\alpha(z+2c)}-1\right)-(Q+1)e^{\gamma(z+c)-\gamma(z)}\left(e^{\alpha(z+c)}-1\right)+Q\left(e^{\alpha(z)}-1\right)=0.$$

That is to say, $y(z)=e^{\alpha(z)}-1$ is a meromorphic solution of equation

$$e^{\gamma(z+2c)-\gamma(z)}y(z+2c)-(Q+1)e^{\gamma(z+c)-\gamma(z)}y(z+c)+Qy(z)=0. \tag{45}$$

Since α cannot be a constant, by $\deg(\beta-\gamma)=\deg\alpha<\deg\gamma$, then $\deg\gamma\geq 2$. Set

$$\gamma(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0,$$

where $k\geq 2$ is an integer, $a_k\neq 0, a_{k-1},\ldots,a_0$ are constant. Then

$$\gamma(z+2c)-\gamma(z)=2kca_kz^{k-1}+\cdots,\qquad \gamma(z+c)-\gamma(z)=kca_kz^{k-1}+\cdots.$$

By these, we see in the equation (45), the coefficient $e^{\gamma(z+2c)-\gamma(z)}$ is of order $k-1$ with type $|2kca_k|$, the coefficient $-(Q+1)e^{\gamma(z+c)-\gamma(z)}$ is of order $k-1$ with type $|kca_k|$. By these and applying Lemma 2.6 to equation (45), we have $\sigma(y)\geq(k-1)+1=k=\deg\gamma$. But $\sigma(y)=\sigma(e^{\alpha}-1)=\deg\alpha=\deg(\beta-\gamma)<\deg\gamma$, a contradiction.

Case 3.3. $\deg(\beta-2\gamma)<\deg\gamma$. Equation (41) can be rewritten as

$$A_4(z)e^{\beta(z)}+A_3(z)e^{\beta(z)-\gamma(z)}+A_0(z)e^{-\gamma(z)}+\left(A_2(z)+A_1(z)e^{\beta(z)-2\gamma(z)}\right)=0. \tag{46}$$

By $\deg\beta=\deg\gamma$ and $\deg(\beta-2\gamma)<\deg\gamma$, we have $\deg(\beta-\gamma)=\deg(\beta+\gamma)=\deg\gamma$. By this and (40), we have

$$\begin{cases} T(r,A_j)=o\{T(r,e^{\beta})\}=o\{T(r,e^{\gamma})\}=o\{T(r,e^{\beta-\gamma})\}=o\{T(r,e^{\beta+\gamma})\} \quad (j=0,3,4)\\ T(r,A_2+A_1e^{\beta-2\gamma})=o\{T(r,e^{\beta})\}=o\{T(r,e^{\gamma})\}=o\{T(r,e^{\beta-\gamma})\}=o\{T(r,e^{\beta+\gamma})\}.\end{cases}$$

Combining this with (46) and Lemma 2.3, it follows

$$A_4(z)\equiv 0,\quad A_3(z)\equiv 0,\ \ A_0(z)\equiv 0,\ \ A_2(z)+A_1(z)e^{\beta(z)-2\gamma(z)}\equiv 0.$$

By $A_0(z)\equiv 0$ and (39), we have

$$-e^{\gamma(z+2c)-\gamma(z)}+(Q(z)+1)e^{\gamma(z+c)-\gamma(z)}-Q(z)\equiv 0. \tag{47}$$

If $\deg\gamma\geq 2$, then $\deg(\gamma(z+2c)-\gamma(z))=\deg(\gamma(z+c)-\gamma(z))=\deg\gamma-1\geq 1$. Set $H(z)=e^{\gamma(z+c)-\gamma(z)}$, then $e^{\gamma(z+2c)-\gamma(z)}=H(z+c)H(z)$. Equation (47) can be written as

$$-H(z+c)H(z)+(Q(z)+1)H(z)-Q(z)=0.$$

For any given meromorphic function $w(z)$, set

$$P(z,w)=-w(z+c)w(z)+(Q(z)+1)w(z)-Q(z).$$

Hence, $P(z, H(z)) = 0$. It is easy to see $P(z, 0) = -Q(z) \not\equiv 0$, by this and Lemma 2.7, we have $m\left(r, \frac{1}{H}\right) = S(r, H)$. Thus, $N\left(r, \frac{1}{H}\right) = T(r, H) + S(r, H)$. But $N\left(r, \frac{1}{H}\right) = N\left(r, \frac{1}{e^{\gamma(z+c)-\gamma(z)}}\right) = 0$, a contradiction.

If $\deg\gamma = 1$, let $\gamma(z) = mz + n_1$, where $m \neq 0, n_1$ are constants. Hence, $\gamma(z + 2c) - \gamma(z) = 2mc$, $\gamma(z + c) - \gamma(z) = mc$, substituting these into (47), we get

$$(e^{mc} - 1)(Q(z) - e^{mc}) = 0.$$

Thus, $e^{mc} = 1$. So, $e^{\gamma(z+c)} = e^{\gamma(z)}$.

By $\deg(\beta - 2\gamma) < \deg\beta = \deg\gamma$, we may assume $\beta(z) = 2mz + n_2$, where n_2 is a constant. Then $e^{\beta(z+c)} = e^{2mz+2mc+n_2} = e^{2mz+n_2} = e^{\beta(z)}$. By $e^{\beta(z+c)} = e^{\beta(z)}, e^{\gamma(z+c)} = e^{\gamma(z)}$ and (32), we see $f(z + c) = f(z)$. Then $\Delta f(z) \equiv 0$, a contradiction again.

Case 3.4. $\deg(\beta + \gamma) = \deg(\beta - \gamma) = \deg(\beta - 2\gamma) = \deg\gamma$. By this and (40), for $j = 0, 1, 2, 3, 4$, we have

$$T(r, A_j) = o\{T(r, e^{\beta})\} = o\{T(r, e^{\gamma})\} = o\{T(r, e^{\beta-\gamma})\} = o\{T(r, e^{\beta+\gamma})\} = o\{T(r, e^{\beta-2\gamma})\}.$$

Combining this with Lemma 2.3, we have

$$A_j(z) \equiv 0, \quad j = 0, 1, 2, 3, 4.$$

By $A_2(z) \equiv 0$ and (37), we also obtain (43).

If $\deg\gamma \geq 2$, using the same method as the above **Case 3.1.1**, we get a contradiction.

If $\deg\gamma = 1$, then $\deg\beta = \deg\gamma = 1$. Let $\gamma(z) = mz + n_1, \beta(z) = nz + n_2$, where $m \neq 0, n \neq 0, n_1, n_2$ are complex constants. Then $\gamma(z + 2c) - \gamma(z + c) = mc, \gamma(z + 2c) - \gamma(z) = 2mc$. Substituting these into (43), we have

$$(e^{mc} - 1)(e^{mc}Q(z) - 1) = 0.$$

Since $Q(z)$ is a nonconstant polynomial, we have $e^{mc} = 1$. Then $e^{\gamma(z+c)} = e^{\gamma(z)}$.

By $A_1(z) \equiv 0$, (38) and $\beta(z + 2c) - \beta(z) = 2nc, \beta(z + c) - \beta(z) = nc$, we have

$$(e^{nc} - 1)(e^{nc} - Q(z)) = 0.$$

Since $Q(z)$ is a nonconstant polynomial, we have $e^{nc} = 1$. Then $e^{\beta(z+c)} = e^{\beta(z)}$. By $e^{\beta(z+c)} = e^{\beta(z)}, e^{\gamma(z+c)} = e^{\gamma(z)}$ and (32), we see $f(z + c) = f(z)$. Then $\Delta f(z) \equiv 0$, a contradiction.

(ii) We second support that $\Delta f(z) \equiv 0$. By checking the proof of Theorem 1.4 (i), we also obtain (30)–(34). Thus, we deduce from (34) and $\Delta f(z) \equiv 0$ that $e^{mc} = 1$, and $mc = 2k_1\pi i$ for some nonzero integer k_1. Therefore, we obtain from (31), $\beta(z) = mz$ and $f(z) = Ae^{mz} + B$ that

$$g(z) = \frac{(b + A)f - b(A + B)}{f - B} = L(f),$$

where $L(f)$ is a Möbius transformation of f. Thus, (ii) holds.

(iii) We third support that $\Delta f(z) \equiv 0$. By checking the proof of **subcase 3.1.2, Case 3.3 and Case 3.4** in the Theorem 1.4 (i), we see $\gamma(z) = mz + n_1, \beta(z) = nz + n_2$, where $mc = 2k_1\pi i, nc = 2k_2\pi i$ for some nonzero integer k_1, k_2. Substituting $\gamma(z) = mz + n_1, \beta(z) = nz + n_2$ into (33), we have

$$f(z) = a + (b - a)\frac{e^{nz+n_2} - 1}{e^{mz+n_1} - 1} = a + (b - a)\frac{Ae^{nz} - 1}{Be^{mz} - 1}, \tag{48}$$

where $A = e^{n_2}, B = e^{n_1}$ are nonzero constants, and $\frac{n}{m} = \frac{k_1}{k_2}$ is a rational number. Substituting (48), $\beta(z) = nz+n_2$ into (31), we have

$$g = b + \frac{(b - a)}{A}\frac{A - Be^{(m-n)z}}{Be^{mz} - 1}.$$

By $\alpha(z) = \beta(z) - \gamma(z)$ cannot be a constant, we see $\frac{n}{m} \neq 1$. Thus, (iii) holds. □

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